

ENERGY FLOW ABOVE THE THRESHOLD OF TUNNEL EFFECT

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ABSTRACT. We consider the Klein-Gordon equation on two half-axes connected at their origins. We add a potential that is constant but different on each branch. In a previous paper, we studied the L^∞ -time decay via Hörmander's version of the stationary phase method. Here we apply these results to show that for initial conditions in an energy band above the threshold of the tunnel effect a fixed portion of the energy propagates between group lines. Further we consider the situation that the potential difference tends to infinity while the energy band of the initial condition is shifted upwards such that the particle stays above the threshold of the tunnel effect. We show that the total transmitted energy as well as the portion between the group lines tend to zero like $a_2^{-1/2}$ in the branch with the higher potential a_2 as a_2 tends to infinity. At the same time the cone formed by the group lines inclines to the t -axis while its aperture tends to zero.

1. INTRODUCTION

In this paper we study the energy flow of waves in two coupled one-dimensional semi-infinite media having different dispersion properties. Results in experimental physics [8, 9], theoretical physics [7] and functional analysis [4, 6] describe phenomena created in this situation by the dynamics of the tunnel effect: the delayed reflection and advanced transmission near nodes issuing two branches. Our purpose is to describe the influence of the height of a potential step on the energy flow of wave packets above the threshold of tunnel effect.

We consider the following setting: let N_1, N_2 be disjoint copies of $(0, +\infty)$. Consider numbers a_k satisfying $0 \leq a_1 \leq a_2 < +\infty$. Find a vector (u_1, u_2) of functions $u_k : [0, +\infty) \times \overline{N_k} \rightarrow \mathbb{C}$ satisfying the Klein-Gordon equations

$$[\partial_t^2 - \partial_x^2 + a_k]u_k(t, x) = 0, \quad k = 1, 2,$$

on N_1, N_2 coupled at zero by usual Kirchhoff conditions and complemented with initial conditions for the functions u_k and their derivatives.

Reformulating this as an abstract Cauchy problem, one is confronted with the self-adjoint operator $A = (-\partial_x^2 + a_1, -\partial_x^2 + a_2)$ in $L^2(N_1) \times L^2(N_2)$, with a domain that incorporates the Kirchhoff transmission conditions at zero. For an exact definition of A , we refer to Section 2. The problem described above can be reformulated as

$$\begin{aligned} \ddot{u}(t) + Au(t) &= 0, \\ u(t) &\in D(A), \end{aligned} \tag{1}$$

for all $t > 0$ together with initial conditions. It is well known that the following expression is invariant with respect to time for solutions of (1):

$$E(u(t, \cdot)) = \frac{1}{2} \left(\|\dot{u}(t, \cdot)\|_H^2 + (Au, u)_H \right). \tag{2}$$

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In Section 2 we recall the solution formula that was proved in [2] by an expansion in generalized eigenfunctions in the more general setting of a star shaped network with semi-infinite branches.

In Section 3 we recall our result on L^∞ -time decay proved in [3]. There we obtained the exact L^∞ -time decay rate $c \cdot t^{-1/2}$ in the group velocity cones together with an expression for the coefficient c for initial conditions in a compact energy band in (a_2, ∞) . In the present work we refine an estimation from below in this context.

In Section 4 we use the preceding results to estimate the L^2 -norm of the outgoing solution on N_2 , which is a part of the total energy given in (2). We suppose that the initial condition belongs to an energy band above the threshold of the tunnel effect, so that the solution propagates in the branch with the higher potential N_2 . We consider the L^2 -norm of the solution at time t both on the whole branch N_2 and inside the cone delimited by the group lines corresponding to the bounds of the energy band. It turns out that the first norm has an upper asymptotic bound and the second one an upper and lower asymptotic bound which behave as $a_2^{-1/2}$, if the height of the potential step a_2 tends to infinity. This implies that the ratio of the energy on the whole branch and the energy between the group lines is time asymptotically confined in a finite interval above 1 which is independent of a_2 .

These results might be interpreted in terms of quantum mechanics as follows: a relativistic, massive particle without spin in a one-dimensional world is submitted to a potential step at the origin. It is supposed to have enough kinetic energy to overcome the step with a fixed remaining energy. Classically, the particle should leave the potential step with a velocity which is independent of the height of the step. Our results show that in the quantum mechanical model, when the height of the potential step tends to infinity, the velocity of the outgoing component of the particle tends to zero while the particle is more and more localized. At the same time the total outgoing energy tends to zero while its ratio to the energy inside the cone delimited by the group lines remains time asymptotically in a constant finite interval above one. Similar estimates should be possible for the other parts of the total energy (2).

The last observation might lead to the idea of viewing the total energy as being subdivided in a part inside the group line cones and a part outside these cones. The ratio of these two energies is quite independent of the experimental configuration, but depends mainly on the chosen energy band.

The results of this paper are related to results in experimental physics, theoretical physics and functional analysis (spectral theory, asymptotic estimates, analysis on networks, cf. [3] and the references cited there). For example in [4] we obtained information on the splitting of the energy flow near zero. In [1] a (not optimal) estimation for the L^∞ -time decay rate has been obtained but without any information on the localization of the energy. In [5] the relation of the eigenvalues of the Laplacian in an L^∞ -setting on infinite, locally finite networks to the adjacency operator of the network is studied. In [10], the authors consider general networks with semi-infinite ends. They give a construction to compute some generalized eigenfunctions but no attempt is made to construct explicit inversion formulas.

2. A SOLUTION FORMULA

The aim of this section is to recall the tools we used in [2] as well as the solution formula of the same paper for a special initial condition and to adapt this formula for the use of the stationary phase method in the next section.

Definition 2.1 (Functional analytic framework).

- i) Let N_1, N_2 be identified with $(0, +\infty)$. Put $N := \overline{N_1} \times \overline{N_2}$, identifying the endpoints 0.

ii) Two transmission conditions are introduced:

$$(T_0): (u_1, u_2) \in C(\overline{N_1}) \times C(\overline{N_2}) \text{ satisfies } u_1(0) = u_2(0).$$

This condition in particular implies that (u_1, u_2) may be viewed as a well-defined function on N .

$$(T_1): (u_1, u_2) \in C^1(\overline{N_1}) \times C^1(\overline{N_2}) \text{ satisfies } \sum_{k=1}^2 \partial_x u_k(0^+) = 0.$$

iii) Define the Hilbert space $H = L^2(N_1) \times L^2(N_2)$ with scalar product

$$(u, v)_H = \sum_{k=1}^2 (u_k, v_k)_{L^2(N_k)}$$

and the operator $A : D(A) \rightarrow H$ by

$$D(A) = \left\{ (u_1, u_2) \in H^2(N_1) \times H^2(N_2) : (u_1, u_2) \text{ satisfies } (T_0) \text{ and } (T_1) \right\},$$

$$A(u_1, u_2) = (A_1 u_1, A_2 u_2) = \left(-\partial_x^2 u_k + a_k u_k \right)_{k=1,2}.$$

Note that, if $a_1 = a_2 = 0$, then A is the Laplacian in the sense of the existing literature, cf. [5, 10].

Definition 2.2 (Fourier-type transform V).

i) For $k = 1, 2$ and $\lambda \in \mathbb{C}$ let

$$\xi_k(\lambda) := \sqrt{\lambda - a_k} \quad \text{as well as} \quad s_1(\lambda) := -\frac{\xi_2(\lambda)}{\xi_1(\lambda)} \quad \text{and} \quad s_2(\lambda) := -\frac{\xi_1(\lambda)}{\xi_2(\lambda)}.$$

Here, and in all what follows, the complex square root is chosen in such a way that $\sqrt{r \cdot e^{i\phi}} = \sqrt{r} e^{i\phi/2}$ with $r > 0$ and $\phi \in [-\pi, \pi)$.

ii) For $\lambda \in \mathbb{C}$ and $j, k \in \{1, 2\}$, we define generalized eigenfunctions $F_\lambda^{\pm, j} : N \rightarrow \mathbb{C}$ of A by $F_\lambda^{\pm, j}(x) := F_{\lambda, k}^{\pm, j}(x)$ with

$$\begin{cases} F_{\lambda, k}^{\pm, j}(x) = \cos(\xi_j(\lambda)x) \pm i s_j(\lambda) \sin(\xi_j(\lambda)x), & \text{for } k = j, \\ F_{\lambda, k}^{\pm, j}(x) = \exp(\pm i \xi_k(\lambda)x), & \text{for } k \neq j. \end{cases}$$

for $x \in \overline{N_k}$.

iii) For $l = 1, 2$ let

$$q_l(\lambda) := \begin{cases} 0, & \text{if } \lambda < a_l, \\ \frac{\xi_l(\lambda)}{|\xi_1(\lambda) + \xi_2(\lambda)|^2}, & \text{if } a_l < \lambda. \end{cases}$$

iv) Considering q_1 and q_2 as weights for our L^2 -spaces, we set $L_q^2 := L^2((a_1, +\infty), q_1) \times L^2((a_2, +\infty), q_2)$. The corresponding scalar product is

$$(F, G)_q := \sum_{k=1}^2 \int_{(a_k, +\infty)} q_k(\lambda) F_k(\lambda) \overline{G_k(\lambda)} d\lambda$$

and its associated norm $|F|_q := (F, F)_q^{1/2}$.

v) For all $f \in L^1(N, \mathbb{C})$ we define $Vf : [a_1, +\infty) \times [a_2, +\infty) \rightarrow \mathbb{C}$ by

$$(Vf)_k(\lambda) := \int_N f(x) \overline{(F_\lambda^{-, k})(x)} dx, \quad k = 1, 2.$$

In [2], we show that V diagonalizes A and we determine a metric setting in which it is an isometry. Let us recall these useful properties of V as well as the fact that the property $u \in D(A^l)$ can be characterized in terms of the decay rate of the components of Vu .

Theorem 2.3. *Endow $C_c^\infty(N_1) \times C_c^\infty(N_2)$ with the norm of $H = L^2(N_1) \times L^2(N_2)$. Then*

- i) $V : C_c^\infty(N_1) \times C_c^\infty(N_2) \rightarrow L_q^2$ is isometric and can be extended to an isometry $\tilde{V} : H \rightarrow L_q^2$, which we shall again denote by V in the following.
- ii) $V : H \rightarrow L_q^2$ is a spectral representation of H with respect to A . In particular, V is surjective.
- iii) The spectrum of the operator A is $\sigma(A) = [a_1, +\infty)$.
- iv) For $l \in \mathbb{N}$ the following statements are equivalent:
 - (a) $u \in D(A^l)$,
 - (b) $\lambda \mapsto \lambda^l(Vu)(\lambda) \in L_q^2$,
 - (c) $\lambda \mapsto \lambda^l(Vu)_k(\lambda) \in L^2((a_k, +\infty), q_k)$, $k = 1, 2$.

We are now interested in the Abstract Cauchy Problem

$$(ACP) : u_{tt}(t) + Au(t) = 0, \quad t > 0, \quad \text{with } u(0) = u_0, \quad u_t(0) = 0.$$

Here, the zero initial condition for the velocity is just chosen for simplicity, as we will not deal with the general case in this contribution.

By the surjectivity of V (cf. Theorem 2.3 (ii)) there exists an initial condition $u_0 \in H$ satisfying

Condition (A): $(Vu_0)_2 \equiv 0$ and $(Vu_0)_1 \in C_c^2((a_2, \infty))$.

Remark 2.4. i) For u_0 satisfying (A) there exist $a_2 < \lambda_{\min} < \lambda_{\max} < \infty$ such that

$$\text{supp}(Vu_0)_1 \subset [\lambda_{\min}, \lambda_{\max}].$$

ii) If $u_0 \in H$ satisfies (A), then $u_0 \in D(A^\infty) = \bigcap_{l \geq 1} D(A^l)$, due to Theorem 2.3 (iv), since

$$\lambda \mapsto \lambda^l(Vu)_m(\lambda) \in L^2((a_m, +\infty), q_m), \quad m = 1, 2 \text{ for all } l \in \mathbb{N} \text{ by the compactness of } \text{supp}(Vu_0)_m.$$

Theorem 2.5 (Solution formula of (ACP) in a special case). *Suppose that u_0 satisfies Condition (A). Then there exists a unique solution u of (ACP) with $u \in C^l([0, +\infty), D(A^{m/2}))$ for all $l, m \in \mathbb{N}$. For $x \in N_2$ we have the representation*

$$u_2(t, x) = \frac{1}{2}(u_+(t, x) + u_-(t, x))$$

with

$$u_\pm(t, x) := \int_{\lambda_{\min}}^{\lambda_{\max}} e^{\pm i\sqrt{\lambda}t} q_1(\lambda) e^{-i\xi_2(\lambda)x} (Vu_0)_1(\lambda) d\lambda. \quad (3)$$

Proof. Since $v_0 = u_t(0) = 0$, we have for the solution of (ACP) the representation

$$u(t) = V^{-1} \cos(\sqrt{\lambda}t) Vu_0.$$

(cf. for example [1, Theorem 5.1]). The expression for V^{-1} given in [2] yields the formula for u_\pm . \square

Remark 2.6. A solution formula for arbitrary initial conditions which is valid on all branches is available in [2]. This general expression is not needed in the following.

3. L^∞ -TIME DECAY

Next, we quote the result on L^∞ -time decay of solutions to the problem (ACP) from [3]. Here, we only consider special initial conditions that are localized in energy in a compact interval contained in $(a_2, a_2 + 1)$. For these it is possible to give very explicit estimates for all the constants that appear in a asymptotic expansion of the solution to the order $t^{1/2}$.

Theorem 3.1. *Let $0 < \alpha < \beta < 1$ and $\psi \in C_c^2((\alpha, \beta))$ with $\|\psi\|_\infty = 1$ be given. Setting $\tilde{\psi}(\lambda) := \psi(\lambda - a_2)$, we choose the initial condition $u_0 \in H$ satisfying $(Vu_0)_2 \equiv 0$ and $(Vu_0)_1 = \tilde{\psi}$. Furthermore, let u_+ be defined as in Theorem 2.5.*

Then there is a constant $C(\psi, \alpha, \beta)$ independent of a_1 and a_2 , such that for all $t \in \mathbb{R}^+$ and all $x \in N_2$ with

$$\sqrt{\frac{a_2 + \beta}{\beta}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha}{\alpha}} \quad (4)$$

we have

$$|u_+(t, x) - H(t, x, u_0) \cdot t^{-1/2}| \leq C(\psi, \alpha, \beta) \cdot t^{-1},$$

where

$$H(t, x, u_0) := e^{-i\varphi(p_0, t, x)} (2i\pi)^{1/2} a_2^{3/4} h_1(t, x) h_2(t, x) (Vu_0)_1(a_2 + p_0^2)$$

with

$$\begin{aligned} \varphi(p, t, x) &:= \sqrt{a_2 + p^2} t - px, & p_0 &:= \sqrt{\frac{a_2 x^2}{(t^2 - x^2)}}, \\ h_1(t, x) &:= \left(\frac{(t/x)^2}{(t/x)^2 - 1} \right)^{3/4}, & h_2(t, x) &:= \frac{\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_1}}{\left(\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_2} + \sqrt{a_2} \right)^2}. \end{aligned}$$

It holds further

$$|H(t, x, u_0)| \leq g(a_1, a_2, \beta) := \sqrt{2\pi} \frac{\sqrt{\beta}(a_2 + \beta)^{3/4}}{\sqrt{a_2}\sqrt{a_2 - a_1 + \beta}} \sim \sqrt{2\pi\beta} a_2^{-1/4} \quad \text{as } a_2 \rightarrow +\infty.$$

Proof. This is contained in Equation (8) from the proof of Theorem 3.2 and in Theorem 4.1 in [3]. \square

Remark 3.2. i) Note that (4) is equivalent to

$$v_{\min} \leq v(t, x) := (t/x)^2 - 1 \leq v_{\max}.$$

$$\text{where } v_{\min} := \frac{a_2}{\lambda_{\max} - a_2} = \frac{a_2}{\beta} \text{ and } v_{\max} := \frac{a_2}{\lambda_{\min} - a_2} = \frac{a_2}{\alpha}.$$

ii) For later use we define $p_{\min} := \xi_2(\lambda_{\min})$ and $p_{\max} := \xi_2(\lambda_{\max})$.

4. ENERGY FLOW

In this section we use the asymptotic expansion from section 3 to estimate the outgoing solution from above and below in the cones given by the group velocities corresponding to the bounds of the energy band of the initial condition. This leads to time independent asymptotic upper (Theorem 4.4) and lower (Theorem 4.1) estimates of the L^2 -norm of the solution on the space interval inside the cones.

Further an upper estimate of the L^2 -norm of the outgoing solution on the whole branch N_2 is obtained using Plancherel's theorem (Theorem 4.2).

These informations lead to our main result (Theorem 4.3), where we give an upper bound for the ratio of the L^2 -norm on the whole branch N_2 and the L^2 -norm inside the cone, which is asymptotically independent of the height of the potential step.

Theorem 4.1. *In the setting of Theorem 3.1 suppose that $\psi(\mu) \geq m > 0$ for $\mu \in [\alpha', \beta']$ with $\alpha < \alpha' < \beta' < \beta$. Then we have*

i) *the lower estimate for the coefficient of $t^{-1/2}$:*

$$\begin{aligned} |H(t, x, u_0)| &\geq f(a_2, \alpha, \beta) \cdot m \\ &:= \sqrt{2\pi} a_2^{3/4} \left(\frac{\beta}{a_2} + 1\right)^{3/4} \frac{1}{\sqrt{a_2}} \sqrt{\frac{a_2 - a_1}{\alpha} + 1} \left(\sqrt{\frac{a_2 - a_1}{\alpha} + 1} + 1\right)^{-2} m. \\ &\sim \sqrt{2\pi\alpha} a_2^{-1/4} m, \text{ as } a_2 \rightarrow \infty \end{aligned}$$

for all (t, x) satisfying

$$\sqrt{\frac{a_2 + \beta'}{\beta'}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha'}{\alpha'}}.$$

ii) *and the lower estimate for the solution*

$$\forall \varepsilon > 0 \exists t_0 > 0 \forall t > t_0 \quad |u_+(t, x)| \geq (f(a_2, \alpha, \beta)m - \varepsilon) t^{-1/2}$$

for all (t, x) satisfying

$$\sqrt{\frac{a_2 + \beta'}{\beta'}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha'}{\alpha'}}.$$

Proof. Note that it is always possible to choose the initial condition in the indicated way, thanks to the surjectivity of V , cf. Theorem 2.3 ii).

(i):

Theorem 3.1 implies

$$|H(t, x, u_0)| = \sqrt{2\pi} a_2^{3/4} h_1(t, x) |h_2(t, x)| \cdot \|(Vu_0)_1\|_\infty.$$

We estimate

$$\begin{aligned} |h_2(t, x)| &= \left| \frac{\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_2}}{(\sqrt{(a_2 - a_1)((t/x)^2 - 1) + a_2} + \sqrt{a_2})^2} \right| \\ &\geq \frac{\sqrt{(a_2 - a_1)v_{\max} + a_2}}{(\sqrt{(a_2 - a_1)v_{\max} + a_2} + \sqrt{a_2})^2} \\ &\geq \frac{\sqrt{(a_2 - a_1)\frac{a_2}{\alpha} + a_2}}{(\sqrt{(a_2 - a_1)\frac{a_2}{\alpha} + a_2} + \sqrt{a_2})^2} \\ &= \frac{1}{\sqrt{a_2}} \frac{\sqrt{\frac{a_2 - a_1}{\alpha} + 1}}{(\sqrt{\frac{a_2 - a_1}{\alpha} + 1} + 1)^2} \\ &\sim \frac{\sqrt{\alpha}}{a_2} \text{ as } a_2 \rightarrow \infty \end{aligned}$$

Here we used, that $b \mapsto \frac{b}{(b+c)^2}$ is decreasing for $b > c \geq 0$. Further

$$|h_1(t, x)| \geq \left(\frac{\frac{\lambda_{\max}}{\lambda_{\max} - a_2}}{\frac{\lambda_{\max}}{\lambda_{\max} - a_2} - 1} \right)^{3/4} = \left(\frac{\beta}{a_2} + 1 \right)^{3/4} \rightarrow 1 \text{ as } a_2 \rightarrow \infty$$

This implies (i).

(ii):

Using the lower triangular inequality we find that $\forall \varepsilon > 0 \exists t_0 > 0 \forall t > t_0$:

$$\begin{aligned} |u_+(t, x)| &\geq |u_+(t, x) - H(t, x, u_0) \cdot t^{-1/2} + H(t, x, u_0) \cdot t^{-1/2}| \\ &\geq \left| |H(t, x, u_0)| \cdot t^{-1/2} - |u_+(t, x) - H(t, x, u_0) \cdot t^{-1/2}| \right| \\ &\geq (f(a_2, \alpha, \beta)m - \varepsilon) t^{-1/2} \end{aligned}$$

for all (t, x) satisfying

$$\sqrt{\frac{a_2 + \beta'}{\beta'}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha'}{\alpha'}}.$$

This shows (ii) □

Theorem 4.2. *Suppose the setting of Theorem 3.1. Then we have*

$$\|u_+(t, \cdot)\|_{L^2(N_2)} \leq \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + \alpha} \sqrt[4]{\alpha}} \|\psi\|_{L^2((\alpha, \beta))} \sim \frac{\sqrt{\beta}}{\sqrt[4]{\alpha}} \|\psi\|_{L^2((\alpha, \beta))} a_2^{-1/2} \text{ as } a_2 \rightarrow \infty$$

Proof. In the expression (3) for u_+ we substitute $p := \xi_2(\lambda)$. This yields

$$u_+(t, x) = 2 \int_{p_{\min}}^{p_{\max}} e^{i\sqrt{s_2 + p^2}t} q_1(a_2 + p^2) e^{-ipx} (Vu_0)_1(a_2 + p^2) p dp.$$

Interpreting this integral as a Fourier transform we find by the Plancherel theorem

$$\|u_+(t, \cdot)\|_{L^2(N_2)} \leq \|u_+(t, \cdot)\|_{L^2(\mathbb{R})} = \|p \mapsto pq_1(a_2 + p^2) (Vu_0)_1(a_2 + p^2) \chi_{[p_{\min}, p_{\max}]}(p)\|_{L^2(\mathbb{R})}$$

Now, we use that $\text{supp}((Vu_0)_1)$ is contained in the interval $[a_2 + \alpha, a_2 + \beta]$, so only the range $\sqrt{\alpha} \leq p \leq \sqrt{\beta}$ is relevant. For these values of p we find

$$pq_1(a_2 + p^2) = p \frac{\sqrt{a_2 - a_1 + p^2}}{(\sqrt{a_2 - a_1 + p^2} + p)^2} \leq \frac{p}{\sqrt{a_2 - a_1 + p^2}} \leq \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + \alpha}}$$

So, we have

$$\|u_+(t, \cdot)\|_{L^2(N_2)} \leq \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + \alpha}} \|(Vu_0)_1(a_2 + p^2)\|_{L^2((p_{\min}, p_{\max}))}$$

and substituting back $\lambda = a_2 + p^2$ we obtain

$$\|(Vu_0)_1(a_2 + p^2)\|_{L^2((p_{\min}, p_{\max}))}^2 = \int_{\lambda_{\min}}^{\lambda_{\max}} |(Vu_0)_1(\lambda)|^2 \frac{d\lambda}{\sqrt{\lambda - a_2}} = \int_{a_2 + \alpha}^{a_2 + \beta} |\tilde{\psi}(\lambda)|^2 \frac{d\lambda}{\sqrt{\lambda - a_2}}.$$

Setting, finally, $\mu = \lambda - a_2$ we end up with

$$\|u_+(t, \cdot)\|_{L^2(N_2)} \leq \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + \alpha}} \left(\int_{\alpha}^{\beta} |\psi(\mu)|^2 \frac{d\mu}{\sqrt{\mu}} \right)^{1/2} \leq \frac{\sqrt{\beta}}{\sqrt{a_2 - a_1 + \alpha} \sqrt[4]{\alpha}} \|\psi\|_{L^2((\alpha, \beta))} \quad \square$$

Theorem 4.3. *In the setting of Theorem 3.1 suppose that $\psi(\mu) \geq m > 0$ for $\mu \in [\alpha', \beta']$ with $\alpha < \alpha' < \beta' < \beta$. Then we have*

i) the estimate from below: $\forall \varepsilon > 0 \exists t_0 > 0 \forall t > t_0$

$$\|u_+(t, \cdot)\|_{L^2(I'_t)}^2 \geq \left(f(a_2, \alpha, \beta)m - \varepsilon\right)^2 \left(\sqrt{\frac{\beta'}{a_2 + \beta'}} - \sqrt{\frac{\alpha'}{a_2 + \alpha'}}\right)$$

where

$$I'_t := \left[t\sqrt{\frac{\alpha'}{a_2 + \alpha'}}, t\sqrt{\frac{\beta'}{a_2 + \beta'}} \right]$$

and f is defined in Theorem 3.1 (i).

ii) the estimate for the ratio of the global and local energy:

$\forall \varepsilon > 0 \exists t_0 > 0 \forall t > t_0$

$$\frac{\|u_+(t, \cdot)\|_{L^2(N_2)}}{\|u_+(t, \cdot)\|_{L^2(I'_t)}} \leq \frac{\sqrt{\beta} \|\psi\|_{L^2((\alpha, \beta))}}{\sqrt{a_1 - a_2 + \alpha} \sqrt[4]{\alpha}} \left(f(a_2, \alpha, \beta)m - \varepsilon\right)^{-1} \left(\sqrt{\frac{\beta'}{a_2 + \beta'}} - \sqrt{\frac{\alpha'}{a_2 + \alpha'}}\right)^{-1/2}$$

iii) and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\|u_+(t, \cdot)\|_{L^2(N_2)}}{\|u_+(t, \cdot)\|_{L^2(I'_t)}} \\ & \leq \frac{\sqrt{\beta} \|\psi\|_{L^2((\alpha, \beta))}}{\sqrt{a_1 - a_2 + \alpha} \sqrt[4]{\alpha}} \left(f(a_2, \alpha, \beta)m\right)^{-1} \left(\sqrt{\frac{\beta'}{a_2 + \beta'}} - \sqrt{\frac{\alpha'}{a_2 + \alpha'}}\right)^{-1/2} \\ & \sim (2\pi)^{-1/2} m^{-1} \beta^{1/2} \alpha^{-3/4} (\sqrt{\beta'} - \sqrt{\alpha'})^{-1/2} \|\psi\|_{L^2((\alpha, \beta))} \text{ as } a_2 \rightarrow \infty \end{aligned}$$

Proof. (i):

Follows from

$$\|u_+(t, \cdot)\|_{L^2(I'_t)}^2 \geq \left(\inf_{x \in I'_t} |u_+(t, x)|\right)^2 \cdot |I'_t|$$

and Theorem 4.1 (ii).

(ii):

Follows from (i) and Theorem 4.2.

(iii):

Direct consequence of (ii) and Theorem 4.1 (i). □

Theorem 4.4. Consider the setting of Theorem 3.1.

i) $\forall \varepsilon > 0 \exists t_1 > 0 \forall t > t_1$:

$$|u_+(t, x)| \leq (g(a_1, a_2, \beta) + \varepsilon) t^{-1/2}$$

for all (t, x) satisfying

$$\sqrt{\frac{a_2 + \beta}{\beta}} \leq \frac{t}{x} \leq \sqrt{\frac{a_2 + \alpha}{\alpha}}.$$

ii) $\forall \varepsilon > 0 \exists t_1 > 0 \forall t > t_1$:

$$\|u_+(t, \cdot)\|_{L^2(I'_t)}^2 \leq (g(a_1, a_2, \beta) + \varepsilon)^2 \left(\sqrt{\frac{\beta}{a_2 + \beta}} - \sqrt{\frac{\alpha}{a_2 + \alpha}}\right)$$

iii)

$$\begin{aligned} \liminf_{t \rightarrow \infty} \|u_+(t, \cdot)\|_{L^2(I_t)}^2 &\leq g(a_1, a_2, \beta)^2 \left(\sqrt{\frac{\beta}{a_2 + \beta}} - \sqrt{\frac{\alpha}{a_2 + \alpha}} \right) \\ &\sim 2\pi\beta(\sqrt{\beta} - \sqrt{\alpha}) a_2^{-1} \text{ as } a_2 \rightarrow \infty \end{aligned}$$

Proof. (i):Theorem 3.1 implies that $\forall \varepsilon > 0 \exists t_1 > 0 \forall t > t_1$:

$$\begin{aligned} |u_+(t, x)| &\leq |u_+(t, x) - H(t, x, u_0)t^{-1/2} + H(t, x, u_0)t^{-1/2}| \\ &\leq C(\psi, \alpha, \beta)t^{-1} + |H(t, x, u_0)|t^{-1/2} \\ &\leq (g(a_1, a_2, \beta) + \varepsilon) t^{-1/2} \end{aligned}$$

for (t, x) in the cone indicated there.

(ii):

Follows from estimating the square of the L^2 -norm against the square of the maximum of the function (using (i)) times the length of the integration interval.

(iii):

Direct consequence of (ii). □

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