Exponential stability of second order evolution equations with structural damping and boundary delay feedback

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Abstract

We consider a stabilization problem for abstract second order evolution equations with boundary feedback laws with a delay and distributed structural damping. We prove an exponential stability result under a suitable condition between the internal damping and the boundary laws. The proof of the main result is based on an identity with multipliers that allows to obtain a uniform decay estimate for a suitable energy functional. Some concrete examples are detailed. Some counterexamples suggest that this condition is optimal.

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1 Introduction

Delay effects arise in many applications and practical problems. On the other hand, it is well–known that an arbitrarily small time delay may destabilize a system which is naturally stable (see e.g. [11], [8], [10]). Here, we consider a damped second order evolution equation with a time delay in the boundary condition.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary Γ. We assume that Γ is divided into two parts Γ_0 and Γ_1, i.e. $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and $meas\Gamma_0 \neq 0$. Note that the condition $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ is only made in order to simplify the presentation, hence our analysis can be performed without this assumption in a similar manner.

For $i = 0$ and 1, we consider in Ω two strongly elliptic operators A_i of order $2m_i$ with a positive integer m_0 and $m_1 = 1$ of the form

$$A_i = \sum_{|\alpha|,|\beta| \leq m_i} (-1)^{|\beta|} D^\beta (a^{(i)}_{\alpha,\beta} D^\alpha),$$

where $a^{(i)}_{\alpha,\beta} = a^{(i)}_{\beta,\alpha}$ are in $C^\infty(\bar{\Omega})$ and such that

$$\sum_{|\alpha|=|\beta|=m_i} a^{(i)}_{\alpha,\beta}(x) \xi^\alpha \xi^\beta \geq \alpha_i |\xi|^{2m_i} \quad \forall \xi \in \mathbb{R}^n,\ x \in \bar{\Omega}.$$

for some positive constant α_i. Accordingly we can introduce the natural bilinear form

$$a_i(u,v) = \sum_{|\alpha|,|\beta| \leq m_i} \int_\Omega a^{(i)}_{\alpha,\beta} D^\alpha u \, D^\beta v \, dx, \forall u, v \in H^{m_i}(\Omega).$$

On Γ we further fix a Dirichlet system $\{D_{0j}\}_{j=0}^{m_0-1}$ of order m_0 with the terminology of [25] (in particular D_{0j} is an operator of order j) and without loss of generality we can assume that D_{00} is equal to the identity operator Id. We also take $D_{10} = Id$. According to section 2.4 of [25], there exists a system $\{T_{ij}\}_{j=0}^{m_i-1}$ with smooth coefficients such that the order of T_{ij} is equal to $2m_i - 1 - j$ and such that the next Green formula holds:

$$a_i(u,v) = \int_\Omega A_i uv \, dx - \sum_{j=0}^{m_i-1} \int_\Gamma T_{ij} u D_{ij} v d\Gamma \quad \forall u \in H^{2m_i}(\Omega),\ v \in H^{m_i}(\Omega). \tag{1.1}$$

In this domain Ω, we consider the initial boundary value problem

$$u_{tt} + A_0 u + A_1 u_t = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{1.2}$$
$$D_{0j} u = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty) \quad \forall j = 0, \cdots, m_0 - 1 \tag{1.3}$$
$$T_{0j} u = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \quad \forall j = 1, \cdots, m_0 - 1 \tag{1.4}$$
$$\mu u_{tt}(x,t) + k u_t(x, t-\tau) = T_{00} u(x,t) + T_{10} u_t(x,t) \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \tag{1.5}$$
$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{1.6}$$
$$u_t(x,t) = f_0(x,t) \quad \text{in } \Gamma_1 \times (-\tau, 0), \tag{1.7}$$

where $\tau > 0$ is the time delay, μ is a nonnegative real numbers, k is a real number and the initial datum (u_0, u_1, f_0) belongs to a suitable space.

In the case $\mu > 0$, (1.5) is a so–called dynamic boundary condition. Dynamic boundary conditions arise in many physical applications, in particular they occur in elastic models. For instance, these conditions appear in modeling dynamic vibrations of linear viscoelastic rods and beams which have attached tip masses at their free ends. See [2, 6, 27] and the references therein for more details.

We are interested in giving an exponential stability result for such a problem under a suitable relation between the operator A_1 and the coefficient k.

Note that the above system is exponentially stable in absence of time delay, that is if $\tau = 0$ (see e.g. [29]). Then, we will investigate the robustness of the boundary feedbacks with respect to (small) time delays.

We will show that, in both cases $\mu = 0$ and $\mu > 0$, under the condition

$$1 > |k| C_P, \tag{1.8}$$

where C_P is a sort of Poincaré constant related to the operator A_1 and described below, the energy of the solution of system (1.2)–(1.7) satisfies a uniform exponential decay estimate. This holds for every delays $\tau > 0$.

Our result applies to several models, in particular wave equation and Kirchoff system with Kelvin-Voight damping (see section 4).

In a previous paper [30] we addressed the problem to contrast the destabilizing effect of a time delay in the boundary feedback considering the system

$$u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.9}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty) \tag{1.10}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -k_1 u_t(x,t) - k_2 u_t(x,t-\tau) \quad \text{on} \quad \Gamma_1 \times (0,+\infty), \tag{1.11}$$

with k_1, k_2 positive constants and initial data in suitable spaces. We proved that under the condition

$$k_1 > k_2 \tag{1.12}$$

the system (1.9)–(1.11) is exponentially stable. This was obtained by introducing a suitable energy functional and by using appropriate observability inequalities. Condition (1.12) is optimal in order to have stability of the above model. Indeed, if $0 \le k_1 \le k_2$ there are unstable solutions, namely solutions whose energy is not decaying to zero.

Analogous problem in one dimension was proposed by Xu, Yung, Li [36] and solved through a careful spectral analysis.

Here, our aim is to contrast the effect of a boundary delay damping with an interior damping. The general idea is that if the dissipative law $A_1 u_t$ in (1.2) is strong enough with respect to the boundary one with a delay, then the system will be exponentially stable. On the contrary, we will show, in dimension $n = 1$, that if this condition (1.8) is not satisfied then the system becomes unstable. We expect that similar phenomena occur in dimension $n \ge 2$ but we were not able to obtain positive results (except in tensor product situations, like a square for instance).

An opposite problem, namely the wave equation with interior delay damping and dissipative not delayed boundary condition, has been studied in [1].

The paper is organized as follows. The second section deals with the well-posedness of the problem obtained by using semigroup theory. Here we need to distinguish the case

$\mu > 0$ to the case $\mu = 0$. In section 3, we prove the exponential stability of the delayed system (1.2)–(1.7) by introducing a suitable energy functional. In section 4 we illustrate some concrete examples of our abstract model: the damped wave equation and the Kirchoff model. Finally section 5 is devoted to some instability examples in dimension one.

2 Well-posedness of the problems

In this section we will give well-posedness results for problem (1.2)–(1.7) using semigroup theory. We have to distinguish the two cases $\mu = 0$ and $\mu > 0$.

We assume that a_0 is strongly elliptic on

$$V := \{u \in H^{m_0}(\Omega) : D_{0j}u = 0 \text{ on } \Gamma_0, \forall j = 0, \cdots, m_0 - 1\}, \tag{2.1}$$

namely we suppose that there exists a positive constant β_0 such that

$$a_0(u, u) \geq \beta_0 \|u\|^2_{H^{m_0}(\Omega)} \quad \forall u \in V. \tag{2.2}$$

Similarly we suppose that there exists a positive constant β_1 such that

$$a_1(u, u) \geq \beta_1 \|u\|^2_{H^1(\Omega)} \quad \forall u \in H^1_{\Gamma_0}(\Omega), \tag{2.3}$$

where, as usual,

$$H^1_{\Gamma_0}(\Omega) = \{ \ u \in H^1(\Omega) \ : \ u = 0 \quad \text{on} \ \ \Gamma_0 \ \}.$$

From this estimate and a trace result, we deduce that the next Poincaré estimate holds: there exists a positive constant C_P such that

$$\int_{\Gamma_1} |v|^2 \, d\Gamma \leq C_P a_1(v, v), \quad \forall v \in H^1_{\Gamma_0}(\Omega). \tag{2.4}$$

In the sequel C_P is defined as the smallest positive constant such that (2.4) holds.

2.1 The case $\mu = 0$

In that case system (1.2)–(1.7) enters in the abstract framework developed in [31]. Indeed using the notations from that paper, it suffices to take $H = L^2(\Omega)$, V given by (2.1), $A = A_0$, $B_1^* = \sqrt{A_1}$ that maps $H^1_{\Gamma_0}(\Omega)$ into H (looking at A_1 as a positive self-adjoint operator from H into itself) so that

$$B_1 B_1^* = A_1$$

maps $H^1_{\Gamma_0}(\Omega)$ into its dual. But since V is continuously and densely embedded into $H^1_{\Gamma_0}(\Omega)$, $B_1 B_1^* = A_1$ also maps V into V'. For B_2^* we take the trace operator up to a multiplicative factor, namely

$$B_2^* : V \to L^2(\Gamma_1) : u \to \sqrt{k}\gamma_1 u,$$

where γ_1 is the trace operator on Γ_1.

With these notations, we now show that (1.2)–(1.7) is equivalent to the system (1.2) of [31], namely

$$\begin{cases} \ddot{u}(t) + Au(t) + B_1 B_1^* \dot{u}(t) + B_2 B_2^* \dot{u}(t-\tau) = 0 \ \text{ in } V',\, t>0 \\ u(0) = u_0, \ \dot{u}(0) = u_1, \\ B_2^* \dot{u}(t-\tau) = f_0(t-\tau), \ \ 0<t<\tau. \end{cases} \tag{2.5}$$

Indeed if u is solution of this last problem then we have

$$\langle \ddot{u}(t) + Au(t) + B_1 B_1^* \dot{u}(t) + B_2 B_2^* \dot{u}(t-\tau), \varphi \rangle_{V'-V} = 0, \forall \varphi \in V. \tag{2.6}$$

Taking first test function $\varphi \in \mathcal{D}(\Omega)$, we find that (1.2) holds in the distributional sense. Coming back to (2.6) and using (1.1) (in a weak form), we find that (1.3)–(1.5) hold (with $\mu = 0$).

Now in order to apply the existence result from Theorem 2.1 of [31] we need to check the assumption (2.5) from that paper that reads in our setting as follows

$$\exists\ 0 < \alpha_0 \le 1, \ \forall u \in V, \ \|B_2^* u\|^2_{L^2(\Gamma_1)} \le \alpha_0 \|B_1^* u\|^2_{L^2(\Omega)}. \tag{2.7}$$

From the definition of B_1^* and B_2^* this is equivalent to

$$\exists\ 0 < \alpha_0 \le 1, \ \forall u \in V, \ |k| \int_{\Gamma_1} |u|^2 \le \alpha_0 a_1(u,u). \tag{2.8}$$

In view of the definition (2.4) this is finally equivalent to

$$|k| C_P \le 1. \tag{2.9}$$

Note that this condition is slightly weaker than (1.8).

At this stage we can apply Theorem 2.1 of [31] and obtain the next existence result, Theorem 2.1 below. Before stating it, in order to write it in a compact form, like in [30] we introduce the auxiliary unknown

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Gamma_1, \ \rho \in (0,1), \ t>0. \tag{2.10}$$

Then if we denote

$$U := (u, u_t, z)^T,$$

problem (1.2)–(1.7) is equivalent to

$$\begin{cases} U' = \mathcal{A} U, \\ U(0) = (u_0, u_1, f_0(\cdot, -\cdot\tau))^T, \end{cases} \tag{2.11}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ -(A_0 u + A_1 v) \\ -\tau^{-1} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned}\mathcal{D}(\mathcal{A}) := \Big\{ & (u,v,z)^T \in V \times V \times L^2(\Gamma_1; H^1(0,1)) \ : \\ & T_{0j}u = 0 \text{ on } \Gamma_1, j = 1, \cdots, m_0 - 1, \\ & A_0 u + A_1 v + B_2 z(\cdot,1) \in L^2(\Omega), \ v = z(\cdot,0) \text{ on } \Gamma_1 \Big\}.\end{aligned} \tag{2.12}$$

This operator is well defined on the Hilbert space

$$\mathcal{H} := V \times L^2(\Omega) \times L^2(\Gamma_1 \times (0,1)). \tag{2.13}$$

From Theorem 2.1 of [31] follows the well–posedness result:

Theorem 2.1 *Assume that* (2.9) *holds. Then for any initial datum* $U_0 \in \mathcal{H}$ *there exists a unique (weak) solution* $U \in C([0,+\infty), \mathcal{H})$ *of problem* (2.11). *Moreover, if* $U_0 \in \mathcal{D}(\mathcal{A})$, *then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}),$$

that is called a strong solution.

2.2 The case $\mu > 0$

As before we use the auxiliary unknown (2.10) hence problem (1.2)–(1.7) is equivalent to

$$u_{tt}(x,t) + A_0 u(x,t) + A_1 u_t(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty), \tag{2.14}$$
$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in} \quad \Gamma_1 \times (0,1) \times (0,+\infty), \tag{2.15}$$
$$D_{0j}u = 0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty) \quad \forall j = 0, \cdots, m_0 - 1, \tag{2.16}$$
$$T_{0j}u(x,t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,+\infty) \quad \forall j = 1, \cdots, m_0 - 1, \tag{2.17}$$
$$u_{tt}(x,t) = \mu^{-1}(-kz(x,1,t) + T_{00}u(x,t) + T_{10}u_t(x,t)) \quad \text{on} \quad \Gamma_1 \times (0,+\infty), \tag{2.18}$$
$$z(x,0,t) = u_t(x,t) \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{2.19}$$
$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{2.20}$$
$$z(x,\rho,0) = f_0(x,-\rho\tau) \quad \text{in} \quad \Gamma_1 \times (0,1). \tag{2.21}$$

The difference with the previous subsection is here to use

$$U := (u, u_t, \gamma_1 u_t, z)^T,$$

as vectorial unknown where, as above, γ_1 is the trace operator on Γ_1. Then the previous problem is formally equivalent to

$$\begin{aligned} U' &:= (u_t, u_{tt}, \gamma_1 u_{tt}, z_t)^T \\ &= (u_t, -A_0 u - A_1 u_t, \mu^{-1}(-kz(\cdot,1,\cdot) + T_{00}u + T_{10}u_t), -\tau^{-1} z_\rho)^T. \end{aligned}$$

Therefore, problem (2.14)–(2.21) can be rewritten as

$$\begin{cases} U' = \mathcal{A}_1 U, \\ U(0) = (u_0, u_1, \gamma_1 u_1, f_0(\cdot, -\cdot\tau))^T, \end{cases} \tag{2.22}$$

where the operator $\mathcal{A}_1$ is defined by

$$\mathcal{A}_1 \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} := \begin{pmatrix} v \\ -A_0 u - A_1 v \\ \mu^{-1}(-kz(\cdot,1) + T_{00}u + T_{10}v) \\ -\tau^{-1} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}_1) := \Big\{ & (u, v, v_1, z)^T \in V \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0,1)) \ : \\ & T_{0j} u = 0 \text{ on } \Gamma_0, j = 1, \cdots, m_0 - 1, \\ & A_0 u + A_1 v \in L^2(\Omega), \\ & T_{00} u + T_{10} v \in L^2(\Gamma_1),\ \gamma_1 v = v_1 = z(\cdot, 0) \text{ on } \Gamma_1 \Big\}, \end{aligned} \tag{2.23}$$

defined as an unbounded operator in the Hilbert space $\mathcal{H}_1$ defined by

$$\mathcal{H}_1 := V \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0,1)). \tag{2.24}$$

Assuming that (2.9) holds we will show that $\mathcal{A}_1$ generates a C_0 semigroup on $\mathcal{H}_1$. From (2.9) there exists a positive real number ξ satisfying

$$|k| \le \frac{\xi}{\tau} \le \frac{2}{C_P} - |k|. \tag{2.25}$$

Hence we define on the Hilbert space $\mathcal{H}_1$ the inner product

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}_1 \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}_1} & := a_0(u, \tilde{u}) + \int_\Omega v(x) \tilde{v}(x) dx \\ & + \mu \int_{\Gamma_1} v_1(x) \tilde{v}_1(x) d\Gamma + \xi \int_{\Gamma_1} \int_0^1 z(x,\rho) \tilde{z}(x,\rho) d\rho d\Gamma. \end{aligned} \tag{2.26}$$

Theorem 2.2 *Assume that* (2.9) *holds. Then for any initial datum* $U_0 \in \mathcal{H}_1$ *there exists a unique (weak) solution* $U \in C([0, +\infty), \mathcal{H}_1)$ *of problem* (2.22). *Moreover, if* $U_0 \in \mathcal{D}(\mathcal{A}_1)$, *then*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}_1)) \cap C^1([0, +\infty), \mathcal{H}_1),$$

that is called a strong solution.

Proof. Take $U = (u, v, v_1, z)^T \in \mathcal{D}(\mathcal{A}_1)$. Then

$$\begin{aligned} \langle \mathcal{A}_1 U, U \rangle_{\mathcal{H}_1} &= a_0(v, u) - \int_\Omega v(x)(A_0 u(x) + A_1 v(x)) dx \\ &+ \int_{\Gamma_1} (T_{00} u + T_{10} v - kz(x,1))\, v(x) d\Gamma - \xi \tau^{-1} \int_{\Gamma_1} \int_0^1 z_\rho(x,\rho) z(x,\rho) d\rho d\Gamma. \end{aligned}$$

So, by Lemmas 2.3 and 2.4 below (generalization of Green's formula (1.1)) and using the definition of $\mathcal{D}(\mathcal{A}_1)$ we obtain

$$\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1} = -a_1(v,v) - k\int_{\Gamma_1} z(x,1)v(x)d\Gamma - \frac{\xi}{2\tau}\int_{\Gamma_1}(z^2(x,1)-v^2(x))d\Gamma. \tag{2.27}$$

Using the Cauchy–Schwarz and the Young inequalities we find

$$\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1} \le -a_1(v,v) + \left(\frac{|k|}{2}+\frac{\xi}{2\tau}\right)\int_{\Gamma_1} v^2(x)d\Gamma + \left(\frac{|k|}{2}-\frac{\xi}{2\tau}\right)\int_{\Gamma_1} z^2(x,1)d\Gamma. \tag{2.28}$$

Using the definition (2.4) of C_P, we deduce that

$$\langle \mathcal{A} U, U\rangle_{\mathcal{H}_1} \le \left(-1 + C_P(\frac{|k|}{2}+\frac{\xi}{2\tau})\right) a_1(v,v) + \left(\frac{|k|}{2}-\frac{\xi}{2\tau}\right)\int_{\Gamma_1} z^2(x,1)d\Gamma. \tag{2.29}$$

Now, observing that from (2.25),

$$-1 + C_P(\frac{|k|}{2}+\frac{\xi}{2\tau}) \le 0, \quad \frac{|k|}{2}-\frac{\xi}{2\tau} \le 0,$$

we obtain $\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1} \le 0$, which means that the operator $\mathcal{A}_1$ is dissipative.

Now, we will show that $\lambda I - \mathcal{A}_1$ is surjective for a fixed $\lambda > 0$. Given $(f, g, g_1, h)^T \in \mathcal{H}_1$, we seek a $U = (u, v, v_1, z)^T \in \mathcal{D}(\mathcal{A}_1)$ solution of

$$(\lambda I - \mathcal{A}_1)\begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ g_1 \\ h \end{pmatrix},$$

that is, verifying

$$\begin{cases} \lambda u - v = f \text{ in } \Omega, \\ \lambda v + A_0 u + A_1 v = g \text{ in } \Omega, \\ \lambda v_1 + \mu^{-1}(kz(\cdot,1) - T_{00}u - T_{10}v) = g_1 \text{ on } \Gamma_1, \\ \lambda z + \tau^{-1} z_\rho = h \text{ on } \Gamma_1 \times (0,1). \end{cases} \tag{2.30}$$

Suppose that we have found u with the appropriate regularity. Then we have

$$v := \lambda u - f \text{ in } \Omega \tag{2.31}$$

as well as

$$v_1 = \gamma_1 v = z(x,0) = \lambda u - f \text{ on } \Gamma_1 \tag{2.32}$$

and, as in the proof of Theorem 2.1 of [30], we find z in the form

$$z(x,\rho) = \lambda u(x)e^{-\lambda\rho\tau} - f(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^\rho h(x,\sigma)e^{\lambda\sigma\tau}d\sigma \quad \text{on } \Gamma_1 \times (0,1), \tag{2.33}$$

and, in particular,

$$z(x,1) = \lambda u(x)e^{-\lambda\tau} + z_0(x), \quad x \in \Gamma_1, \tag{2.34}$$

with $z_0 \in L^2(\Gamma_1)$ defined by

$$z_0(x) = -f(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\sigma\tau}d\sigma, \quad x \in \Gamma_1. \tag{2.35}$$

By (2.31), (2.32), the function u verifies

$$\lambda^2 u + A_0 u + \lambda A_1 u = g + \lambda f + A_1 f \text{ in } \Omega, \tag{2.36}$$

with the boundary condition (at least formally):

$$\mu\lambda^2 u - T_{00}u - \lambda T_{10}u = \mu(g_1 + \lambda f) - kz(\cdot,1) - T_{10}f \quad \text{on} \quad \Gamma_1. \tag{2.37}$$

Multiplying the equation (2.36) by $w \in V$ and integrating in Ω, we find

$$\lambda^2\int_\Omega uw\,dx + < A_0u, w > +\lambda < A_1u, w >= \int_\Omega (g+\lambda f)w\,dx + < A_1 f, w > \quad \forall w \in V.$$

Hence a formal application of Green's formula (1.1) leads to

$$\lambda^2\int_\Omega uw\,dx + a_0(u,w) + \lambda a_1(u,w) + \int_{\Gamma_1}(T_{00}u + \lambda T_{10}u)w\,d\Gamma = \int_\Omega (g+\lambda f)w\,dx$$
$$+a_1(f,w) + \int_{\Gamma_1} T_{10}fw\,d\Gamma \quad \forall w \in V.$$

By taking into account (2.37) we get

$$\lambda^2\int_\Omega uw\,dx + a_0(u,w) + \lambda a_1(u,w) = \int_\Omega (g+\lambda f)w\,dx + a_1(f,w)$$
$$+\int_{\Gamma_1}[\mu(g_1+\lambda f) - kz(\cdot,1) - \mu\lambda^2 u]w\,d\Gamma \quad \forall w \in V.$$

Finally using (2.34) we arrive at

$$a_0(u,w) + \lambda a_1(u,w) + \lambda^2\int_\Omega uw\,dx + \int_{\Gamma_1}[\lambda k e^{-\lambda\tau}u + \mu\lambda^2 u]w\,d\Gamma \tag{2.38}$$
$$= \int_\Omega (g+\lambda f)w\,dx + a_1(f,w) + \int_{\Gamma_1}[\mu(g_1+\lambda f) - kz_0]w\,d\Gamma \quad \forall w \in V.$$

As the left-hand side of (2.38) is coercive on V, and the right-hand side defines a continuous linear form on V, the Lax–Milgram lemma guarantees the existence and uniqueness of a solution $u \in V$ of (2.38). Once we have obtained $u \in V$, we define v by (2.31) and z by (2.33). We can notice that v belongs to V since u and f are in V, while z belongs to $L^2(\Gamma_1; H^1(0,1)$ by the regularity of g, the fact that $u, v \in V$ and a trace theorem. In particular $z(\cdot,1)$ belongs to $L^2(\Gamma_1)$. With these choices, we can equivalently rewrite (2.38) as follows:

$$a_0(u,w)+a_1(v,w)+\lambda\int_\Omega vw\,dx=\int_\Omega gw\,dx \tag{2.39}$$
$$+\int_{\Gamma_1}[\mu(g_1-\lambda v)-kz(\cdot,1)]w\,d\Gamma \quad \forall w\in V.$$

If we consider $w\in\mathcal{D}(\Omega)$ in (2.39), we see that u,v solve in $\mathcal{D}'(\Omega)$

$$A_0u+A_1v+\lambda v=g, \tag{2.40}$$

which is nothing else than the second identity of (2.30). Note that this identity also guarantees that

$$A_0u+A_1v=g-\lambda v\in L^2(\Omega).$$

It then remains to check the natural boundary conditions, namely the third identity of (2.30) and

$$T_{0j}u=0 \text{ on } \Gamma_0,\quad \forall j=1,\cdots,m_0-1. \tag{2.41}$$

For that purpose, we distinguish the case $m_0=1$ or $m_0\geq 2$:
i) If $m_0\geq 2$, we notice that A_1v belongs to $L^2(\Omega)$ because v belongs to $H^2(\Omega)$ (V being included into $H^2(\Omega)$) and therefore $u\in E(A_0;L^2(\Omega))$ (with the notations introduced below). Then in that case we apply Lemma 2.3 below for the first term of the right-hand side of (2.39), while we can directly apply (1.1) to the second term of the right-hand side of (2.39). This yields

$$\int_\Omega A_0uw\,dx-\sum_{j=0}^{m_0-1}\langle T_{0j}u;D_{0j}w\rangle+\int_\Omega A_1vv\,dx-\int_{\Gamma_1}T_{10}vwd\Gamma$$
$$+\lambda\int_\Omega vw\,dx=\int_\Omega gw\,dx+\int_{\Gamma_1}\mu(g_1-\lambda v-kz(\cdot,1))w\,d\Gamma \quad \forall w\in V.$$

Due to (2.40), this is equivalent to

$$-\sum_{j=0}^{m_0-1}\langle T_{0j}u;D_{0j}w\rangle-\int_{\Gamma_1}T_{10}vwd\Gamma=\int_{\Gamma_1}[\mu(g_1-\lambda v)-kz(\cdot,1)]w\,d\Gamma \quad \forall w\in V.$$

This proves that the third identity of (2.30) and (2.41) are satisfied because the mapping

$$w\to(D_{0j}w)_{j=0}^{m_0-1}$$

is continuous and surjective from V into $\prod_{j=0}^{m_0-1}H^{m_0-j-1/2}(\Gamma_1)$.
ii) If $m_0=1$, we directly apply Lemma 2.4 below to the pair (u,v). This yields

$$\int_\Omega(A_0+A_1)uw\,dx-\langle T_{00}u+T_{10}v;w\rangle+\lambda\int_\Omega vw\,dx=\int_\Omega gw\,dx$$
$$+\int_{\Gamma_1}[\mu(g_1-\lambda v)-kz(\cdot,1)]w\,d\Gamma \quad \forall w\in V.$$

Due to (2.40), this is now equivalent to

$$-\langle T_{00}u + T_{10}v; w\rangle = \int_{\Gamma_1} [\mu(g_1 - \lambda v) - kz(\cdot, 1)]w \, d\Gamma \quad \forall w \in V.$$

This proves that the third identity of (2.30) holds (while (2.41) is meaningless).

The well-posedness result follows from the Lumer-Phillips theorem. ∎

Let us now recall the next technical results proved in Theorem 4.2 of [25] (see also Theorems 2.3 and 3.2 of [26]).

Lemma 2.3 *Let us set*

$$E(A_0; L^2(\Omega)) := \{u \in H^{m_0}(\Omega) : A_0 u \in L^2(\Omega)\}.$$

Then for any $u \in E(A_0; L^2(\Omega))$ and any $j = 0, \ldots, m_0 - 1$, $T_{0j}u$ belongs to $H^{m_0-j-1/2}(\Gamma)'$ and the next Green's formula holds

$$a_0(u, w) = \int_\Omega A_0 u w \, dx - \sum_{j=0}^{m_0-1} \langle T_{0j}u; D_{0j}w\rangle \quad \forall w \in H^{m_0}(\Omega),$$

where $\langle T_{0j}u; D_{0j}w\rangle$ means the duality bracket between $H^{m_0-j-1/2}(\Gamma)' = H^{-m_0+j+1/2}(\Gamma)$ and $H^{m_0-j-1/2}(\Gamma)$.

In a similar manner we now prove the next technical results.

Lemma 2.4 *Assume that $m_0 = m_1 = 1$. Let us set*

$$E(A_0, A_1; L^2(\Omega)) := \{(u, v) \in H^1(\Omega)^2 : A_0 u + A_1 v \in L^2(\Omega)\},$$

which is a Hilbert space for the norm

$$\|(u, v)\|_E^2 = \|u\|_{H^1(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 + \|A_0 u + A_1 v\|_{L^2(\Omega)}^2.$$

Then for any $(u, v) \in E(A_0, A_1; L^2(\Omega))$, $T_{00}u + T_{10}v$ belongs to $H^{-1/2}(\Gamma)'$ and the next Green's formula holds

$$a_0(u, w) + a_1(v, w) = \int_\Omega (A_0 u + A_1 v) w \, dx - \langle T_{00}u + T_{10}v; w\rangle_\Gamma \quad \forall w \in H^1(\Omega), \tag{2.42}$$

where $\langle \cdot; \cdot\rangle_\Gamma$ means the duality bracket between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Proof. We follows the lines of the proof of Theorem 1.5.3.10 of [13]. In a first step we prove that $\mathcal{D}(\bar{\Omega})^2$ is dense in $E(A_0, A_1; L^2(\Omega))$. For that purpose we fix an extension operator P from $H^1(\Omega)$ into $H^1(\mathbb{R}^n)$. Thus for every continuous linear form l on $E(A_0, A_1; L^2(\Omega))$, there exist $f_0, f_1 \in H^{-1}(\mathbb{R}^n)$ and $g \in L^2(\Omega)$ such that

$$l(u, v) = \langle f_0; Pu\rangle + \langle f_1; Pv\rangle + \int_\Omega (A_0 u + A_1 v) g \, dx \quad \forall (u, v) \in E(A_0, A_1; L^2(\Omega)). \tag{2.43}$$

Moreover since l depends only on u and v in Ω and not on Pu and Pv in Ω^c, we deduce that supp $f_0 \subset \bar{\Omega}$ and supp $f_1 \subset \bar{\Omega}$.

The density result will be proved if we can show that any l that vanishes on $\mathcal{D}(\bar{\Omega})^2$ is identically equal to 0. Hence consider l such that

$$l(u,v) = 0 \quad \forall\ (u,v) \in \mathcal{D}(\bar{\Omega})^2.$$

From the previous considerations this is equivalent to have

$$\langle f_0; U\rangle + \langle f_1; V\rangle + \int_{\mathbb{R}^n} (\tilde{A}_0 U + \tilde{A}_1 V)\tilde{g}\, dx = 0 \quad \forall\ (U,V) \in \mathcal{D}(\mathbb{R}^n)^2,$$

where $\tilde{g}$ means the extension of g by zero outside Ω and for $i = 1, 2$, $\tilde{A}_i$ is an extension of A_i in Ω^c that keeps the same properties than A_i. But this is equivalent to (recall that $A_i^* = A_i$)

$$\tilde{A}_0\tilde{g} = -f_0 \text{ and } \tilde{A}_1\tilde{g} = -f_1 \text{ in } H^{-1}(\mathbb{R}^n).$$

Consequently due to the ellipticity assumption we get that $\tilde{g}$ belongs to $H^1(\mathbb{R}^n)$ or equivalently g belongs to $H^1_0(\Omega)$. Using these informations in (2.43) we get

$$l(u,v) = -\langle \tilde{A}_0\tilde{g}; Pu\rangle - \langle \tilde{A}_1\tilde{g}; Pv\rangle + \int_\Omega (A_0 u + A_1 v) g\, dx \quad \forall (u,v) \in E(A_0, A_1; L^2(\Omega)).$$

Now we take a sequence $g_n \in \mathcal{D}(\Omega)$ such that

$$g_n \to g \text{ in } H^1_0(\Omega) \text{ as } n \to \infty.$$

Then for $(u,v) \in E(A_0, A_1; L^2(\Omega))$,

$$l(u,v) = \lim_{n\to\infty} \Big(-\langle \tilde{A}_0\tilde{g}_n; Pu\rangle - \langle \tilde{A}_1\tilde{g}_n; Pv\rangle + \int_\Omega (A_0 u + A_1 v) g_n\, dx \Big).$$

But since g_n is smooth we may write

$$\begin{aligned} -\langle \tilde{A}_0\tilde{g}_n; Pu\rangle - \langle \tilde{A}_1\tilde{g}_n; Pv\rangle + \int_\Omega (A_0 u + A_1 v) g_n\, dx &= -\int_\Omega A_0 g_n u\, dx \\ -\int_\Omega A_1 g_n v\, dx &+ \int_\Omega (A_0 u + A_1 v) g_n\, dx = 0, \end{aligned}$$

by an application of Green's formula. These two identities show that l is zero on $E(A_0, A_1; L^2(\Omega))$.

With the density result in hands, we consider the mapping

$$T : \mathcal{D}(\bar{\Omega})^2 \to L^2(\Gamma) : (u,v) \to T_{00}u + T_{10}v.$$

Fix for a moment $(u,v) \in \mathcal{D}(\bar{\Omega})^2$. Then by Green's formula for any $w \in H^1(\Omega)$, we have

$$\int_\Gamma T(u,v) w\, d\Gamma = \int_\Omega (A_0 u + A_1 v) w\, dx - a_0(u,w) - a_1(v,w). \tag{2.44}$$

Therefore by Cauchy-Schwarz's inequality we obtain

$$|\int_\Gamma T(u,v)w\, d\Gamma| \le C\|(u,v)\|_E\|w\|_{H^1(\Omega)},$$

for some positive constant C that does not depend on u, v, w. This shows that $T(u,v)$ belongs to $H^{-1/2}(\Gamma)$ because the trace operator

$$w \to w_{|\Gamma}$$

is continuous and surjective from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$.

Since $\mathcal{D}(\bar{\Omega})^2$ is dense in $E(A_0, A_1; L^2(\Omega))$, we can extend T by density to the whole of $E(A_0, A_1; L^2(\Omega))$. Finally we obtain (2.42) by passing to the limit in the standard Green's formula (2.44), the left-hand side being transformed into a duality bracket after the limit process. ■

3 Stability result

In this section, we will prove an exponential stability result for problem (1.2)–(1.7) under the assumption (1.8).

Fix a positive constant α such that

$$\alpha > |k| \quad \text{and} \quad \alpha C_P < 2 - |k|C_P. \tag{3.1}$$

Note that such a α exists due to the assumption (1.8). Now, let us introduce the energy of the system

$$\begin{aligned} E(t) = \frac{1}{2}(\int_\Omega u_t^2(x,t)dx + a_0(u,u)) + \frac{\mu}{2}\int_{\Gamma_1} u_t^2(x,t)d\Gamma \\ +\frac{\alpha}{2}\int_{t-\tau}^t\int_{\Gamma_1} u_t^2(x,s)d\Gamma ds, \end{aligned} \tag{3.2}$$

which is the standard energy for wave type equation plus an integral term due to the presence of a time delay and, in the case $\mu > 0$, a term due to the dynamical boundary condition.

Proposition 3.1 *Assume* (1.8). *For any strong solution of problem* (1.2) − (1.7) *the energy is decreasing and there exists a positive constant C such that*

$$E'(t) \le -C\left\{a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x, t-\tau)d\Gamma\right\}. \tag{3.3}$$

Proof. Differentiating (3.2) we obtain

$$E'(t) = \int_\Omega u_t u_{tt} dx + a_0(u, u_t) + \frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t) d\Gamma$$
$$-\frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma + \mu \int_{\Gamma_1} u_t(x,t) u_{tt}(x,t) d\Gamma.$$

Using (1.2) and taking into account the regularity of u, we get

$$E'(t) = -\int_\Omega u_t(A_0 u + A_1 u_t)\, dx + a_0(u, u_t) + \frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t) d\Gamma$$
$$-\frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma + \mu \int_{\Gamma_1} u_t(x,t) u_{tt}(x,t) d\Gamma.$$

Applying Lemmas 2.3 and 2.4 and using the boundary condition (1.4), we obtain

$$E'(t) = -a_1(u_t, u_t) + \langle T_{00} u + T_{10} u_t; u_t\rangle_{\Gamma_1} + \frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t) d\Gamma$$
$$-\frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma + \mu \int_{\Gamma_1} u_t(x,t) u_{tt}(x,t) d\Gamma.$$

By the feedback law (1.5), we arrive at

$$E'(t) = -a_1(u_t, u_t) - k\int_{\Gamma_1} u_t(x,t) u_t(x,t-\tau) d\Gamma$$
$$+\frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t) d\Gamma - \frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma.$$

Then, from Cauchy-Schwarz inequality,

$$\begin{aligned} E'(t) = -a_1(u_t, u_t) + \frac{|k|}{2}\int_{\Gamma_1} u_t^2(x,t) d\Gamma + \frac{|k|}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma \\ +\frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t) d\Gamma - \frac{\alpha}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma, \end{aligned} \tag{3.4}$$

and so

$$E'(t) \le -\left(1 - \frac{|k|+\alpha}{2} C_P\right) a_1(u_t, u_t) - \frac{\alpha - |k|}{2}\int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma,$$

where in this estimate we have used the Poincaré's estimate (2.4). Since the constant α satisfies (3.1), estimate (3.3) is proved. ∎

Proposition 3.2 *Assume* (1.8). *There exists a time* $\overline{T} > 0$ *such that for all times* $T > \overline{T}$ *there exists a positive constant* C_0 *(depending on* T*) for which*

$$E(T) \le C_0 \int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x,t-\tau) d\Gamma \right\} dt, \tag{3.5}$$

for any strong solution u *of problem* (1.2) – (1.7).

Proof. Multiplying equation (1.2) by u and integrating in space and time, we have

$$\int_0^T \int_\Omega (u_{tt} + A_0 u + A_1 u_t) u dx dt = 0 \, .$$

So, integrating by parts, i.e., using Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned} \Big[\int_\Omega u_t(t) u(t) dx \Big]_0^T - \int_0^T \int_\Omega u_t^2(t) dx dt + \int_0^T a_0(u,u) dt \\ + \int_0^T a_1(u_t, u) dt + \int_0^T \langle T_{00} u + T_{10} u_t ; u \rangle_{\Gamma_1} dt = 0 \, , \end{aligned} \tag{3.6}$$

where we used also the boundary conditions (1.3), (1.4).

From (3.6) using the boundary condition (1.5) we have

$$\begin{aligned} \int_0^T a_0(u,u) dt = - \Big[\int_\Omega u_t(t) u(t) dx \Big]_0^T + \int_0^T \int_\Omega u_t^2(t) dx dt \\ - \int_0^T \int_{\Gamma_1} u(t) (\mu u_{tt}(t) + k u_t(t-\tau)) d\Gamma dt - \int_0^T a_1(u_t, u) dt. \end{aligned} \tag{3.7}$$

Then, integrating by parts in time,

$$\begin{aligned} \int_0^T a_0(u,u) dt = - \Big[\int_\Omega u_t(t) u(t) dx \Big]_0^T - \mu \Big[\int_{\Gamma_1} u(t) u_t(t) d\Gamma \Big]_0^T + \int_0^T \int_\Omega u_t^2(t) dx dt \\ + \mu \int_0^T \int_{\Gamma_1} u_t^2(t) d\Gamma dt - k \int_0^T \int_{\Gamma_1} u(t) u_t(t-\tau) d\Gamma dt - \int_0^T a_1(u_t, u) dt. \end{aligned} \tag{3.8}$$

Since (2.3) guarantees that $a_1(u,v)$ is an inner product on $H^1_{\Gamma_0}(\Omega)$ by Cauchy-Schwarz inequality we obtain

$$|a_1(u_t, u)| \le a_1(u_t, u_t)^{1/2} a_1(u,u)^{1/2}.$$

Now by the continuity of a_1 and (2.2) we find a constant $C > 0$ such that

$$|a_1(u_t, u)| \le C a_1(u_t, u_t)^{1/2} a_0(u,u)^{1/2},$$

and finally by Young's inequality we obtain

$$|a_1(u_t, u)| \le C^2 a_1(u_t, u_t) + \frac{1}{4} a_0(u,u). \tag{3.9}$$

On the other hand using a trace result there exists $C_1 > 0$ such that

$$| \int_{\Gamma_1} u(t) u_t(t-\tau) d\Gamma | \le C_1 \|u\|_{H^1(\Omega)} \left(\int_{\Gamma_1} u_t^2(t-\tau) d\Gamma \right)^{1/2} .$$

Hence using again (2.2) and Young's inequality we get

$$|\int_{\Gamma_1} u(t)u_t(t-\tau)d\Gamma| \le C_1^2 \int_{\Gamma_1} u_t^2(t-\tau)d\Gamma + \frac{1}{4}a_0(u,u). \tag{3.10}$$

From (3.8), (3.9), (3.10) and (2.2) and (2.4) we deduce the following inequality:

$$\begin{aligned} \frac{1}{2}\int_0^T a_0(u,u)dt \le \tilde{C}(E(T)+E(0)) + C\int_0^T a_1(u_t,u_t)dt \\ +C\int_0^T\int_{\Gamma_1} u_t^2(t-\tau)d\Gamma dt, \end{aligned} \tag{3.11}$$

for suitable positive constants $C, \tilde{C}$. In the following we will denote by C any suitable positive constant, while $\tilde{C}$ is the constant in (3.11). Note that the constant $\tilde{C}$ is independent of T.

By adding $\frac{1}{2}\int_0^T\int_\Omega u_t^2 dxdt$ and $\frac{\mu}{2}\int_0^T\int_{\Gamma_1} u_t^2 d\Gamma dt$ to both sides of (3.11) and using once more (2.2), we obtain

$$\begin{aligned} \int_0^T E_S(t)dt + \frac{\mu}{2}\int_0^T\int_{\Gamma_1} u_t^2(t)d\Gamma dt \le \tilde{C}(E(T)+E(0)) \\ +C\int_0^T \left\{a_1(u_t,u_t) + \int_{\Gamma_1} u_t^2(t-\tau)d\Gamma\right\} dt, \end{aligned} \tag{3.12}$$

where $E_S(\cdot)$ denotes the standard energy for our system, that is $E_S(t) := \frac{1}{2}\int_\Omega u_t^2 dx + a_0(u,u)$.

Now observe that, for $T > \tau$, we have

$$\begin{aligned} \int_0^T\int_{t-\tau}^t\int_{\Gamma_1} u_t^2(x,s)d\Gamma dsdt \le \int_0^T\int_{-\tau}^T\int_{\Gamma_1} u_t^2(x,s)d\Gamma dsdt \\ = \int_0^T\int_0^{T+\tau}\int_{\Gamma_1} u_t^2(x,s-\tau)d\Gamma dsdt \le T\int_0^{T+\tau}\int_{\Gamma_1} u_t^2(x,t-\tau)d\Gamma dt \end{aligned}$$

Adding $\frac{\alpha}{2}\int_0^T\int_{t-\tau}^t\int_{\Gamma_1} u_t^2(x,s)d\Gamma dsdt$ to both sides of (3.12) and using the above estimate we deduce

$$\int_0^T E(t)dt \le \tilde{C}(E(T)+E(0)) + C\int_0^{T+\tau}\left\{a_1(u_t,u_t) + \int_{\Gamma_1} u_t^2(t-\tau)d\Gamma\right\} dt\,. \tag{3.13}$$

From (3.4), using a trace result and (2.3), we have

$$E(0) \le E(T) + C\int_0^T\left\{a_1(u_t,u_t) + \int_{\Gamma_1} u_t^2(t-\tau)d\Gamma\right\} dt\,,$$

and so, using this estimate in (3.13) and the fact that the energy $E(\cdot)$ is decreasing,

$$TE(T) \le \int_0^T E(t)dt \le 2\tilde{C}E(T) + C\int_0^{T+\tau}\left\{a_1(u_t,u_t) + \int_{\Gamma_1} u_t^2(t-\tau)d\Gamma\right\} dt\,. \tag{3.14}$$

Therefore, we can estimate

$$(T-2\tilde{C})E(T+\tau) \le (T-2\tilde{C})E(T) \le C\int_0^{T+\tau}\left\{a_1(u_t,u_t)+\int_{\Gamma_1}u_t^2(t-\tau)d\Gamma\right\}dt\,.$$

Since the constant $\tilde{C}$ is independent of T (while C depends on T), then for T sufficiently large estimate (3.5) is proved. ■

Theorem 3.3 *Assume* (1.8). *For any strong solution of problem* (1.2) – (1.7)

$$E(t) \le C_1 E(0)e^{-C_2 t}, \quad t>0,$$

with constants C_1, C_2 independent of the initial data.

Proof. From (3.3), we have

$$E(T)-E(0) \le -C\int_0^T\left\{a_1(u_t,u_t)+\int_{\Gamma_1}u_t^2(x,t-\tau)d\Gamma\right\}dt,$$

or equivalently

$$\int_0^T\left\{a_1(u_t,u_t)+\int_{\Gamma_1}u_t^2(x,t-\tau)d\Gamma\right\}dt \le C^{-1}(E(0)-E(T)). \tag{3.15}$$

By (3.15) and the estimate (3.5), we obtain

$$E(T) \le C_0\int_0^T\left\{a_1(u_t,u_t)+\int_{\Gamma_1}u_t^2(x,t-\tau)d\Gamma\right\} \le C_0C^{-1}(E(0)-E(T)),$$

then

$$E(T) \le \tilde{C}E(0),$$

with $\tilde{C}<1$. This easily implies the exponential stability estimate, since our system (1.2)–(1.7) is invariant by translation and the energy $E(\cdot)$ is decreasing. ■

Remark 3.4 Analogous arguments apply if we have more than one delay term in the boundary feedback, that is if condition (1.5) is substituted by

$$\mu u_{tt}(x,t)+\sum_{i=1}^r k_i u_t(x,t-\tau_i) = T_{00}u(x,t)+T_{10}u_t(x,t) \quad \text{on} \quad \Gamma_1\times(0,+\infty),$$

with $\tau_i, i=1,\dots,r$, positive parameters and real numbers $k_i, i=1,\dots,r,$. In this case, the right energy for our problem is

$$E(t) := \frac{1}{2}\int_\Omega u_t^2(x,t)dx+\frac{1}{2}a_1(u,u)+\frac{\mu}{2}\int_{\Gamma_1}u_t^2(x,t)d\Gamma+\sum_{i=1}^r\frac{\alpha_i}{2}\int_{t-\tau_i}^t\int_{\Gamma_1}u_t^2(x,s)d\Gamma ds,$$

with suitable positive constants $\alpha_i, i = 1, \dots, r$. Indeed, if

$$1 > C_P \sum_{i=1}^{r} |k_i|,$$

choosing α_i such that

$$\alpha_i > |k_i|, \ i = 1, \dots, r, \quad \text{and} \quad C_P \sum_{i=1}^{r} \alpha_i < 2 - C_p \sum_{i=1}^{r} |k_i|,$$

we can prove that the energy is exponentially decaying to zero.

4 Some examples

4.1 The damped wave equation

As a first example we can take $A_0 = -\Delta$ and $A_1 = -a\Delta$ with associated forms

$$a_0(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad a_1(u, v) = a a_0(u, v),$$

where a is a fixed positive parameter. In that case $V = H^1_{\Gamma_0}(\Omega)$ and (2.2) and (2.3) hold by the Poincaré-Friedrichs inequality.

With that choice (1.1) holds with

$$T_{00}u = -\frac{\partial u}{\partial \nu} \quad T_{10}u = -a\frac{\partial u}{\partial \nu},$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\dfrac{\partial u}{\partial \nu}$ is the normal derivative.

In this situation problem (1.2)–(1.7) reduces to

$$u_{tt}(x, t) - \Delta u(x, t) - a\Delta u_t(x, t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{4.1}$$

$$u(x, t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty) \tag{4.2}$$

$$\mu u_{tt}(x, t) + \frac{\partial u}{\partial \nu}(x, t) = -a\frac{\partial u_t}{\partial \nu}(x, t) - k u_t(x, t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \tag{4.3}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \tag{4.4}$$

$$u_t(x, t) = f_0(x, t) \quad \text{in } \Gamma_1 \times (-\tau, 0). \tag{4.5}$$

For $\mu = 0$, the above model, with boundary condition on Γ_1,

$$\frac{\partial u}{\partial \nu}(x, t) = -k u_t(x, t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \tag{4.6}$$

instead of (4.3), has been introduced and investigated in [9]. It can be shown that system (4.1), (4.2), (4.6) with initial data in suitable spaces is exponentially stable if $\tau = 0$, that is in absence of delay. We refer also to [3, 20, 21, 23, 17, 18, 19, 24, 37] for the more studied case $a = 0$. However, the feedback law (4.6) is not stable with respect to small time delays. Indeed, by direct eigenvalue calculations, in [9] it is proved that for any $a, k > 0$ and any arbitrarily small delay, system (4.1), (4.2), (4.6) admits solutions whose energy is not decaying to zero. Hence, it is important to look for feedback laws which are robust with respect to (small) time delays.

The 1–d version of the above model with $\mu = 0$ in the boundary condition (4.3) has been considered by Morgül [29] who proposed a class of dynamic boundary controllers to solve the stability robustness problem.

In the case of dynamic boundary condition, that is $\mu > 0$ in (4.3), the above model without delay (e.g. $\tau = 0$) has been proposed in one dimension by Pellicer and Sòla–Morales [32] as an alternative model for the classical spring–mass damper system. It is well-known that in absence of delay the system is exponentially stable. Then, we are interested in conditions ensuring the robustness with respect to small delays in the boundary feedback.

For this purpose, note that (2.4) may be rephrased as follows

$$\int_{\Gamma_1} |v|^2 \, d\Gamma \le \frac{\tilde{C}_P}{a} a \int_\Omega \nabla v \cdot \nabla v \, dx, \quad \forall v \in H^1_{\Gamma_0}(\Omega), \tag{4.7}$$

where $\tilde{C}_P$ is the smallest positive constant such that

$$\int_{\Gamma_1} |v|^2 \, d\Gamma \le C \int_\Omega \nabla v \cdot \nabla v \, dx, \quad \forall v \in H^1_{\Gamma_0}(\Omega)$$

holds. This means that $C_P = \frac{\tilde{C}_P}{a}$ and (1.8) is equivalent to

$$a > k\tilde{C}_P \, . \tag{4.8}$$

Therefore, Theorem 3.3 implies exponential stability of system (4.1)–(4.5) under the assumption (4.8). Some counterexamples when this condition is not satisfied are illustrated in section 5.

4.2 The damped Kirchoff model

Here we reduce to the one-dimensional or two-dimensional case, i.e. we let $\Omega \subset \mathbb{R}^n$, with $n = 1$ or 2. As operator A_0 we take the biharmonic operator

$$A_0 = \Delta^2,$$

with associated bilinear form

$$a_0(u, v) = \int_\Omega u'' v'' dx, \tag{4.9}$$

in dimension 1, while in dimension 2, we take

$$a_0(u, v) = \int_\Omega \Big\{ \Delta u \Delta v - (1 - \tilde{\nu}) \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \right) \Big\} dx dy, \tag{4.10}$$

where $\tilde{\nu}$ is the so–called Poisson coefficient, that depends on the constitutive material of the plate Ω and is a real parameter that belongs to $(-1, \frac{1}{2})$. As operator D_{0j}, $j = 0, 1$, we take

$$D_{00} = Id \text{ and } D_{01} = \frac{\partial}{\partial \nu}.$$

With that choice, we have

$$V = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\},$$

and (2.2) holds still by the Poincaré-Friedrichs inequality.

With this choice, we know that (1.1) holds for $i = 0$ with

$$T_{00}u = Nu = \frac{\partial u''}{\partial \nu}, \quad T_{01}u = -Mu = -u''$$

in dimension 1, while

$$T_{00}u = Nu = \frac{\partial \Delta u}{\partial \nu} + (1 - \tilde{\nu}) \frac{\partial^3 u}{\partial \nu \partial \tau^2}, \quad T_{01}u = -Mu = -\tilde{\nu} \Delta u + (1 - \tilde{\nu}) \frac{\partial^2 u}{\partial \nu^2}.$$

in dimension 2.

We make the same choice as before for the operator A_1 and the form a_1.

For that choice, problem (1.2)–(1.7) reduces to

$$u_{tt}(x,t) + \Delta^2 u(x,t) - a\Delta u_t(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{4.11}$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty) \tag{4.12}$$

$$Mu(x,t) = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \tag{4.13}$$

$$\mu u_{tt}(x,t) = Nu(x,t) - a\frac{\partial u_t}{\partial \nu}(x,t) - ku_t(x, t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty) \tag{4.14}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega \tag{4.15}$$

$$u_t(x,t) = f_0(x,t) \quad \text{in } \Gamma_1 \times (-\tau, 0). \tag{4.16}$$

This system with $a = \mu = \tau = 0$ has been studied in section 9.4 of [18] where it is proved that the system is exponentially stable under some standard geometrical conditions on Γ_0 and Γ_1 (for the use of two feedbacks, we refer to [22, 33, 15]). According to [8, 10], instability phenomena occur when $a = \mu = 0$ and $\tau > 0$.

The system with $a = \tau = 0$ is extensively studied in the literature and corresponds to some SCOLE models, where some exponential or polynomial stability results are proved in [4, 5, 6, 7, 27, 28, 34, 35] (for nonlinear problems, see for instance [12, 14]). As in the previous section, the term $a\Delta u_t$ with $a > 0$ is introduced in order to restitute a stability result independently of the delay.

Since the operator A_1 has not changed with respect to the previous subsection, Theorem 3.3 implies exponential stability of system (4.11)–(4.16) under the assumption (4.8).

5 Some instability examples

In this section we will give some instability examples, in dimension one, for problem (4.1)–(4.5) if a is close enough to zero and hence if condition (4.8) is no more valid.

For that purpose, let us consider the spectral problem for the system (4.1)–(4.5). Namely we look for a solution u of this system in the form

$$u(x,t) = e^{\lambda t}\varphi(x), \quad \lambda \in \mathbb{C}.$$

Then, φ has to be a solution of the eigenvalue problem

$$\left\{ \begin{array}{l} \lambda^2\varphi - (1+a\lambda)\Delta\varphi = 0 \quad \text{in} \quad \Omega, \\ \varphi = 0 \quad \text{on} \quad \Gamma_0, \\ (1+a\lambda)\frac{\partial\varphi}{\partial\nu} + (\mu\lambda^2 + k\lambda e^{-\lambda\tau})\varphi = 0 \quad \text{on} \quad \Gamma_1. \end{array} \right. \tag{5.1}$$

5.1 The case $\mu = 0$

Take $\Omega = (0,1)$, $\Gamma_0 = \{0\}$ and $\Gamma_1 = \{1\}$. For the sake of simplicity, we fix $k = 1$. In that situation, the constant $\tilde{C}_P = 1$. Indeed by the identity,

$$v(1) = \int_0^1 v'(x)\,dx,$$

valid for all $v \in H^1_{\Gamma_0}(\Omega)$, we obtain by Cauchy-Schwarz's inequality that

$$|v(1)|^2 \leq \int_0^1 |v'|^2\,dx, \forall v \in H^1_{\Gamma_0}(\Omega),$$

which shows that (2.4) holds with $\tilde{C}_P \leq 1$. But the choice $v(x) = x$ implies that $\tilde{C}_P = 1$ because in that case the above inequality becomes an equality.

Now with the above choices, the eigenvalue problem (5.1) reduces to find $\lambda \in \mathbb{C}$ and $\varphi \in H^2(0,1)$ solution of

$$\left\{ \begin{array}{l} \lambda^2\varphi - (1+a\lambda)\varphi'' = 0 \quad \text{in} \quad (0,1), \\ \varphi(0) = 0, \\ (1+a\lambda)\varphi'(1) + \lambda e^{-\lambda\tau}\varphi(1) = 0. \end{array} \right. \tag{5.2}$$

Hence for $\Re\lambda \geq 0$ and $\lambda \neq 0$, φ takes the form

$$\varphi(x) = \alpha \sinh\left(\frac{\lambda x}{\sqrt{1+a\lambda}}\right),$$

for some constant α. The boundary condition at 1 is then equivalent to

$$\alpha\Big(\sqrt{1+a\lambda}\cosh\left(\frac{\lambda}{\sqrt{1+a\lambda}}\right) + e^{-\lambda\tau}\sinh\left(\frac{\lambda}{\sqrt{1+a\lambda}}\right)\Big) = 0.$$

Therefore any non zero eigenvalue λ of problem (5.2) such that $\Re\lambda \geq 0$ is a solution of the equation

$$\cosh(\frac{\lambda}{\sqrt{1+a\lambda}}) + \frac{e^{-\lambda\tau}}{\sqrt{1+a\lambda}} \sinh(\frac{\lambda}{\sqrt{1+a\lambda}}) = 0. \tag{5.3}$$

This characteristic equation is similar to the equation (3.15) of [9] but unfortunately the analysis performs in that paper to find solutions λ with a non negative real part cannot be adapted to our situation. Nevertheless some numerical results presented in Figures 2 to 4 for different values of τ show that the equation (5.3) has indeed solution λ such that $\Re\lambda \geq 0$ if a is small enough. For $\tau = 6$, Figure 2 shows even that for all $a \in [0, 1]$, there exists a solution λ such that $\Re\lambda \geq 0$ and therefore the system is not stable for $\tau = 6$.

In order to obtain analytic results we use a perturbation argument. Indeed using Theorem 5.1 below, we know that if the equation (5.3) with $a = 0$ has a solution λ_0 such that $\Re\lambda_0 > 0$, then the equation (5.3) has a solution λ_a such that $\Re\lambda_a > 0$ if $a > 0$ is small enough. Hence we are reduced to study the equation (5.3) with $a = 0$, namely

$$\cosh\lambda + e^{-\lambda\tau} \sinh\lambda = 0. \tag{5.4}$$

In a first attempt we look at τ in the form

$$\tau = 2n,$$

with $n \in \mathbb{N}$. In that case, (5.4) becomes

$$(e^{\lambda} + e^{-\lambda}) + e^{-2n\lambda}(e^{\lambda} - e^{-\lambda}) = 0.$$

Hence by setting $y = e^{-2\lambda}$, we find the polynomial equation

$$y^{n+1} - y^n - y - 1 = 0. \tag{5.5}$$

Now setting $p(y) = y^{n+1} - y^n - y - 1$, we notice that $p(0) = -1$ and $p(-1) = 2$ if n is odd. Hence for n odd, we deduce that p has a root $y_0 \in (-1, 0)$ and coming back to λ, we find a solution λ_0 of (5.4) given by

$$-2\lambda_0 = \ln(-y_0) \pm i\pi,$$

or equivalently

$$\lambda_0 = -\frac{1}{2}\ln(-y_0) \pm i\frac{\pi}{2}.$$

Since $y_0 \in (-1, 0)$, $\ln(-y_0)$ is negative and therefore the real part of λ_0 is positive. Figures 1 and 2 show the real part of the roots of (5.3) for the values $\tau = 2$ and $\tau = 6$ respectively and for $a \in (0, 2)$. Red lines correspond to real roots, while blue lines give the real part of complex roots. These figures confirm the above considerations.

To get similar results for small values of delays, we now take τ in the form

$$\tau = \frac{2}{n},$$

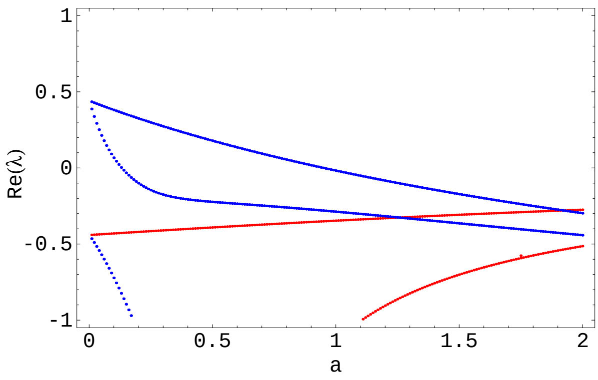

Figure 1: The case $\tau = 2$ and $a \in [0, 2]$

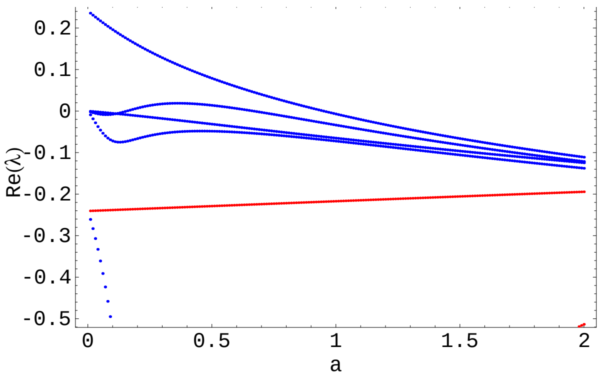

Figure 2: The case $\tau = 6$ and $a \in [0, 2]$

with $n \in \mathbb{N}$ odd. In that case, (5.4) becomes

$$(e^{\lambda} + e^{-\lambda}) + e^{-\frac{2\lambda}{n}}(e^{\lambda} - e^{-\lambda}) = 0.$$

Hence by setting $y = e^{-\frac{2\lambda}{n}}$, we find again the polynomial equation (5.5). Therefore we find a solution λ_0 of (5.4) given by

$$-\frac{2\lambda_0}{n} = \ln(-y_0) \pm i\pi,$$

or equivalently

$$\lambda_0 = -\frac{n}{2}\ln(-y_0) \pm i\frac{n\pi}{2}.$$

Since $\ln(-y_0)$ is negative and the real part of λ_0 is positive. Figures 3 and 4 show the real part of the roots of (5.3) for the values $\tau = 0.4$ and $\tau = 2/3$ respectively and for $a \in (0, 2)$. As before these figures confirm the above considerations.

In summary we have shown that there exist arbitrarily small or large delays for which the system (4.1)–(4.5) in 1d becomes unstable if a is too small.

Note also that we can repeat the above argument with the substitution $y = e^{-\frac{2\lambda}{m}}$, $m \in \mathbb{N}$, for both sequences $\tau = \frac{2}{n}$ and $\tau = 2n$ obtaining, for m, n odd, for arbitrarily small or large delays, solutions with arbitrarily large real part (cfr. [9]).

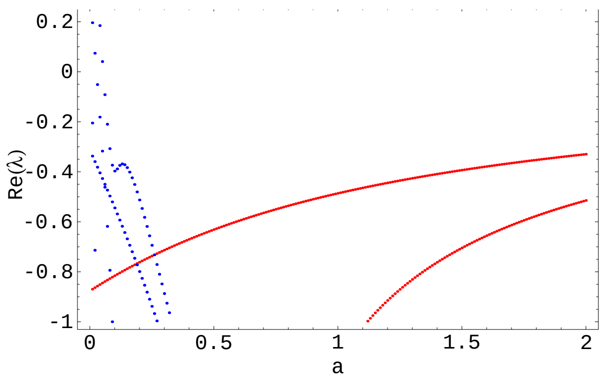

Figure 3: The case $\tau = 0.4$ and $a \in [0, 2]$

5.2 A perturbation result

In the previous section we have seen that we need to show that the eigenvalues of problem (5.1) approach the ones corresponding to $a = 0$ as $a > 0$ but going to zero. This is a natural perturbation result that we shall show by using some classical perturbation results from Kato. For that purpose, we notice that φ is a solution of (5.1) if and only if the vector

$$U := \left(\varphi, \lambda\varphi, \lambda e^{-\lambda\tau\rho}\varphi\right)^T$$

belongs to $\mathcal{D}(\mathcal{A})$ and satisfies

$$\mathcal{A}U = \lambda U.$$

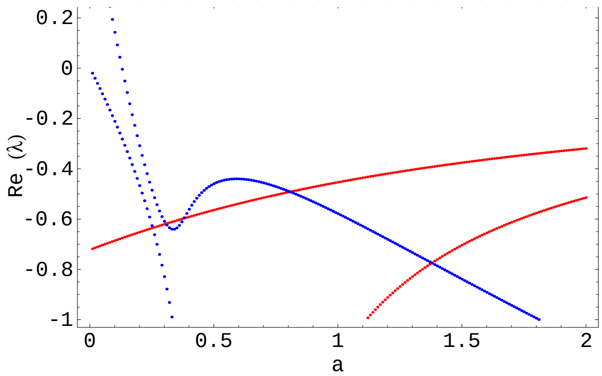

Figure 4: The case $\tau = 2/3$ and $a \in [0, 2]$

Hence we only need to study the dependence of the eigenvalue of the operator $\mathcal{A}$ with respect to a. For convenience we need to specify the dependency of $\mathcal{A}$ with respect to a, that is denoted by $\mathcal{A}_a$.

We now state and proof the following result.

Theorem 5.1 *The operator $\mathcal{A}_a$ tends to $\mathcal{A}_0$ in the generalized sense of Kato (cf. [16, section IV.2.6]) or equivalently*

$$(\lambda - \mathcal{A}_a)^{-1} \to (\lambda - \mathcal{A}_0)^{-1} \text{ in norm as } a \to 0, \forall \lambda \in \rho(\mathcal{A}_0).$$

Consequently if λ_0 is an eigenvalue of $\mathcal{A}_0$ of algebraic multiplicity k, then for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $a \in (0, \delta_\epsilon)$, $\mathcal{A}_a$ has k eigenvalues in the open ball $B(\lambda_0, \epsilon)$.

Proof. Fix a positive real number λ. By Theorem 2.1 for all $F := (f, g, h)^T \in \mathcal{H}$, there exists a unique $U_a = (u_a, v_a, z_a)^T \in \mathcal{D}(\mathcal{A}_a)$ solution of

$$(\lambda I - \mathcal{A}_a) \begin{pmatrix} u_a \\ v_a \\ z_a \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}.$$

According to the proof of Theorem 2.2, they are given by

$$z_a(x, \rho) = \lambda u_a(x) e^{-\lambda\rho\tau} - f(x) e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho h(x, \sigma) e^{\lambda\sigma\tau} d\sigma \quad \text{on } \Gamma_1 \times (0, 1), \quad (5.6)$$

$$v_a = \frac{\lambda}{1 + \lambda a} s_a - \frac{1}{1 + \lambda a} f, \quad (5.7)$$

$$u_a = \frac{1}{1 + \lambda a} s_a + \frac{a}{1 + \lambda a} f, \quad (5.8)$$

where $s_a \in H^1_{\Gamma_0}(\Omega)$ is the unique solution of

$$b_a(s_a, w) = \int_\Omega (g + \frac{\lambda}{1 + \lambda a} f) w dx - \int_{\Gamma_1} (\frac{k a \lambda}{1 + \lambda a} e^{-\lambda\tau} f + k z_0) w d\Gamma \quad \forall w \in H^1_{\Gamma_0}(\Omega). \quad (5.9)$$

with

$$b_a(s,w) = \int_\Omega (\frac{\lambda^2}{1+\lambda a} sw + \nabla s \cdot \nabla w) dx + \int_{\Gamma_1} \frac{k\lambda}{1+\lambda a} e^{-\lambda\tau} sw d\Gamma.$$

Since

$$b_a(s,s) = \int_\Omega (\frac{\lambda^2}{1+\lambda a} s^2 + |\nabla s|^2) dx + \int_{\Gamma_1} \frac{k\lambda}{1+\lambda a} e^{-\lambda\tau} s^2 d\Gamma,$$

we deduce that there exists $a_0 > 0$ small enough and a positive constant α_0 (independent of a) such that for all $a \in [0, a_0]$, one has

$$b_a(s,s) \geq \alpha_0 (\|s\|^2_{H^1(\Omega)} + \|s\|^2_{L^2(\Gamma_1)}) \quad \forall s \in H^1_{\Gamma_0}(\Omega). \tag{5.10}$$

For the remainder of the proof a is arbitrary in $[0, a_0]$, and $\beta_0 > 0$ is a positive constant independent of a (that may depend on λ) and that varies from place to place.

The identity (5.9), the estimate (5.10) and Cauchy-Schwarz's inequality yield

$$\|s_a\|_{H^1(\Omega)} + \|s_a\|_{L^2(\Gamma_1)} \leq \beta_0 \|F\|_{\mathcal{H}}, \tag{5.11}$$

where $\|\cdot\|_{\mathcal{H}}$ is the natural norm of $\mathcal{H}$, i.e.,

$$\| (u,v,z)^\top \|^2_{\mathcal{H}} = \int_\Omega \{|\nabla u(x)|^2 + v(x)^2\} dx + \int_{\Gamma_1} \int_0^1 z(x,\rho)^2 d\rho d\Gamma.$$

Now for an arbitrary $w \in H^1_{\Gamma_0}(\Omega)$, by (5.9) and the definition of b_a, we may write

$$\begin{aligned} b_0(s_a - s_0, w) &= b_a(s_a, w) - b_0(s_0, w) + (b_0 - b_a)(s_a, w) &(5.12)\\ &= \frac{\lambda^2 a}{1+\lambda a} \int_\Omega f w dx + \frac{ka\lambda}{1+\lambda a} e^{-\lambda\tau} \int_{\Gamma_1} f w d\Gamma &(5.13)\\ &+ \frac{\lambda^3 a}{1+\lambda a} \int_\Omega s_a w dx + \frac{ka\lambda^2}{1+\lambda a} e^{-\lambda\tau} \int_{\Gamma_1} s_a w d\Gamma. &(5.14)\end{aligned}$$

Taking $w = s_a - s_0$ and using (5.10), we obtain for all $a \in [0, a_0]$:

$$\|s_a - s_0\|_{H^1(\Omega)} + \|s_a - s_0\|_{L^2(\Gamma_1)} \leq a\beta_0 (\|f\|_{0,\Omega} + \|f\|_{L^2(\Gamma_1)} + \|s_a\|_{0,\Omega} + \|s_a\|_{L^2(\Gamma_1)}).$$

Hence (5.11) and a trace theorem yield

$$\|s_a - s_0\|_{H^1(\Omega)} + \|s_a - s_0\|_{L^2(\Gamma_1)} \leq a\beta_0 \|F\|_{\mathcal{H}_0}. \tag{5.15}$$

Now using (5.7), we have

$$\|v_a - v_0\|_{0,\Omega} \leq \lambda \|s_a - s_0\|_{0,\Omega} + \frac{\lambda^2 a}{1+\lambda a} \|s_a\|_{0,\Omega} + \frac{\lambda a}{1+\lambda a} \|f\|_{0,\Omega}.$$

By (5.15) and (5.11) we deduce that

$$\|v_a - v_0\|_{0,\Omega} \leq a\beta_0 \|F\|_{\mathcal{H}_0}. \tag{5.16}$$

Similarly using (5.8), (5.15) and (5.11) we get

$$\|u_a - u_0\|_{H^1(\Omega)} \leq a\beta_0 \|F\|_{\mathcal{H}_0}. \tag{5.17}$$

Finally thanks to (5.6) we have

$$\|z_a - z_0\|_{L^2(\Gamma_1\times(0,1))} \leq \beta_0 \|u_a - u_0\|_{L^2(\Gamma_1)}$$

and by a trace theorem and (5.17), we deduce that

$$\|z_a - z_0\|_{L^2(\Gamma_1\times(0,1))} \leq a\beta_0 \|F\|_{\mathcal{H}_0}. \tag{5.18}$$

The estimates (5.16), (5.17) and (5.18) show that

$$\|(\lambda - \mathcal{A}_a)^{-1}F - (\lambda - \mathcal{A}_0)^{-1}F\|_{\mathcal{H}_0} \leq a\beta_0 \|F\|_{\mathcal{H}_0},$$

which implies that

$$(\lambda - \mathcal{A}_a)^{-1} \to (\lambda - \mathcal{A}_0)^{-1} \text{ in norm as } a \to 0.$$

The reminder of the statements of the Theorem follow from Theorems IV.2.25 and IV.3.16 of [16]. ∎

5.3 The case $\mu > 0$

In the case $\mu > 0$, with the above choices, the eigenvalue problem (5.1) reduces to find $\lambda \in \mathbb{C}$ and $\varphi \in H^2(0,1)$ solution of

$$\left\{\begin{array}{l} \lambda^2\varphi - (1 + a\lambda)\varphi'' = 0 \quad \text{in} \quad (0,1), \\ \varphi(0) = 0, \\ (1 + a\lambda)\varphi'(1) + (\lambda e^{-\lambda\tau} + \lambda^2\mu)\varphi(1) = 0. \end{array}\right. \tag{5.19}$$

Hence for $\Re\lambda \geq 0$ and $\lambda \neq 0$, φ takes the form

$$\varphi(x) = \alpha \sinh(\frac{\lambda x}{\sqrt{1 + a\lambda}}),$$

for some constant α. The boundary condition at 1 is then equivalent to

$$\alpha\Big(\lambda\sqrt{1 + a\lambda}\cosh(\frac{\lambda}{\sqrt{1 + a\lambda}}) + (\lambda e^{-\lambda\tau} + \lambda^2\mu)\sinh(\frac{\lambda}{\sqrt{1 + a\lambda}})\Big) = 0.$$

Hence any non zero eigenvalue λ of problem (5.19) such that $\Re\lambda \geq 0$ is a solution of the equation

$$\cosh(\frac{\lambda}{\sqrt{1 + a\lambda}}) + \frac{e^{-\lambda\tau} + \lambda\mu}{\sqrt{1 + a\lambda}}\sinh(\frac{\lambda}{\sqrt{1 + a\lambda}}) = 0. \tag{5.20}$$

We rewrite (5.20) as

$$F(\lambda) + G(\lambda) = 0, \tag{5.21}$$

where

$$F(\lambda) = \cosh(\frac{\lambda}{\sqrt{1+a\lambda}}) + \frac{e^{-\lambda\tau}}{\sqrt{1+a\lambda}} \sinh(\frac{\lambda}{\sqrt{1+a\lambda}}),$$

$$G(\lambda) = \frac{\lambda\mu}{\sqrt{1+a\lambda}} \sinh(\frac{\lambda}{\sqrt{1+a\lambda}})\,.$$

We know from the previous analysis that, for small a and for arbitrarily small (large) delays, the equation $F(\lambda) = 0$ admits a solution $\overline{\lambda}$ with positive real part. Now, consider a ball $B \subset \{\lambda \in \mathbb{C} \, : \, Re\lambda > 0\}$ centered at $\overline{\lambda}$ and that does not contain other zeroes of F. Then, we have

$$|F(\lambda)| \geq \epsilon > 0 \quad \forall \ \ \lambda \in \partial B, \tag{5.22}$$

and

$$|G(\lambda)| \leq \mu \max_{\lambda\in\partial B} \left| \frac{\lambda}{\sqrt{1+a\lambda}} \sinh(\frac{\lambda}{\sqrt{1+a\lambda}}) \right| \,, \quad \forall \ \lambda \in \partial B\,. \tag{5.23}$$

So, for μ sufficiently small,

$$|G(\lambda)| < |F(\lambda)|\,, \quad \forall \ \lambda \in B\,,$$

and therefore, from Rouché's Theorem the equation (5.21) has a zero in the ball B.

In conclusion, we have also found unstable solutions of problem (4.1)–(4.5), if $a > 0$ and $\mu > 0$ are small.

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