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INTERNAL STABILIZATION OF A MINDLIN-TIMOSHENKO MODEL BY INTERIOR FEEDBACKS

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Abstract. A Mindlin-Timoshenko model with non constant and non smooth coefficients set in a bounded domain of $\mathbb{R}^d, d \geq 1$ with some internal dissipations is proposed. It corresponds to the coupling between the wave equation and the dynamical elastic system. If the dissipation acts on both equations, we show an exponential decay rate. On the contrary if the dissipation is only active on the elasticity equation, a polynomial decay is shown; a similar result is proved in one dimension if the dissipation is only active on the wave equation.

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1. **Introduction.**

In this paper we consider the internal stabilization of the following Mindlin-Timoshenko (beam/plate) model set in a bounded domain Ω of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, with a Lipschitz boundary Γ (for a simpler model, see Chapter 5 of [18], Chapters 2 and 4 of [17] and [11])

$$\begin{cases} Jw_{tt} = \operatorname{div}(K(\nabla w + u)) - aw_t, \\ \tilde{\rho} u_{tt} = \operatorname{div} C\epsilon(u) - K(\nabla w + u) - bu_t, \end{cases} \qquad \text{in } \Omega \times (0, +\infty), \tag{1}$$

with the boundary conditions

$$u = 0, \quad w = 0 \quad \text{ on } \Gamma \times (0, +\infty), \tag{2}$$

and, finally, the initial conditions

$$u(x, 0) = u^0(x),\, u_t(x, 0) = u^1(x), \quad w(x, 0) = w^0(x),\, w_t(x, 0) = w^1(x), \quad \text{ in } \Omega. \tag{3}$$

If $d = 1$ (resp. $d = 2$) the scalar variable w represents the displacement of the beam (resp. plate) in the vertical direction, while the vectorial variable $u = (u_i)_{i=1}^d$ is the angles of rotation of a filament of the beam (resp. plate).

The coefficients $\tilde{\rho}$ and J are in $L^\infty(\Omega)$ (hence do not depend of the time variable) and are positive definite in the sense that there exist positive constants ρ_0 and J_0 such that

$$\tilde{\rho}(x) \geq \rho_0, \quad J(x) \geq J_0 \text{ for a.e. } x \in \Omega;$$

K belongs to $L^\infty(\Omega)^{d\times d}$, is symmetric and positive definite, i.e., there exists a positive constant k_0 such that

$$X^\top K(x)X \geq k_0, \quad \forall X \in \mathbb{R}^d, \text{ for a.e. } x \in \Omega.$$

Similarly $C = (c_{ijk\ell})$ is a tensor such that

$$c_{ijk\ell} = c_{ji\ell k} = c_{k\ell ij} \in L^\infty(\Omega), \tag{4}$$

all indices running over the integers $1, \cdots, d$. These quantities are related to the constitutive materials of the beam/plate.

As usual for $u = (u_i)_{i=1}^d$, $\epsilon(u)$ is the linear strain tensor defined by

$$\epsilon(u) = (\epsilon_{ij}(u))_{i,j=1}^d \quad \text{ with } \epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

For a $d\times d$ matrix $\epsilon = (\epsilon_{ij})_{i,j=1}^d$ the product $C\epsilon = ((C\epsilon)_{ij})_{i,j=1}^d$ is the $d\times d$ matrix given by

$$(C\epsilon)_{ij} = \sum_{k,\ell=1}^d c_{ijk\ell}\epsilon_{k\ell}.$$

Finally for a (smooth enough) vector valued function $v : \Omega \to \mathbb{R}^d$, $\operatorname{div} v$ is its standard divergence, namely

$$\operatorname{div} v = \sum_{j=1}^d \partial_j v_j,$$

while for a (smooth enough) matrix-valued function $w = (w_{ij}) : \Omega \to \mathbb{R}^{d\times d}$, $\operatorname{div} w$ is its divergence line by line, i.e.,

$$\operatorname{div} w = \Big(\sum_{j=1}^d \partial_j w_{ij}\Big)_{i=1}^d.$$

As usual we assume that C is positive definite in the sense that there exists $\mu_0 > 0$ such that

$$C(x)\epsilon : \epsilon \geq \mu_0|\epsilon|^2, \forall \epsilon \in \mathbb{R}^{d\times d}, \text{ for a.e. } x \in \Omega. \tag{5}$$

Finally the coefficients a and b are also in $L^\infty(\Omega)$ (while the case of a matrix valued b can also be considered) and are nonnegative, i.e.,

$$a(x) \geq 0, \quad b(x) \geq 0 \text{ for a.e. } x \in \Omega.$$

System (1) can be viewed as a coupling between the wave equation in w with the dynamical elastic system in u, the terms $-aw_t$ and $-bu_t$ are internal feedback laws.

In dimension $d = 2$ a standard choice is the case of an homogeneous and isotropic plate of thickness h (see [18, 17]) for which the above parameters are given by

$$\tilde{\rho} = \frac{\rho h^3}{12}, J = \rho h, K = \frac{kEh}{2(1+\mu)},$$

and

$$c_{1111} = c_{2222} = D = \frac{Eh^3}{12(1-\mu^2)}, c_{1122} = D\mu, c_{1212} = 2D(1-\mu),$$

the other coefficients $c_{ijk\ell}$ are either equal to one of these values using the relation (4) or are equal to zero. Here k is called the shear correction coefficient, K is the shear modulus, D is the modulus of flexural rigidity and as usual $E > 0$ is the Young modulus, $\mu \in (0, 1/2)$ is the Poisson ratio and $\rho > 0$ is the mass density per unit volume. With these choices, system (1) takes the form:

$$\left\{\begin{array}{l} \rho h w_{tt} = K \operatorname{div}(\nabla w + u) - a w_t, \\ \frac{\rho h^3}{12} u_{tt} = D(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \operatorname{div} u) - K(\nabla w + u) - b u_t, \end{array}\right. \quad \text{in } \Omega \times (0, +\infty). \tag{6}$$

Under the above assumptions, we easily prove that the system (1)-(3) is well-posed using standard semigroup theory.

The main aim of this paper is to extend to the d-dimensional situation some earlier stabilization results obtained for constant coefficients in dimension 1 (case of a beam) and in dimension 2 only for system (6). To our best knowledge the only existing result in dimension > 1 is for system (6) in dimension 2. For results in one dimension, let us quote the following references: Exponential decay rate obtained in [29] with two interior damping terms, exponential or polynomial decay rates with one interior damping term in [32, 22, 34, 1, 23, 35]. Let us also mention that an alternative approach is to use dissipation on the boundary, we refer to [16, 36, 21, 33] for one-dimensional results. As mentioned in [11], in dimension greather than 2, the coupling is weaker than in dimension 1 and therefore stability results are quite chalenging.

Since in the d-dimensional situation and for discontinuous coefficients, the wave speeds are never equal (see below), we prove the polynomial stability of (1)-(3) if only one feedback is used, i.e., if b is positive definite (a could be zero or non negative). The converse case $b = 0$ and a positive definite is also treated in dimension 1, the multi-dimensional case remains an open (and difficult) problem. Finally if a and b are positive definite we prove the exponential decay rate.

Quite recently the case of feedbacks with delays get an increasing interest, we refer to [24, 26, 37] for the wave equation and to [30] for the Timoshenko system (in one dimension). We then finish this paper by showing that the addition of delayed feedback terms lead to similar decay rates when the additional terms are small enough.

The paper is organized as follows: In section 2 we show that our problem is well-posed by using semi-group theory. The strong stability of the system is analyzed in section 3 by using Benchimol's result [7]. Sections 4 and 5 are devoted to the exponential or polynomial decay of the energy under appropriate sufficient conditions on the damping terms. Finally in section 6 we look at the situation when some delay terms are added.

Let us finish this introduction with some notation used in the remainder of the paper: The $L^2(\Omega)$-inner product (resp. norm) will be denoted by $(\cdot,\cdot)$ (resp. $\|\cdot\|$). The usual norm and semi-norm of $H^s(\Omega)$ $(s \geq 0)$ are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. For $s = 0$ we drop the index s.

2. **Well-posedness of the system.**

We consider the Hilbert space

$$\mathcal{H} = H_0^1(\Omega)^d \times L^2(\Omega)^d \times H_0^1(\Omega) \times L^2(\Omega),$$

equipped with the inner product

$$\langle U, U^* \rangle_{\mathcal{H}} = \int_\Omega \Big(C\epsilon(u) : \epsilon(\bar{u}^*) + \tilde{\rho} v \cdot \bar{v}^* + J y \bar{y}^* + K(\nabla w + u)\cdot(\nabla \bar{w}^* + \bar{u}^*) \Big)\, dx$$

with $U = (u, v, w, y)^\top$, $U^* = (u^*, v^*, w^*, y^*)^\top \in \mathcal{H}$. It is indeed an inner product on $\mathcal{H}$ using Korn's and Poincaré's inequalities, its associated norm being equivalent to the natural norm of $\mathcal{H}$.

By a standard reduction order method, (1)-(3) can be rewritten as the first order evolution equation

$$\left\{\begin{array}{l} U' = \mathcal{A}U, \\ U(0) = U_0 = (u^0, u^1, w^0, w^1)^\top, \end{array}\right. \tag{7}$$

where U is the vector $U = (u, u_t, w, w_t)^\top$ and the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} := \begin{pmatrix} v \\ \tilde{\rho}^{-1}\Big(\operatorname{div} C\epsilon(u) - K(\nabla w + u) - bv\Big) \\ y \\ J^{-1}\Big(\operatorname{div}(K(\nabla w + u)) - ay\Big) \end{pmatrix} \tag{8}$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{(u, v, w, y) \in \mathcal{H}; v \in H_0^1(\Omega)^d, y \in H_0^1(\Omega), \operatorname{div} C\epsilon(u) \in L^2(\Omega)^d, \operatorname{div}(K(\nabla w + u)) \in L^2(\Omega)\right\}.$$

We now prove that the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions over $\mathcal{H}$. For that purpose we need the two following lemmas.

Lemma 2.1. *The operator $\mathcal{A}$ is dissipative and satisfies, for all $U = (u, v, w, y)^\top \in \mathcal{D}(\mathcal{A})$,*

$$\Re \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\int_\Omega (a|y|^2 + b|v|^2)dx \le 0. \tag{9}$$

Proof. Take $U = (u, v, w, y)^\top \in \mathcal{D}(\mathcal{A})$. Then, by the definition of $\mathcal{A}$ we may write

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \int_\Omega \Big(C\epsilon(v) : \epsilon(\bar{u}) + (\operatorname{div} C\epsilon(u) - K(\nabla w + u) - bv) \cdot \bar{v}$$
$$+(\operatorname{div}(K(\nabla w + u)) - ay)\bar{y} + K(\nabla y + v) \cdot (\nabla \bar{w} + \bar{u})\Big)\, dx.$$

By a generalized Green formula (see formula (I.2.18) in [12]), we have

$$\int_\Omega \operatorname{div} C\epsilon(u) \cdot \bar{v}\, dx = -\int_\Omega C\epsilon(u) : \epsilon(\bar{v})\, dx,$$

as well as

$$\int_\Omega \operatorname{div}(K(\nabla w + u))\bar{y}\, dx = -\int_\Omega K(\nabla w + u) \cdot \nabla \bar{y}\, dx.$$

Using these two identities in the previous one directly leads to the conclusion. □

Lemma 2.2. *For all non negative real number λ, $\lambda Id - \mathcal{A}$ is surjective.*

Proof. Fix a non negative real number λ and let $F = (f, g, h, j)^\top \in \mathcal{H}$. We look for $U = (u, v, w, y)^\top \in \mathcal{D}(\mathcal{A})$ solution of

$$\lambda U - \mathcal{A}U = F,$$

or equivalently

$$\left\{\begin{array}{l} \lambda u - v = f \in H_0^1(\Omega)^d, \\ \lambda v - \tilde{\rho}^{-1}\Big(\operatorname{div} C\epsilon(u) - K(\nabla w + u) - bv\Big) = g \in L^2(\Omega)^d, \\ \lambda w - y = h \in H_0^1(\Omega), \\ \lambda y - J^{-1}\Big(\operatorname{div}(K(\nabla w + u)) - ay\Big) = j \in L^2(\Omega). \end{array}\right. \tag{10}$$

Eliminating v and y in these last equations and taking into account the Dirichlet boundary conditions, we are first looking for $u \in H_0^1(\Omega)^d$ and $w \in H_0^1(\Omega)$ solutions of

$$\left\{\begin{array}{l} \lambda^2 u - \tilde{\rho}^{-1}\Big(\operatorname{div} C\epsilon(u) - K(\nabla w + u) - b\lambda u\Big) = g_\lambda \in L^2(\Omega)^d, \\ \lambda^2 w - J^{-1}\Big(\operatorname{div}(K(\nabla w + u)) - a\lambda w\Big) = j_\lambda \in L^2(\Omega), \end{array}\right. \tag{11}$$

where

$$g_\lambda = g + \lambda(1+b)f, \quad j_\lambda = j + \lambda(1+a)h.$$

Multiplying the first identity by a test function $\bar{u}^*$ and the second identity by a test function $\bar{w}^*$, integrating in space and using formal integration by parts, we obtain the weak formulation

$$a_\lambda((u,w),(u^*,w^*)) = \int_\Omega (\tilde{\rho} g_\lambda \cdot \bar{u}^* + J j_\lambda \bar{w}^*)\,dx \quad \forall (u^*,w^*) \in H_0^1(\Omega)^d \times H_0^1(\Omega), \tag{12}$$

where

$$a_\lambda((u,w),(u^*,w^*)) = \int_\Omega \Big(C\epsilon(u) : \epsilon(\bar{u}^*) + K(\nabla w + u)\cdot(\nabla \bar{w}^* + \bar{u}^*) + (b+\tilde{\rho}\lambda)\lambda u\cdot\bar{u}^* + (a+J\lambda)\lambda w \bar{w}^* \Big)\,dx.$$

Since the sesquilinear form a_λ is continuous and coercive on $H_0^1(\Omega)^d \times H_0^1(\Omega)$ (for $\lambda = 0$ this a consequence of Korn's and Poincaré's inequality) and since the right-hand side of (12) defines a continuous linear form on this Hilbert space, by Lax-Milgram's lemma problem (12) has a unique solution $(u,w) \in H_0^1(\Omega)^d \times H_0^1(\Omega)$. This solution is a solution of (11) by taking test functions in the form $(u^*,0)$ with $u^* \in \mathcal{D}(\Omega)^d$ and $(0,w^*)$ with $w^* \in \mathcal{D}(\Omega)$. This leads to the conclusion by seting $v = \lambda u - f$, $y = \lambda w - h$ and remarking that

$$\operatorname{div} C\epsilon(u) = \tilde{\rho}(\lambda^2 u - g_\lambda) + K(\nabla w + u) + b\lambda u$$

belongs to $L^2(\Omega)^d$ as well as

$$\operatorname{div}(K(\nabla w + u)) = J(\lambda^2 w - j_\lambda) + a\lambda w$$

belongs to $L^2(\Omega)$. □

Semigroup theory yields that problem (1)-(3) is well-posed in $\mathcal{H}$:

Theorem 2.3. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions over $\mathcal{H}$, and thus for an initial datum $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C([0,+\infty), \mathcal{H})$ to problem (7). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}).$$

Proof. Theorem I.4.6 of [27], Lemmas 2.1 and 2.2 imply that the domain of $\mathcal{A}$ is dense in $\mathcal{H}$. It then suffices to apply Lumer-Philips's Theorem (see Theorem I.4.3 of [27]). □

3. **Strong stability .** It is proved in [22] in dimension 1 that the system (1)-(3) (and different boundary conditions) is not exponentially stable if $a = 0$ and if the speeds of propagation of the first equation and of the second one are different. Since in the two-dimensional situation, the first equation has two speeds of propagation, we cannot expect to obtain an exponential stability, see [11]. Nevertheless we may hope a strong stability or even better a polynomial stability. For the simpler model (6) with $a = 0$ (in dimension 2), this was proved in [11]. In this section, we concentrate on strong stability results, stronger results are postponed to the next sections.

For that purpose we define the energy of (1)-(3) by

$$E(t) = \frac{1}{2}\int_\Omega \Big(C\epsilon(u) : \epsilon(\bar{u}) + |K^{1/2}(\nabla w + u)|^2 + \tilde{\rho}\,|u_t|^2 + J\,|w_t|^2 \Big)\,dx, \tag{13}$$

which corresponds to the norm of (u, u_t, w, w_t) in $\mathcal{H}$ (up to the factor 1/2).

Proposition 3.1. *The solution (u,w) of (1)-(3) with initial datum in $\mathcal{D}(\mathcal{A})$ satisfies*

$$E'(t) = -\int_\Omega (a\,|w_t|^2 + b\,|u_t|^2)dx \le 0.$$

Therefore the energy is non increasing.

Proof. It suffices to derive the energy (13) for regular solutions and to use systems (1)-(3). The calculations are analogous to those of the proof of the dissipativeness of $\mathcal{A}$ in Lemma 2.1, and then, are left to the reader. □

To get strong stability results, we make use of the following result due to Benchimol [7]:

Theorem 3.2. *Let L be a maximal dissipative linear operator in a complex Hilbert space H. Assume that L has a compact resolvent and that L has no eigenvalues on the imaginary axis. Then e^{tL} is strongly stable, i.e.,*

$$e^{tL}x \to 0 \text{ in } H \text{ as } t \to \infty, \forall x \in H.$$

In view of this theorem we now need to characterize the spectrum of $\mathcal{A}$ on the imaginary axis. This is the aim of the next Lemmas.

Lemma 3.3. *Assume that a and b are positive definite, i.e., there exists $\gamma > 0$ such that*

$$a(x) \geq \gamma, \quad b(x) \geq \gamma, \text{ for a.e. } x \in \Omega.$$

Then $\mathcal{A}$ has no eigenvalues on the imaginary axis.

Proof. Since in Lemma 2.2 we have already shown that 0 belongs to the resolvent set of $\mathcal{A}$, we only need to look at the possible non zero eigenvalue $\lambda \in \mathbb{C} \setminus \{0\}$ of $\mathcal{A}$. For that purpose let $U = (u, v, w, y)^\top \in \mathcal{D}(\mathcal{A})$ be a non trivial solution of

$$\mathcal{A}U = \lambda U,$$

or equivalently (see (10))

$$\begin{cases} \lambda u - v = 0, \\ \lambda v - \tilde{\rho}^{-1}(\operatorname{div} C\epsilon(u) - K(\nabla w + u) - bv) = 0, \\ \lambda w - y = 0, \\ \lambda y - J^{-1}(\operatorname{div}(K(\nabla w + u)) - ay) = 0. \end{cases} \tag{14}$$

From the proof of Lemma 2.2, $(u, w) \in H_0^1(\Omega)^d \times H_0^1(\Omega)$ is the unique solution of (see (12))

$$a_\lambda((u, w), (u^*, w^*)) = 0 \quad \forall (u^*, w^*) \in H_0^1(\Omega)^d \times H_0^1(\Omega).$$

In particular (u, v) satisfies

$$a_0((u, w), (u, w)) + \lambda\alpha(u, w) + \lambda^2\beta(u, w) = 0, \tag{15}$$

where

$$\alpha(u, w) = \int_\Omega (a|w|^2 + b|u|^2)\, dx, \beta(u, w) = \int_\Omega (J|w|^2 + \tilde{\rho}|u|^2)\, dx.$$

Since (u, w) is different from zero (otherwise U is zero), we can normalize it by taking $\beta(u, w) = 1$. In that case

$$\alpha(u, w) \geq \gamma \min\{\frac{1}{\sup_{x\in\Omega} \tilde{\rho}(x)}, \frac{1}{\sup_{x\in\Omega} J(x)}\} > 0.$$

With this normalization, (15) reduces to

$$a_0((u, w), (u, w)) + \lambda\alpha(u, w) + \lambda^2 = 0. \tag{16}$$

The unique solutions of this quadratic equation in λ are then given by

$$\lambda_\pm = \frac{-\alpha(u, w) \pm \sqrt{\alpha(u, w)^2 - 4a_0((u, w), (u, w))}}{2}.$$

We first notice that $a_0((u,w),(u,w)) > 0$ and therefore we distinguish two cases:
1. $\alpha(u,w)^2 - 4a_0((u,w),(u,w)) < 0$. In that case $\lambda_\pm$ are complex with

$$\Re\lambda_\pm = -\frac{\alpha(u,w)}{2},$$

which is negative.
2. $\alpha(u,w)^2 - 4a_0((u,w),(u,w)) \geq 0$. In that case $\lambda_\pm$ are real but negative because in this situation

$$\sqrt{\alpha(u,w)^2 - 4a_0((u,w),(u,w))} < \alpha(u,w).$$

In both cases, we have shown that the real part of any eigenvalue of $\mathcal{A}$ is negative. □

We now want to treat the more interesting cases $a = 0$ or $b = 0$. We start with the first case.

Lemma 3.4. *Assume that b is positive definite. Then $\mathcal{A}$ has no eigenvalues on the imaginary axis.*

Proof. As before let $U = (u,v,w,y)^\top \in \mathcal{D}(\mathcal{A})$ be a non trivial solution of (14) with $\lambda \in \mathbb{C} \setminus \{0\}$. Assume that $\alpha(u,w) = 0$, then $u = 0$ and by the first identity of (14), $v = 0$. Using the second identity of (14), we see that $\nabla w = 0$ and consequently $w = 0$ (due to the Dirichlet boundary conditions on w). This is a contradiction and therefore $\alpha(u,w)$ is always positive. Hence we can use the arguments of the previous Lemma to conclude that $\Re\lambda < 0$. □

Lemma 3.5. *Assume that $b = 0$, that a is positive definite and that all nonzero eigenvectors $u \in H_0^1(\Omega)^d$ with eigenvalue μ^2 ($\mu^2 > 0$) of the elasticity system, i.e., solution of*

$$-\operatorname{div} C\epsilon(u) + Ku = \mu^2\tilde{\rho}u, \tag{17}$$

does not satisfy

$$\operatorname{div}(Ku) = 0.$$

Then $\mathcal{A}$ has no eigenvalues on the imaginary axis.

Proof. Again let $U = (u,v,w,y)^\top \in \mathcal{D}(\mathcal{A})$ be a non trivial solution of (14) with $\lambda \in \mathbb{C} \setminus \{0\}$. Assume that $\alpha(u,w) = 0$, then $w = 0$ and by the third identity of (14), $y = 0$. Using the last identity of (14), we see that $\operatorname{div}(Ku) = 0$. Since the second identity means that u is solution of

$$-\operatorname{div} C\epsilon(u) + Ku = -\lambda^2\tilde{\rho}u,$$

we deduce that $\lambda = i\mu$ for some real number μ. By our assumption we deduce that $u = 0$, which is a contradiction. Therefore $\alpha(u,w)$ is always positive and we conclude as before. □

Remark 3.6. In Lemma 3.5, the assumption on the elasticity system always holds in dimension one, since in that case, $\operatorname{div}(Ku) = 0$ implies that $u = 0$. The situation is more complicated in higher dimension but can be checked on some particular cases. Indeed for the simpler system (6) in the square $(0,\pi)^2$ ($d = 2$), a solution of (17) which is divergence free would be an eigenvector of the (vectorial) Laplace operator with Dirichlet boundary conditions. Hence it should be of the form

$$\sum_{m_1,m_2\in\mathbb{N}^*: m_1^2+m_2^2=n_1^2+n_2^2} a_{(m_1,m_2)} \sin(m_1x_1)\sin(m_2x_2),$$

with $a_{(m_1,m_2)} \in \mathbb{C}^2$, for some $n_1, n_2 \in \mathbb{N}^*$. It is easy to check that the divergence free property implies that all $a_{(m_1,m_2)}$ are zero and therefore this assumption holds for system (6) in a square.

Remark 3.7. The above Lemmas do not give any information about the distance from the eigenvalues to the imaginary axis. In one dimension with different boundary conditions this distance is zero if the speed of propagation are different as suggested by [22]. A similar phenomenon occurs for the simpler model (6) with different boundary conditions, we refer to [11].

The above lemmas show that there is no eigenvalue of $\mathcal{A}$ on the imaginary axis, hence using Benchimol's theorem we deduce the next stability result.

Corollary 3.8. *Assume that $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$ and that the assumptions of Lemma 3.3 or Lemma 3.4 or Lemma 3.5 are satisfied. Then system (1)-(3) is strongly stable.*

The assumption that $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$ is very weak and holds in the following particular cases:
1. Assume that the coefficients are smooth ($C^2(\bar{\Omega})$ is sufficient). If the boundary of Ω is $C^{1,1}$, or if Ω is convex, then

$$D(\mathcal{A}) = H_0^1(\Omega)^d \times (H_0^1(\Omega) \cap H^2(\Omega)^d) \times H_0^1(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \tag{18}$$

and is consequently compactly embedded into $\mathcal{H}$. Indeed in that case, for $(u, v, w, y) \in D(\mathcal{A})$, the conditions

$$\operatorname{div} C\epsilon(u) \in L^2(\Omega)^d, \ \operatorname{div}(K(\nabla w + u)) \in L^2(\Omega),$$

are equivalent to

$$\operatorname{div} C\epsilon(u) \in L^2(\Omega)^d, \ \operatorname{div}(K\nabla w) \in L^2(\Omega),$$

and by standard elliptic regularity results we deduce that $u \in H^2(\Omega)^d$ and $w \in H^2(\Omega)$.
2. Assume that the coefficients are smooth. If Ω is a polygonal domain of the plane ($d = 2$), then

$$D(\mathcal{A}) = H_0^1(\Omega)^2 \times (H_0^1(\Omega)^2 \cap H^s(\Omega)^2) \times H_0^1(\Omega) \times (H_0^1(\Omega) \cap H^s(\Omega)), \tag{19}$$

for some $s \in (1, 2)$ and is consequently compactly embedded into $\mathcal{H}$. As before for $(u, v, w, y) \in D(\mathcal{A})$, the conditions

$$\operatorname{div} C\epsilon(u) \in L^2(\Omega)^2, \ \operatorname{div}(K(\nabla w + u)) \in L^2(\Omega),$$

are equivalent to

$$\operatorname{div} C\epsilon(u) \in L^2(\Omega)^2, \ \operatorname{div}(K\nabla w) \in L^2(\Omega),$$

and by regularity results from [13, 14] we deduce that $u \in H^{1+\epsilon}(\Omega)^2$ and $w \in H^{1+\epsilon}(\Omega)$ for some $\epsilon > 0$.
3. Let Ω be again a polygonal domain of the plane and assume that the coefficients are piecewise smooth, in the sense that Ω is partitioned into a finite number of polygonal subdomains Ω_i such that the coefficients are smooth on each $\bar{\Omega}_i$. Suppose further that $K = kId$ with k piecewise smooth (and positive definite). Then the embedding (19) still holds for some $s \in (1, 2)$. For the elastic part we use similar arguments, namely we still have

$$\operatorname{div} C\epsilon(u) \in L^2(\Omega)^2,$$

and by applying Theorem 4.2 of [25], we deduce that $u \in H^{1+\epsilon}(\Omega)^2$ for some $\epsilon > 0$. The situation is a little bit more delicate for the diffusion part, because our assumptions do not guarantee that $\operatorname{div}(Ku)$ belongs to $L^2(\Omega)$. Hence we introduce the auxiliary unknown

$$v = \nabla w + u,$$

that satisfies

$$\operatorname{div}(kv) \in L^2(\Omega), \operatorname{rot} v \in L^2(\Omega), v \cdot t = 0 \text{ on } \Gamma.$$

Hence by Theorem 3.5 of [10] and Theorem 4.2 of [25], we obtain the regularity

$$v \in H^{\epsilon}(\Omega)^2$$

for some $\epsilon > 0$. From its definition, we deduce that

$$\nabla w \in H^{\epsilon}(\Omega)^2$$

for some $\epsilon > 0$.

4. **Exponential stability.** In this section we want to prove the exponential decay of the energy of solutions of (1)-(3) when a and b are positive definite. For that purpose we use the following result (see [28] or [15]):

Lemma 4.1. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space H is exponentially stable, i.e., satisfies*

$$||e^{t\mathcal{L}}U_0|| \leq C\, e^{-\omega t}||U_0||_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if

$$\rho(\mathcal{L}) \supset \big\{ i\beta \bigm| \beta \in \mathbb{R} \big\} \equiv i\mathbb{R}, \tag{20}$$

and

$$\sup_{\beta \in \mathbb{R}} \|(i\beta - \mathcal{L})^{-1}\| < \infty, \tag{21}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

According to this Lemma we need to check the property (21) (recall that (20) was already studied in the previous section). It is shown in the following lemma.

Lemma 4.2. *Assume that a and b are positive definite. Then the resolvent operator of $\mathcal{A}$ satisfies condition (21).*

Proof. We use a contradiction argument, i.e., we suppose that (21) is false. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (u_n, v_n, w_n, y_n)^\top$ in $\mathcal{D}(\mathcal{A})$ with $\|z_n\|_{\mathcal{H}} = 1$ such that

$$\|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty. \tag{22}$$

By (8), this directly implies that

$$\|i\beta_n u_n - v_n\|_{1,\Omega} \to 0, \tag{23}$$

$$\|i\beta_n w_n - y_n\|_{1,\Omega} \to 0, \tag{24}$$

$$\|i\beta_n \tilde{\rho} v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n\| \to 0, \tag{25}$$

$$\|i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + ay_n\| \to 0. \tag{26}$$

We first notice that

$$\Re\,\langle(i\beta_n - \mathcal{A})z_n, z_n\rangle_{\mathcal{H}} \leq \|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}}\, \|z_n\|_{\mathcal{H}} = \|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}} \tag{27}$$

and, by (9),

$$\Re\,\langle(i\beta_n - \mathcal{A})z_n, z_n\rangle_{\mathcal{H}} = \int_\Omega \left(a|y_n|^2 + b|v_n|^2\right) dx.$$

By the assumptions $a \geq \gamma > 0$ and $b \geq \gamma > 0$ a.e. in Ω, we deduce that

$$v_n \to 0 \text{ in } L^2(\Omega)^d, y_n \to 0 \text{ in } L^2(\Omega). \tag{28}$$

By (23) and (24) we directly get

$$\beta_n u_n \to 0 \text{ in } L^2(\Omega)^d, \beta_n w_n \to 0 \text{ in } L^2(\Omega). \tag{29}$$

This property and (25) imply that

$$(i\beta_n\tilde{\rho}v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n, u_n) \to 0.$$

By Green's formula this is equivalent to

$$i(\tilde{\rho}v_n, \beta_n u_n) + (C\epsilon(u_n), \epsilon(u_n)) + (Ku_n, u_n) + (K\nabla w_n, u_n) + (bv_n, u_n) \to 0.$$

By (28) and (29), the first and last terms tend to zero, while the term

$$(K\nabla w_n, u_n)$$

also tends to zero thanks to (29) and the fact that $\|\nabla w_n\| \lesssim 1$ (due to $\|z_n\|_{\mathcal{H}} = 1$). We then deduce that

$$(C\epsilon(u_n), \epsilon(u_n)) + (Ku_n, u_n) \to 0,$$

which, by Korn's inequality, means that

$$u_n \to 0 \text{ in } H_0^1(\Omega)^d. \tag{30}$$

We now use (26) and again $\|w_n\| \lesssim 1$ to write

$$(i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + ay_n, w_n) \to 0$$

Again using Green's formula this is equivalent to

$$i(Jy_n, \beta_n w_n) + (K\nabla w_n, \nabla w_n) + (Ku_n, \nabla w_n) + (ay_n, w_n) \to 0.$$

By the previous properties, the terms $(Jy_n, \beta_n w_n)$, $(Ku_n, \nabla w_n)$ and (ay_n, w_n) go to zero and therefore we deduce that

$$(K\nabla w_n, \nabla w_n) \to 0.$$

By Poincaré's inequality we obtain

$$w_n \to 0 \text{ in } H_0^1(\Omega). \tag{31}$$

In conclusion, by (28), (30) and (31) we have obtained that

$$\|z_n\|_{\mathcal{H}} \to 0,$$

which contradicts $\|z_n\|_{\mathcal{H}} = 1$. □

Since the two hypotheses of Lemma 4.1 are proved in Lemma 3.3 and Lemma 4.2 we deduce the main result of this section.

Theorem 4.3. *If a and b are positive definite, the system (1)-(3) is exponentially stable.*

Remark 4.4. A natural question is the following: is the assumption that a and b are positive definite on the whole domain needed? In dimension one, a positive answer is given in [31], but in higher dimension it looks like a problem of propagation of waves and some kind of geometric control conditions could be expected. Since our system is a coupling between the dynamical elasticity system and the wave equation, we should combine the arguments from [2, 3, 9] for the wave equation with the ones from [4] for the elasticity system. This is not an easy task that is outside the scope of this paper.

5. **Polynomial stability.** Our main goal is here to prove the polynomial decay of the energy of solutions of (1)-(3) in some particular but quite general situations. For that purpose we use the following result from Theorem 2.4 of [8] (see also [5, 6, 19] for weaker variants).

Lemma 5.1. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space satisfies*

$$||e^{t\mathcal{L}}U_0|| \leq C\, t^{-\frac{1}{l}}||U_0||_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}), \quad \forall t > 1,$$

as well as

$$||e^{t\mathcal{L}}U_0|| \leq C\, t^{-1}||U_0||_{\mathcal{D}(\mathcal{L}^l)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}^l), \quad \forall t > 1,$$

for some constant $C > 0$ and for some positive integer l if (20) holds and if

$$\limsup_{|\beta|\to\infty} \frac{1}{\beta^l} \,\|(i\beta - \mathcal{L})^{-1}\| < \infty. \tag{32}$$

As (20) was already studied in section 3, it remains to check the property (32).

5.1. **The case b positive definite.** This is the commonly used assumption made in the literature. In our general situation, the next lemma show that (32) holds with $\mathcal{L} = \mathcal{A}$ and $l = 6$.

Lemma 5.2. *Assume that b is positive definite and that $K \in C^1(\bar{\Omega})^{d\times d}$. Then the resolvent operator of $\mathcal{A}$ satisfies condition (32) for $l = 6$.*

Proof. As before we use a contradiction argument, i.e., we suppose that (32) is false for $l \geq 6$. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (u_n, v_n, w_n, y_n)^\top$ in $\mathcal{D}(\mathcal{A})$ with $\|z_n\|_{\mathcal{H}} = 1$ such that

$$\beta_n^l \,\|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty. \tag{33}$$

By (8), this directly implies that

$$\beta_n^l\|i\beta_n u_n - v_n\|_{1,\Omega} \to 0, \tag{34}$$

$$\beta_n^l\|i\beta_n w_n - y_n\|_{1,\Omega} \to 0, \tag{35}$$

$$\beta_n^l\|i\beta_n\tilde{\rho}v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n\| \to 0, \tag{36}$$

$$\beta_n^l\|i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + ay_n\| \to 0. \tag{37}$$

By (27) and by (9), we get

$$\beta_n^l \Re\,\langle(i\beta_n - \mathcal{A})z_n, z_n\rangle_{\mathcal{H}} = \beta_n^l \int_\Omega \left(a|y_n|^2 + b|v_n|^2\right) dx.$$

By the assumption $b \geq b_0 > 0$ a.e. in Ω, we deduce that

$$\beta_n^{l/2}\|v_n\| \to 0. \tag{38}$$

By (34) we directly get

$$\beta_n^j v_n - i\beta_n^{j+1}u_n \to 0 \text{ in } L^2(\Omega)^d, \forall j \leq l.$$

Then by taking $j = l/2$ and by (38) we obtain

$$\beta_n^{l/2+1}u_n \to 0 \text{ in } L^2(\Omega)^d. \tag{39}$$

This property and (36) imply that (since $3l/2 + 1 \geq 4$)

$$\beta_n^4(i\beta_n\tilde{\rho}v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n, u_n) \to 0.$$

By Green's formula this is equivalent to

$$i(\beta_n^2\tilde{\rho}v_n, \beta_n^3 u_n) + \beta_n^4(C\epsilon(u_n), \epsilon(u_n)) + \beta_n^4(Ku_n, u_n) + (K\nabla w_n, \beta_n^4 u_n) + (b\beta_n^2 v_n, \beta_n^2 u_n) \to 0.$$

By (38) and (39), the first and last terms tend to zero, while the term

$$(K\nabla w_n, \beta_n^4 u_n)$$

tend also to zero thanks to (39) and the fact that $\|\nabla w_n\| \lesssim 1$ (due to $\|z_n\|_{\mathcal{H}} = 1$). We then deduce that

$$\beta_n^4 (C\epsilon(u_n), \epsilon(u_n)) + \beta_n^4 (Ku_n, u_n) \to 0,$$

which, by Korn's inequality, means that

$$\beta_n^2 u_n \to 0 \text{ in } H_0^1(\Omega)^d. \tag{40}$$

Now using (36) and the fact that $\|w_n\| \lesssim 1$ we may write

$$\beta_n^2 (i\beta_n \tilde{\rho} v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + b v_n, m w_n) \to 0,$$

where m is the multiplier $m(x) = K^{-1}(x)x$. Again by Green's formula this is equivalent to

$$i(\beta_n^2 \tilde{\rho} v_n, m\beta_n w_n) + (\beta_n^2 C\epsilon(u_n), \epsilon(m w_n)) + (\beta_n^2 K u_n, m w_n) + (b\beta_n^2 v_n, m w_n) + \beta_n^2 (K\nabla w_n, m w_n) \to 0.$$

Since $\|w_n\|_{1,\Omega} \lesssim 1$, $\beta_n \|w_n\| \lesssim 1$ and using (38), (39) and (40), we deduce that

$$\beta_n^2 (\nabla w_n, x w_n) \to 0. \tag{41}$$

But a direct application of Green's formula yields

$$(\nabla w_n, x w_n) = -\frac{d}{2} \int_\Omega |w_n|^2 \, dx.$$

Hence (41) is equivalent to

$$\beta_n w_n \to 0 \text{ in } L^2(\Omega). \tag{42}$$

This property and (35) directly yield

$$\|y_n\|^2 \to 0. \tag{43}$$

We now use (37) and again $\|w_n\| \lesssim 1$ to write

$$(i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + a y_n, w_n) \to 0$$

Again using Green's formula this is equivalent to

$$i(J y_n, \beta_n w_n) + (K\nabla w_n, \nabla w_n) - (K u_n, \nabla w_n) + (a y_n, w_n) \to 0.$$

By the previous properties, the terms $(J y_n, \beta_n w_n)$, $(K u_n, \nabla w_n)$ and $(a y_n, w_n)$ go to zero and therefore we deduce that

$$(K\nabla w_n, \nabla w_n) \to 0.$$

By Poincaré's inequality we obtain

$$w_n \to 0 \text{ in } H_0^1(\Omega). \tag{44}$$

In conclusion, by (38), (40), (44) and (43) we obtain

$$\|z_n\|_{\mathcal{H}} \to 0,$$

which contradicts $\|z_n\|_{\mathcal{H}} = 1$. □

Since the two hypotheses of Lemma 5.1 are proved in Lemma 3.4 and Lemma 5.2 we deduce the main result of this paper.

Theorem 5.3. *If b is positive definite and that $K \in C^1(\bar{\Omega})^{d\times d}$, then there exists $C > 0$ such that for all $U_0 \in \mathcal{D}(\mathcal{A}^6)$, the solution of system (1)-(3) satisfies the following estimate*

$$E(t) \leq C\, t^{-2}\, \|U_0\|^2_{\mathcal{D}(\mathcal{A}^6)}, \forall t > 1. \tag{45}$$

Remark 5.4. The previous Lemma improves the result from Theorem 4.2 of [11] where the obtained decay rate is t^{-1} for data in $D(\mathcal{A}^4)$. Indeed using Proposition 3.1 of [5] , the decay (45) is equivalent to

$$E(t) \leq C(\gamma)\, t^{-2\gamma}\ \|U_0\|^2_{\mathcal{D}(\mathcal{A}^{6\gamma})}\,, \forall t > 1,$$

for all $\gamma > 0$.

5.2. **The case $b = 0$ in one dimension.** According to Remark 3.6, the case $b = 0$ is difficult to treat in dimension higher than one. We conjecture that in that case no polynomial stability holds but we were not able to prove it. In order to give a positive result, we have treated the case $d = 1$.

Lemma 5.5. *Assume that $d = 1$, $b = 0$, a is positive definite and $K \in C^1(\bar{\Omega})$. Then the resolvent operator of $\mathcal{A}$ satisfies condition (32) for $l = 6$.*

Proof. We still use a contradiction argument, i.e., we suppose that (32) is false for $l \geq 6$. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (u_n, v_n, w_n, y_n)^\top$ in $\mathcal{D}(\mathcal{A})$ with $\|z_n\|_{\mathcal{H}} = 1$ satisfying (33) or equivalently (34-(37). As before the assumption $a > a_0 > 0$ a.e. in Ω yields

$$\beta_n^{l/2}\|y_n\| \to 0, \tag{46}$$

and hence

$$\beta_n^{l/2+1} w_n \to 0 \text{ in } L^2(\Omega). \tag{47}$$

This property and (37) imply that

$$\beta_n^4(i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + a y_n, w_n) \to 0.$$

By Green's formula this is equivalent to

$$i\beta_n^5(J y_n, w_n) + \beta_n^4(K\nabla w_n, \nabla w_n) + \beta_n^4((K u_n)', w_n) + \beta_n^4(a y_n, w_n) \to 0.$$

By the previous results, all the terms except the second one tend to zero, and therefore we deduce that

$$\beta_n^4(K\nabla w_n, \nabla w_n) \to 0,$$

that implies that

$$\beta_n^2 w_n \to 0 \text{ in } H_0^1(\Omega). \tag{48}$$

Now multiplying (37) by xKu_n and integration yields

$$\beta_n^2(i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + a y_n, xKu_n) \to 0.$$

By Green's formula we obtain equivalently

$$i\beta_n^3(J y_n, xKu_n) + \beta_n^2(K w_n', (xKu_n)') - \beta_n^2((K u_n)', xKu_n) + (a y_n, xKu_n) \to 0.$$

Again all the terms except the third one tend to zero, and therefore we deduce that

$$\beta_n^2((K u_n)', xKu_n) \to 0.$$

But Green's formula leads to

$$((K u_n)', xKu_n) = -(K u_n, (xKu_n)') = -(K u_n, K u_n) - (K u_n, (K u_n)'),$$

which shows that (here we need the assumption $d = 1$, see Remark 5.6)

$$((K u_n)', xKu_n) = -\frac{1}{2}(K u_n, K u_n).$$

From these properties, we have obtained that

$$\beta_n u_n \to 0 \text{ in } L^2(\Omega). \tag{49}$$

Hence multiplying (36) by u_n and integrating we have

$$(i\beta_n\tilde{\rho}v_n - (Cu_n')' + K(w_n' + u_n), u_n) \to 0,$$

or equivalently

$$i\beta_n(\tilde{\rho}v_n, u_n) + (Cu_n', u_n') + (Ku_n, u_n) + (Kw_n', u_n) \to 0.$$

The first and last terms tending to zero, we deduce that

$$(Cu_n', u_n') + (Ku_n, u_n) \to 0,$$

which yields

$$u_n \to 0 \text{ in } H_0^1(\Omega). \tag{50}$$

Finally using (34) we have

$$(i\beta_n u_n - v_n, v_n) \to 0,$$

and with the help of (49) we deduce that

$$v_n \to 0 \text{ in } L^2(\Omega). \tag{51}$$

The properties (46), (48), (50) and (51) furnish the contradiction. □

Remark 5.6. The above arguments fail in dimension $d \geq 2$ since in that case

$$(\operatorname{div}(Ku_n), Ku_n \cdot x) \neq -\frac{1}{2}(Ku_n, Ku_n).$$

In fact in dimension $d \geq 2$, (48) is still valid and therefore one can show that

$$\beta_n^2 \operatorname{div}(Ku_n) \to 0 \text{ in } H^{-1}(\Omega).$$

Unfortunately this property does not give any information on the convergence of u_n to 0 in $L^2(\Omega)^d$.

As before, the two hypotheses of Lemma 5.1 being proved in Lemma 3.5 and Lemma 5.5 we deduce the following result.

Theorem 5.7. *If $d = 1$, $b = 0$, a is positive definite and $K \in C^1(\bar{\Omega})$, then there exists $C > 0$ such that for all $U_0 \in \mathcal{D}(\mathcal{A}^6)$, the solution of system (1)-(3) satisfies the estimate (45).*

Remark 5.8. As in the previous section, we may ask if the positiveness of a and/or b on the whole domain is also needed? In dimension one, we refer to [34, 35] for the treatment of this question. In higher dimension the question is more delicate, see [20] for the wave equation in a square.

6. **Addition of delay feedback terms.** We finish this paper by looking at the same problem as before but with additional feedback terms with delay. Namely given a delay $\tau > 0$ we consider the problem

$$\left\{ \begin{array}{l} Jw_{tt} = \operatorname{div}(K(\nabla w + u)) - aw_t - a_0 w_t(t-\tau), \\ \tilde{\rho}u_{tt} = \operatorname{div} C\epsilon(u) - K(\nabla w + u) - bu_t - b_0 u_t(t-\tau), \end{array} \right. \quad \text{in } \Omega \times (0, +\infty), \tag{52}$$

with the boundary conditions

$$u = 0, \quad w = 0 \quad \text{on } \Gamma \times (0, +\infty), \tag{53}$$

and, finally, the initial conditions

$$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \quad w(x,0) = w^0(x),\ w_t(x,0) = w^1(x), \quad \text{in } \Omega, \tag{54}$$

$$u_t(x,t-\tau) = f^0(x,t-\tau),\ w_t(x,t-\tau) = f^1(x,t-\tau), \quad \text{in } \Omega \times (0,\tau). \tag{55}$$

The hypotheses on the coefficients are the same as before and for the moment a_0 and b_0 are also in $L^\infty(\Omega)$ and non negative.

6.1. **Existence results.** The existence of a solution of this new system (52)-(55) is by now standard and is proved by introducing the new unknowns (see [24, 26, 37]):

$$z_1(\rho,x,t)=u_t(x,t-\rho\tau),\quad z_2(\rho,xt)=w_t(x,t-\rho\tau),\forall(x,\rho,t)\in\Omega\times(0,1)\times(0,\tau).$$

Hence formally problem (52) is equivalent to

$$\left\{\begin{array}{l} Jw_{tt}=\operatorname{div}(K(\nabla w+u))-aw_t-a_0z_2(1,\cdot,\cdot),\\ \tilde\rho u_{tt}=\operatorname{div}C\epsilon(u)-K(\nabla w+u)-bu_t-b_0z_1(1,\cdot,\cdot),\end{array}\right. \quad \text{in } \Omega\times(0,+\infty), \tag{56}$$

$$\left\{\begin{array}{l} \tau\partial_t z_1+\partial_\rho z_1=0,\\ \tau\partial_t z_2+\partial_\rho z_2=0,\end{array}\right. \quad \text{in } (0,1)\times\Omega\times(0,+\infty), \tag{57}$$

with the boundary conditions (53) and initial conditions (54) as well as

$$z_1(0,x,t)=u_t(x,t),\quad z_2(0,x,t)=w_t(x,t),\forall(x,t)\in\Omega\times(0,+\infty). \tag{58}$$

$$z_1(\rho,x,0)=f^0(x,-\tau\rho),\ z_2(\rho,x,0)=f^1(x,-\tau\rho),\quad \text{in } (0,1)\times\Omega. \tag{59}$$

This last problem is well-posed in

$$\mathcal{H}_1=H_0^1(\Omega)^d\times L^2(\Omega)^d\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(0,1;L^2(\Omega)^d)\times L^2(0,1;L^2(\Omega)),$$

equipped with the inner product

$$\begin{aligned}\langle U,U^*\rangle_{\mathcal{H}_1} &= \int_\Omega\Big(C\epsilon(u):\epsilon(\bar u^*)+\tilde\rho v\cdot\bar v^*+Jy\bar y^*+K(\nabla w+u)\cdot(\nabla\bar w^*+\bar u^*)\Big)\,dx\\ &+\int_0^1\int_\Omega\Big(\xi_1(x)z_1(\rho,x)\cdot z_1^*(\rho,x)+\xi_2(x)z_2(\rho,x)z_2^*(\rho,x)\Big)\,dx\end{aligned}$$

with $U=(u,v,w,y,z_1,z_2)^\top$, $U^*=(u^*,v^*,w^*,y^*,z_1^*,z_2^*)^\top\in\mathcal{H}$, for some positive definite function $\xi_1,\xi_2\in L^\infty(\Omega)$ fixed below.

Indeed by introducing the vectorial unknown $U=(u,u_t,w,w_t,z_1,z_2)^\top$ we see that problem (56)-(59) is equivalent to

$$\left\{\begin{array}{l} U'=\mathcal{A}_1U,\\ U(0)=U_0=(u^0,u^1,w^0,w^1,f^0(\cdot,-\tau\cdot),f^1(\cdot,-\tau\cdot))^\top,\end{array}\right. \tag{60}$$

where the operator $\mathcal{A}_1:\mathcal{D}(\mathcal{A}_1)\to\mathcal{H}_1$ is defined by

$$\mathcal{A}_1\begin{pmatrix}u\\ v\\ w\\ y\\ z_1\\ z_2\end{pmatrix}:=\begin{pmatrix} v\\ \tilde\rho^{-1}\Big(\operatorname{div}C\epsilon(u)-K(\nabla w+u)-bv-b_0z_1(1,\cdot)\Big)\\ y\\ J^{-1}\Big(\operatorname{div}(K(\nabla w+u))-ay-a_0z_2(1,\cdot)\\ -\tau^{-1}\partial_\rho z_1\\ -\tau^{-1}\partial_\rho z_2\end{pmatrix}$$

with domain

$$\begin{aligned}\mathcal{D}(\mathcal{A}_1):=\Big\{&(u,v,w,y,z_1,z_2)\in\mathcal{H}_1; v\in H_0^1(\Omega)^d, y\in H_0^1(\Omega), \operatorname{div}C\epsilon(u)\in L^2(\Omega)^d,\\ &\operatorname{div}(K(\nabla w+u))\in L^2(\Omega),\\ &z_1(0,x)=v(x),\quad z_2(0,x)=y(x),\forall x\in\Omega\Big\}.\end{aligned}$$

Hence using the arguments from section 2.2 of [24] we can prove the next results.

Theorem 6.1. *If*

$$a_0 \leq a, \quad b_0 \leq b \text{ a. e. in } \Omega, \tag{61}$$

then by choosing

$$\xi_1 = \tau b, \quad \xi_2 = \tau a, \tag{62}$$

the operator $\mathcal{A}_1$ is maximal dissipative in $\mathcal{H}_1$. Therefore $\mathcal{A}_1$ generates a C_0-semigroup of contractions over $\mathcal{H}_1$.

Note that the dissipativeness of $\mathcal{A}_1$ comes from the identity

$$\begin{aligned} \Re \langle \mathcal{A}_1 U, U \rangle_{\mathcal{H}_1} &= \int_\Omega \Big((\frac{\xi_2}{2\tau} - a)|y|^2 - a_0 z_2(1,\cdot)\bar{y} - \frac{\xi_2}{2\tau}|z_2(1,\cdot)|^2 \Big)\, dx \\ &+ \int_\Omega \Big((\frac{\xi_1}{2\tau} - b)|v|^2 - a_0 z_1(1,\cdot)\bar{v} - \frac{\xi_1}{2\tau}|z_1(1,\cdot)|^2 \Big)\, dx. \end{aligned} \tag{63}$$

Hence by Young's inequality we get

$$\begin{aligned} \Re \langle \mathcal{A}_1 U, U \rangle_{\mathcal{H}_1} &\leq \int_\Omega \Big((\frac{\xi_2}{2\tau} - a + \frac{a_0}{2})|y|^2 + (\frac{a_0}{2} - \frac{\xi_2}{2\tau}) z_2(1,\cdot)^2 \Big)\, dx \\ &+ \int_\Omega \Big((\frac{\xi_1}{2\tau} - b + \frac{b_0}{2})|v|^2 + (\frac{b_0}{2} - \frac{\xi_1}{2\tau}) z_1(1,\cdot)^2 \Big)\, dx. \end{aligned} \tag{64}$$

With the choice (62), we see that this last right-hand side is non positive.

6.2. **Stability results.** For the sake of shortness we here only treat the case b positive definite.

Here the energy of our system is defined by

$$\begin{aligned} E(t) &= \frac{1}{2} \int_\Omega \Big(C\epsilon(u) : \epsilon(\bar{u}) + K|\nabla w + u)|^2 + \tilde{\rho}\, |u_t|^2 + J\, |w_t|^2 \Big)\, dx \\ &+ \int_0^1 \int_\Omega \Big(\xi_1(x)|u_t(x, t-\rho\tau)|^2 + \xi_2(x)|w_t(x, t-\rho\tau)|^2 \Big)\, dx, \end{aligned}$$

which corresponds to the norm of $(u, u_t, w, w_t, z_1, z_2)$ in $\mathcal{H}_1$ (up to the factor $1/2$). By the estimate (64) and the choice (62), we readily check that

$$E'(t) \leq 0, \forall t > 0.$$

To prove polynomial decay, we again make use of Lemma 5.1. Hence we first look at possible eigenvalues of $\mathcal{A}_1$ on the imaginary axis.

Lemma 6.2. *Assume that b is positive definite, that (61) holds and that $b - b_0$ is also positive definite, i.e. there exists $\beta > 0$ such that*

$$\beta < b - b_0 \text{ a. e. in } \Omega.$$

Then $\mathcal{A}_1$ has no eigenvalues on the imaginary axis.

Proof. Let $\lambda = \imath\zeta$ with $\zeta \in \mathbb{R}$ and $U = (u, v, w, y, z_1, z_2)^\top \in \mathcal{D}(\mathcal{A}_1)$ be a non trivial solution of

$$\mathcal{A}_1 U = \imath\zeta U.$$

Then we get equivalently

$$\left\{\begin{array}{l} \imath\zeta u - v = 0, \\ \imath\zeta v - \tilde{\rho}^{-1}(\operatorname{div} C\epsilon(u) - K(\nabla w + u) - bv - b_0 z_1(1,\cdot)) = 0, \\ \imath\zeta w - y = 0, \\ i\zeta y - J^{-1}(\operatorname{div}(K(\nabla w + u)) - ay - a_0 z_2(1,\cdot)) = 0, \\ i\zeta z_1 + \tau^{-1}\partial_\rho z_1 = 0, \\ i\zeta z_2 + \tau^{-1}\partial_\rho z_2 = 0. \end{array}\right. \tag{65}$$

Then these last two equations and the conditions $z_1(0,\cdot) = v$, $z_2(0,\cdot) = y$ lead to

$$z_1 = ve^{-\imath\zeta\tau\rho}, \quad z_2 = ye^{-\imath\zeta\tau\rho}.$$

Eliminating v, y, $z_1(1,\cdot)$ and $z_2(1,\cdot)$ in the second and fourth equations, we get

$$-\zeta^2\tilde{\rho}v - \operatorname{div} C\epsilon(u) + K(\nabla w + u) + (b + b_0 e^{-\imath\zeta\tau})u = 0,$$
$$-\zeta^2 Jy - \operatorname{div}(K(\nabla w + u)) + (a + a_0 e^{-\imath\zeta\tau})w = 0.$$

Multiplying the first equation by $\bar{u}$ and the second one by $\bar{w}$ and integrating in Ω, we get (with the notations of the proof of Lemma 3.3)

$$a_0((u,w),(u,w)) + \imath\zeta\alpha_\zeta(u,w) - \zeta^2\beta(u,w) = 0,$$

where

$$\alpha_\zeta(u,w) = \int_\Omega \left((a + a_0 e^{-\imath\zeta\tau})|w|^2 + (b + b_0 e^{-\imath\zeta\tau})|u|^2\right) dx.$$

If $\zeta = 0$, we directly get that $a_0((u,w),(u,w)) = 0$ which is impossible. On the other hand if $\zeta \neq 0$, then by taking the imaginary part of the above identity we find that

$$\Re \int_\Omega \left((a + a_0 e^{-\imath\zeta\tau})|w|^2 + (b + b_0 e^{-\imath\zeta\tau})|u|^2\right) dx = 0,$$

or equivalently

$$\int_\Omega \left((a + a_0\cos(\zeta\tau))|w|^2 + (b + b_0\cos(\zeta\tau))|u|^2\right) dx = 0.$$

By our assumption (61) this implies that

$$\int_\Omega (b - b_0)|u|^2\, dx = 0.$$

As $b - b_0$ is positive definite, we deduce that $u = 0$. Hence $v = z_1 = 0$ and coming back to the second identity of (65) we obtain

$$K\nabla w = 0.$$

Hence $w = 0$. This leads to a contradiction and to the conclusion. □

Lemma 6.3. *Under the assumptions of Lemma 6.2, the resolvent operator of $\mathcal{A}_1$ satisfies condition (32) for $l \geq 6$.*

Proof. We again use a contradiction argument, i.e., we suppose that (32) is false for $l \geq 6$. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (u_n, v_n, w_n, y_n, z_{1,n}, z_{2,n})^\top$ in $\mathcal{D}(\mathcal{A}_1)$ with $\|z_n\|_{\mathcal{H}_1} = 1$ such that

$$\beta_n^l \,\|(i\beta_n - \mathcal{A}_1)z_n\|_{\mathcal{H}_1} \to 0 \text{ as } n \to \infty. \tag{66}$$

By (5), this directly implies that

$$\beta_n^l \|i\beta_n u_n - v_n\|_{1,\Omega} \to 0, \tag{67}$$

$$\beta_n^l \|i\beta_n w_n - y_n\|_{1,\Omega} \to 0, \tag{68}$$

$$\beta_n^l \|i\beta_n \tilde{\rho} v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n + b_0 z_{1,n}(1,\cdot)\| \to 0, \tag{69}$$

$$\beta_n^l \|i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + a y_n + a_0 z_{2,n}(1,\cdot)\| \to 0, \tag{70}$$

$$\beta_n^l \|i\beta_n z_{1,n} + \tau^{-1}\partial_\rho z_{1,n}\|_{(0,1)\times\Omega} \to 0, \tag{71}$$

$$\beta_n^l \|i\beta_n z_{2,n} + \tau^{-1}\partial_\rho z_{2,n}\|_{(0,1)\times\Omega} \to 0. \tag{72}$$

By (27), (64), the choice (62) and the hypothesis $\beta < b - b_0$ a. e. in Ω, we get

$$\beta_n^l \Re \langle (i\beta_n - \mathcal{A}_1) z_n, z_n \rangle_{\mathcal{H}} \geq \beta_n^l \frac{\beta}{2} \int_\Omega \left(|v_n|^2 + |z_{1,n}(1,\cdot)|^2\right) dx.$$

Hence we deduce that

$$\beta_n^{l/2} (\|v_n\| + \|z_{1,n}(1,\cdot)\|) \to 0. \tag{73}$$

As before by (67) and (73) we directly get (39). This property and (69) imply that

$$\beta_n^4 (i\beta_n \tilde{\rho} v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n + b_0 z_{1,n}(1,\cdot), u_n) \to 0.$$

By Green's formula this is equivalent to

$$i(\beta_n^2 \tilde{\rho} v_n, \beta_n^3 u_n) + \beta_n^4 (C\epsilon(u_n), \epsilon(u_n)) + \beta_n^4 (K u_n, u_n) + (K\nabla w_n, \beta_n^4 u_n) + (\beta_n^2 (bv_n + b_0 z_{1,n}(1,\cdot)), \beta_n^2 u_n) \to 0.$$

By (73) and (39), the first and last terms tend to zero, while the term

$$(K\nabla w_n, \beta_n^4 u_n)$$

tend also to zero thanks to (39) and the fact that $\|\nabla w_n\| \lesssim 1$ (due to $\|z_n\|_{\mathcal{H}} = 1$). We then deduce that

$$\beta_n^4 (C\epsilon(u_n), \epsilon(u_n)) + \beta_n^4 (K u_n, u_n) \to 0,$$

which, by Korn's inequality, implies (40).

Now using (69) and the fact that $\|w_n\| \lesssim 1$ we may write

$$\beta_n^2 (i\beta_n \tilde{\rho} v_n - \operatorname{div} C\epsilon(u_n) + K(\nabla w_n + u_n) + bv_n + b_0 z_{1,n}(1,\cdot), m w_n) \to 0,$$

where again m is the multiplier $m(x) = K^{-1}(x)x$. Again by Green's formula this is equivalent to

$$i(\beta_n^2 \tilde{\rho} v_n, m\beta_n w_n) + (\beta_n^2 C\epsilon(u_n), \epsilon(m w_n)) + (\beta_n^2 K u_n, m w_n)$$
$$+ (\beta_n^2 (bv_n + b_0 z_{1,n}(1,\cdot)), m w_n) + \beta_n^2 (K\nabla w_n, m w_n) \to 0.$$

Since $\|w_n\|_{1,\Omega} \lesssim 1$, $\beta_n \|w_n\| \lesssim 1$ and using (73), (39) and (40), we deduce that (41) holds. By a direct application of Green's formula we deduce that (42) is still valid.

The property (42) and (68) directly yield (43).

We now use (70) and again $\|w_n\| \lesssim 1$ to write

$$(i\beta_n J y_n - \operatorname{div}(K(\nabla w_n + u_n)) + a y_n + a_0 z_{2,n}(1,\cdot), w_n) \to 0$$

Again using Green's formula this is equivalent to

$$i(J y_n, \beta_n w_n) + (K\nabla w_n, \nabla w_n) - (K u_n, \nabla w_n) + (a y_n + a_0 z_{2,n}(1,\cdot), w_n) \to 0.$$

By the previous properties, the terms $(J y_n, \beta_n w_n)$, $(K u_n, \nabla w_n)$ and $(a y_n + a_0 z_{2,n}(1,\cdot), w_n)$ go to zero and therefore we deduce that

$$(K\nabla w_n, \nabla w_n) \to 0.$$

By Poincaré's inequality we obtain (44).

It remains to estimate the L^2 norm of $z_{1,n}$ and $z_{2,n}$. Let us make all the details for $z_{1,n}$. Denote

$$h_n = i\beta_n z_{1,n} + \tau^{-1}\partial_\rho z_{1,n}.$$

Then by (71) we see that

$$\|h_n\|_{(0,1)\times\Omega} \to 0. \tag{74}$$

But an explicit resolution of the above differential equation yields

$$z_{1,n}(\rho,x) = z_{1,n}(0,x)e^{-\imath\beta_n\tau\rho} + \tau\int_0^\rho e^{-\imath\beta_n\tau(\rho-r)}h_n(r,x)\,dr.$$

As $z_n = (u_n, v_n, w_n, y_n, z_{1,n}, z_{2,n})^\top$ belongs to $\mathcal{D}(\mathcal{A}_1)$, we have

$$z_{1,n}(0,x) = v_n(x),$$

and therefore

$$z_{1,n}(\rho,x) = v_n(x)e^{-\imath\beta_n\tau\rho} + \tau\int_0^\rho e^{-\imath\beta_n\tau(\rho-r)}h_n(r,x)\,dr.$$

This identity implies that

$$\|z_{1,n}\|_{(0,1)\times\Omega} \le \|v_n\| + \tau\|h_n\|_{(0,1)\times\Omega}.$$

By (73) and (74), we deduce that

$$\|z_{1,n}\|_{(0,1)\times\Omega} \to 0. \tag{75}$$

In the same manner using (72) and (43), we prove that

$$\|z_{2,n}\|_{(0,1)\times\Omega} \to 0. \tag{76}$$

In conclusion, by (73), (40), (44), (43), (75) and (76) we obtain

$$\|z_n\|_{\mathcal{H}_1} \to 0,$$

which contradicts $\|z_n\|_{\mathcal{H}} = 1$. □

By Lemmas 5.1, 6.2 and 6.3 we deduce the next result.

Theorem 6.4. *Under the assumptions of Lemma 6.2, there exists $C > 0$ such that for all $U_0 \in \mathcal{D}(\mathcal{A}_1^6)$, the solution of system (52)-(55) satisfies the following estimate*

$$E(t) \le C\,t^{-2}\,\|U_0\|^2_{\mathcal{D}(\mathcal{A}_1^6)}\,, \forall t > 1. \tag{77}$$

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