Singular behavior of the solution of the periodic-Dirichlet heat equation in weighted L^p-Sobolev spaces

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Abstract

We consider the heat equation in a polygonal domain Ω of the plane in weighted L^p-Sobolev spaces

$$\begin{aligned} \partial_t u - \Delta u = h, &\quad \text{in } \Omega \times]-\pi, \pi[, \\ u = 0, &\quad \text{on } \partial\Omega \times [-\pi, \pi], \\ u(\cdot, -\pi) = u(\cdot, \pi), &\quad \text{in } \Omega. \end{aligned} \tag{0.1}$$

Here h belongs to $L^p(-\pi, \pi; L^p_\mu(\Omega))$, where $L^p_\mu(\Omega) = \{v \in L^p_{loc}(\Omega) : r^\mu v \in L^p(\Omega)\}$, with a real parameter μ and $r(x)$ the distance from x to the set of corners of Ω. We give sufficient conditions on μ, p and Ω that guarantee that problem (0.1) has a unique solution $u \in L^p(-\pi, \pi; L^p_\mu(\Omega))$ that admits a decomposition into a regular part in weighted L^p-Sobolev spaces and an explicit singular part.

The classical Fourier transform techniques do not allow to handle such a general case. Hence we use the theory of sums of operators.

Keywords: heat equation, singular behavior, nonsmooth domains.
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1 Introduction

This paper is the second one of a large program of research devoted to the study of (nonlinear) heat equation in nonsmooth domains in weighted L^p-Sobolev spaces. Our final goal requires precise information about the solution of the linear heat equation

$$\begin{aligned} \partial_t u - \Delta u = h, &\quad \text{in } \Omega \times]-\pi, \pi[, \\ u = 0, &\quad \text{on } \partial\Omega \times [-\pi, \pi], \\ u(\cdot, -\pi) = u(\cdot, \pi), &\quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

in weighted L^p-Sobolev spaces. In particular its decomposition into a regular part and an explicit singular part is needed. Although this theory is well developed in weighted L^2-Sobolev spaces [13, 16, 15, 4] or in L^p-Sobolev spaces [14], to our best knowledge such a result does not exist in the framework of weighted L^p-Sobolev spaces. The first class of papers are based on the Fourier analysis, while the second one uses the theory of sums of operators. For maximal regularity type results in weighted L^p-Sobolev spaces, we refer to [6, 19, 23, 20, 22]; here different techniques like estimates of the Green function, the theory of sum of operators or blowing up can be used.

According to the approach of [14], the study of the linear heat equation in non-hilbertian Sobolev spaces can be performed with the help of the theory of sums of operators. Hence the goal of this paper is to make this analysis in $L^p(-\pi, \pi; L^p_\mu(\Omega))$ for a large range of values of μ and p. Our results extend the ones from [13, 14] to the $L^p_\mu(\Omega)$ setting.

This theory also requires, in a first step, to obtain uniform estimates of the solution of the Helmholtz equation

$$\begin{aligned} -\Delta u + zu = g &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where z is a complex number. This was performed in the companion paper [8].

For the sake of simplicity we have restricted ourselves to two-dimensional domains Ω. The results of this paper can be easily extended to the case of domains with conical points.

The paper is organized as follows: In section 2 we recall some results on the sums of operators in Banach spaces of Da Prato-Grisvard [6] and of Dore-Venni [9]; we also state some basic results from [8] used later on. In section 3 we apply the approach of Da Prato-Grisvard to obtain a decomposition but with non-optimal regularity informations. Section 4 is devoted to the proof of the regularity of $(\partial_t - \Delta)S$, where S is the singular part of the solution obtained before. The use of the approach of Dore-Venni and the results of section 4 allows to get the optimal regularity result obtained in section 5.

In the whole paper the notation $a \lesssim b$ means the existence of a positive constant C, which is independent of the quantities a, b (and eventually the above parameter z) under consideration such that $a \leq Cb$.

2 Preliminary results

Results on the sums of operators in Banach spaces, such as the result of G. Da Prato and P. Grisvard [6] and of G. Dore and A. Venni [9], can be fruitfully used to prove the singular behaviour of elliptic problems in non-Hilbertian Sobolev spaces as in [14]. Let us recall these results.

Fix a complex Banach space E and a pair of closed linear densely defined operators $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$. Hence we can define their sum

$$L : D(L) := D(A) \cap D(B) \subset E \to E : x \mapsto Lx := Ax + Bx.$$

For an operator C we denote by $\sigma(C)$ and $\rho(C)$ respectively its spectrum and its resolvent set.

2.1 First strategy

Assumptions on A and B:
(H_1) There exist $M \geq 0$, $R \geq 0$, $\theta_A \in]0, \pi]$, $\theta_B \in]0, \pi]$ such that

$$\begin{aligned} \theta_A + \theta_B &> \pi, \\ S_A := \{\lambda \mid |\lambda| \geq R, |\arg \lambda| \leq \theta_A\} &\subset \rho(-A), \\ S_B := \{\lambda \mid |\lambda| \geq R, |\arg \lambda| \leq \theta_B\} &\subset \rho(-B), \end{aligned}$$

and, for all $\lambda \in S_A$ and all $\mu \in S_B$,

$$\|(A + \lambda\, I)^{-1}\| \leq \frac{M}{|\lambda|}, \qquad \|(B + \mu\, I)^{-1}\| \leq \frac{M}{|\mu|};$$

(H_2) $\sigma(-A) \cap \sigma(B) = \emptyset$;

(H_3) The resolvent of A and B commute, i.e., for every $\lambda \in \rho(-A)$ and every $\mu \in \rho(-B)$,

$$(A+\lambda\, I)^{-1}(B+\mu\, I)^{-1} = (B+\mu\, I)^{-1}(A+\lambda\, I)^{-1}.$$

Theorem 2.1. [**6**] *Under assumptions (H_1), (H_2) and (H_3), the operator L has an invertible closure.*

Definition 2.1. *The* closure *of L is defined by $x \in D(\bar{L})$ and $\bar{L}x = y$ if there exists a sequence $(x_n)_n \subset D(L)$, which satisfies $x_n \to x$ and $Lx_n \to y$.*

A solution of $\bar{L}x = y$ is called a strong solution *of $Lx = y$.*

The inverse of $\bar{L}$ is obtained as the integral

$$(\bar{L})^{-1} = \frac{1}{2i\pi}\int_\gamma (A+\lambda\, I)^{-1}(\lambda\, I - B)^{-1} d\lambda,$$

where γ is a path which separates $\sigma(-A)$ and $\sigma(B)$ and joins $\infty e^{-i\theta_\gamma}$ to $\infty e^{i\theta_\gamma}$ where θ_γ is chosen so that $\pi - \theta_B < \theta_\gamma < \theta_A$.

2.2 Second strategy

Assumptions on A, B and E:

(H_4) E is a U.M.D. space;

(H_5) $]-\infty, 0] \subset \rho(A) \cap \rho(B)$ and there exists $M \geq 0$ such that, for every $t \geq 0$,

$$\|(A+t\, I)^{-1}\| \leq \frac{M}{t+1}, \qquad \|(B+t\, I)^{-1}\| \leq \frac{M}{t+1};$$

This allows to define the complex power of A and B by setting, for $\Re(z) < 0$,

$$A^z = -\frac{\sin(\pi z)}{\pi}\int_0^{+\infty} t^z (A+t\, I)^{-1} dt.$$

This definition can be extended to $\Re(z) = 0$ by taking limits when they exist.

(H_6) For every $s \in \mathbb{R}$, the complex power A^{is} and B^{is} exist and are bounded operators. In addition there exist $K > 0$, $\tau_A > 0$, $\tau_B > 0$ such that

$$\tau_A + \tau_B < \pi,$$

and, for all $s \in \mathbb{R}$,

$$\|A^{is}\| \leq K e^{|s|\tau_A}, \qquad \|B^{is}\| \leq K e^{|s|\tau_B}.$$

Theorem 2.2. [**9**] *Under assumptions (H_3), (H_4), (H_5) and (H_6), the operator L is invertible.*

2.3 Results on the Helmholtz equation

In this paper, we work with a polygonal domain Ω of $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, in the following sense.

Definition 2.2. *Let Ω be a bounded domain of $\mathbb{R}^2$. We say that Ω is a* polygonal domain *if its boundary is the union of a finite number of line segments $\bar{\Gamma}_j$, $j \in \{1, \ldots, J\}$ (Γ_j being supposed to be open). Hence we do not assume that Ω is a Lipschitz domain, that is we include the presence of cracks.*

Denote by $S_j, j = 1, \cdots, J$ the vertices of $\partial\Omega$ enumerated clockwise. Without loss of generality we may assume that $B(S_j, 1) \cap \Omega$ does not contain any other vertex of Ω. For $j \in \{1, 2, \cdots, J\}$, let ψ_j be the interior angle of Ω at the vertex S_j, $\lambda_j = \frac{\pi}{\psi_j}$ and (r_j, θ_j) the polar coordinates centered at S_j such that

$$B(S_j, 1) \cap \Omega = \{(r_j \cos\theta_j, r_j \sin\theta_j) \mid 0 < r_j < 1,\, 0 < \theta_j < \psi_j\} =: D_j.$$

For $\vec{\mu} = (\mu_j)_{j=1}^J$, we define the spaces $L^p_{\vec{\mu}}(\Omega) = \{f \in L^p_{loc}(\Omega) \mid wf \in L^p(\Omega)\}$ with

$$w = 1 + \sum_{j=1}^{J} \eta_j (r_j^{\mu_j} - 1), \tag{2.1}$$

where $r_j(x)$ is the distance from x to the vertex S_j and $\eta_j \in \mathcal{D}(\mathbb{R}^2)$ are such that

$$\eta_j \equiv 1 \text{ in } D_j(1/2), \quad \eta_j \equiv 0 \text{ on } \Omega \setminus D_j(1),$$

where $D_j(r)$ is the truncated cone $D_j(r) = \Omega \cap B(S_j, r)$.

The space $L^p_{\vec{\mu}}(\Omega)$ is a Banach space for the norm

$$\|f\|_{L^p_{\vec{\mu}}(\Omega)} = \left(\int_\Omega |f(x)|^p\, w^p(x)\; dx \right)^{1/p}.$$

$V^{k,p}_{\vec{\mu}}(\Omega)$ is defined as the closure of

$$\mathcal{C}^\infty_S(\Omega) = \{v \in \mathcal{C}^\infty(\overline{\Omega}) \mid S_j \notin \text{ supp } v\}$$

with respect to the norm

$$\|u\|_{V^{k,p}_{\vec{\mu}}(\Omega)} = \left(\sum_{|\gamma| \le k} \int_\Omega |D^\gamma u(x)|^p\, w^p(x)\, r^{(|\gamma|-k)p}(x)\, dx \right)^{1/p}.$$

We use the following notation for the semi-norm

$$|u|_{V^{k,p}_{\vec{\mu}}(\Omega)} = \left(\sum_{|\gamma|=k} \int_\Omega |D^\gamma u(x)|^p \, w^p(x) \, r^{(|\gamma|-k)p}(x) \, dx \right)^{1/p}.$$

In $H^1_0(\Omega)$ we will denote the norms in the following way

$$|u|^2_{H^1_0} = \int_\Omega |\nabla u|^2 \quad \text{and} \quad \|u\|^2_{H^1_0} = \int_\Omega (|\nabla u|^2 + |u|^2).$$

For $\vec{\mu}$ and $\vec{\gamma}$, we write $\vec{\mu} > \vec{\gamma}$ in case, for all $j \in \{1, \dots, J\}$, $\mu_j > \gamma_j$.

Let us finish this subsection by stating two theorems obtained in [8] that concern uniform regularity results for the Helmholtz equation in weighted Sobolev spaces.

Theorem 2.3. [8] *Let $R > 0$, $p \geq 2$ and Ω be a bounded polygonal domain of $\mathbb{R}^2$. Denote $\vec{\lambda} = (\lambda_j)_{1 \leq j \leq J}$.*

Let $\vec{\mu} > -\vec{\lambda}$ satisfies, for all $j = 1, \dots, J$,

$$\begin{gathered} \mu_j < \tfrac{2p-2}{p}, \quad \text{if } p > 2, \qquad \mu_j \leq 1, \quad \text{if } p = 2, \\ 4(p-1)\lambda_j^2 - \mu_j^2 p^2 > 0 \end{gathered} \tag{2.2}$$

and, for all $k \in \mathbb{Z}^*$ and all $j \in \{1, 2, \cdots, J\}$, $2 - \frac{2}{p} - \mu_j \neq k\lambda_j$.

Then, there exists $\theta_A \in]\frac{\pi}{2}, \pi[$ such that, for all $g \in L^p_{\vec{\mu}}(\Omega)$, all $z \in \pi^+ \cup S_A$, with

$$\pi^+ = \{z \in \mathbb{C} \mid \Re(z) \geq 0\}, \quad S_A = \{z \in \mathbb{C} \mid |z| \geq R \text{ and } |\arg z| \leq \theta_A\},$$

the problem

$$\begin{cases} -\Delta u + zu = g, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $u \in H^1_0(\Omega)$. Moreover this solution is in $D(\Delta_{p,\vec{\mu}}) := \{u \in H^1_0(\Omega) \mid \Delta u \in L^p_{\vec{\mu}}(\Omega)\}$ and admits the decomposition

$$u = u_R + \sum_{j=1}^J \eta_j \sum_{\substack{0 < \lambda'_j < 2 - \frac{2}{p} - \mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} c_{\lambda'_j}(z) \, P_{j,\lambda'_j}(r\sqrt{z}) e^{-r\sqrt{z}} \, r^{\lambda'_j} \sin(\lambda'_j \theta), \tag{2.3}$$

with $u_R \in V^{2,p}_{\vec\mu}(\Omega)$, $c_{\lambda'_j}(z) \in \mathbb{C}$ *and* $P_{j,\lambda'_j}(s) = \sum_{i=0}^{l_{j,\lambda'_j}-1} \frac{s^i}{i!}$ *with* $l_{j,\lambda'_j} > 2 - \mu_j - \frac{2}{p} - \lambda'_j$.

Moreover, the following inequalities are satisfied

(a) $|u_R|_{V^{2,p}_{\vec\mu}(\Omega)} + |u_R|_{V^{1,p}_{\vec\mu-1}(\Omega)} + \|u_R\|_{L^p_{\vec\mu-2}(\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$;

(b) $|u_R|_{V^{2,p}_{\vec\mu}(\Omega)} + |z|^{1/2}\, |u_R|_{V^{1,p}_{\vec\mu}(\Omega)} + |z|\, \|u_R\|_{L^p_{\vec\mu}(\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$;

(c) $\sum_{j=1}^{J} \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N},\lambda'_j=k\lambda_j}} |c_{\lambda'_j}(z)|\, (1+|z|^{1-\frac{1}{p}-\frac{\mu_j+\lambda'_j}{2}}) \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$.

Theorem 2.4. [8] *Under the assumptions of Theorem 2.3,* $D(\Delta_{p,\vec\mu}) \subset L^p_{\vec\mu}(\Omega)$ *and we have*

(a) If $z \in \mathbb{C}$ *satisfies* $\Re(z) \geq 0$ *then* $\Re(z)\, \|u\|_{L^p_{\vec\mu}(\Omega)} \leq \|g\|_{L^p_{\vec\mu}(\Omega)}$;

(b) If $z \in \mathbb{C}$ *satisfies* $|\arg z| \leq \theta_A$ *then* $(1+|z|)\|u\|_{L^p_{\vec\mu}(\Omega)} \lesssim \|g\|_{L^p_{\vec\mu}(\Omega)}$.

Theorem 2.3 can be rephrased as follows. The operator $(-\Delta + z\, I)^{-1}$ can be decomposed as

$$(-\Delta + z\, I)^{-1} = R(z) + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N},\lambda'_j=k\lambda_j}} T_{\lambda'_j}(z) \otimes \tilde\psi_{\lambda'_j,z} \tag{2.4}$$

where we use the linear and continuous operators

$$R(z) : L^p_{\vec\mu}(\Omega) \to V^{2,p}_{\vec\mu}(\Omega) : g \mapsto u_R,$$
$$T_{\lambda'_j}(z) : L^p_{\vec\mu}(\Omega) \to \mathbb{C} : g \mapsto c_{\lambda'_j}(z) = \langle T_{\lambda'_j}(z), g\rangle$$

and the function $\tilde\psi_{\lambda'_j,z}(r,\theta) = P_{j,\lambda'_j}(r\sqrt{z})e^{-r\sqrt{z}}\, r^{\lambda'_j} \sin(\lambda'_j\theta)$. Recall that

$$(T_{\lambda'_j}(z) \otimes \tilde\psi_{\lambda'_j,z})(g) = \langle T_{\lambda'_j}(z), g\rangle\, \tilde\psi_{\lambda'_j,z}.$$

Moreover, for all $z \in \pi^+ \cup S_A$, we have

$$\|R(z)\|_{L^p_{\vec\mu}(\Omega)\to V^{2,p}_{\vec\mu}(\Omega)} + |z|^{1/2}\|R(z)\|_{L^p_{\vec\mu}(\Omega)\to V^{1,p}_{\vec\mu}(\Omega)} + |z|\, \|R(z)\|_{L^p_{\vec\mu}(\Omega)\to L^p_{\vec\mu}(\Omega)} \lesssim 1, \tag{2.5}$$

and

$$\|T_{\lambda'_j}(z)\|_{(L^p_{\vec\mu}(\Omega))'} \lesssim \frac{1}{1+|z|^{(1-\frac{1}{p})-\frac{\mu_j+\lambda'_j}{2}}}. \tag{2.6}$$

3 Application of the first strategy

Let us assume in the future that the assumptions of Theorem 2.3 are satisfied.

Consider the problem (1.1) with $h \in L^p(I; L^p_{\vec{\mu}}(\Omega))$ with $I =]-\pi, \pi[$. In that case, h admits the decomposition

$$h(x,t) = g_1(x) + g(x,t) \quad \text{with, for a.e. } x \in \Omega, \int_{-\pi}^{\pi} g(x,t)\,dt = 0,$$

$g_1 \in L^p_{\vec{\mu}}(\Omega)$ and $g \in L^p(I; L^p_{\vec{\mu}}(\Omega))$. To obtain such a decomposition, we just have to define

$$g_1(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x,s)\,ds.$$

Hence u is a solution of (1.1) if and only if $u(x,t) = \bar{u}(x) + v(x,t)$ with $\bar{u}$ solution of

$$\begin{aligned} -\Delta \bar{u} &= g_1(x), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

and $v(x,t)$ solution of

$$\begin{aligned} \partial_t v - \Delta v &= g(x,t), && \text{in } \Omega \times]-\pi, \pi[, \\ v &= 0, && \text{on } \partial\Omega \times [-\pi, \pi], \\ v(\cdot, -\pi) &= v(\cdot, \pi), && \text{in } \Omega, \\ \int_{-\pi}^{\pi} v(x,t)\,dt &= 0, && \text{for all } x \in \Omega. \end{aligned} \tag{3.2}$$

By [18] and as in [8], $\bar{u}$ admits the decomposition

$$\bar{u} = \bar{u}_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} \bar{c}_{\lambda'_j}\, r^{\lambda'_j} \sin(\lambda'_j \theta) \tag{3.3}$$

with $\bar{u}_R \in V^{2,p}_{\vec{\mu}}(\Omega)$,

$$\|\bar{u}_R\|_{V^{2,p}_{\vec{\mu}}(\Omega)} \lesssim \|g_1\|_{L^p_{\vec{\mu}}(\Omega)} \qquad \text{and} \qquad |\bar{c}_{\lambda'_j}| \lesssim \|g_1\|_{L^p_{\vec{\mu}}(\Omega)}.$$

Hence we concentrate on (3.2).

We shall apply the First Strategy (Theorem 2.1) on the space

$$E = \{h \in L^p(I; L^p_{\vec{\mu}}(\Omega)) \mid \text{ for a.e. } x \in \Omega, \int_{-\pi}^{\pi} h(x,t)\,dt = 0\}.$$

In the future, we will use the index m to denote the fact that the functions h of the space satisfy, for a.e. $x \in \Omega$, $\int_{-\pi}^{\pi} h(x,t)\,dt = 0$. In that way $E =: L^p_m(I; L^p_{\vec{\mu}}(\Omega))$.

We consider the operators

$$\begin{gathered} A : D(A) \subset E \to E : u \mapsto -\Delta u, \quad \text{with} \\ D(A) = L^p_m(I; D(\Delta_{p,\vec{\mu}})) \text{ where } D(\Delta_{p,\vec{\mu}}) = \{u \in H^1_0(\Omega) \,|\, \Delta u \in L^p_{\vec{\mu}}(\Omega)\}, \end{gathered}$$

and

$$\begin{aligned} B_0 : D(B_0) \subset E \to E : u \mapsto \partial_t u, \quad \text{with} \\ D(B_0) &= W^{1,p}_{2\pi,m}(I; L^p_{\vec{\mu}}(\Omega)) \\ &= \{u \in E \mid \partial_t u \in L^p(I; L^p_{\vec{\mu}}(\Omega)), u(\cdot,-\pi) = u(\cdot,\pi)\}. \end{aligned}$$

Proposition 3.1. *Under the assumptions of Theorem 2.3, the operator $A + B_0$ has an inverse closure i.e., for all $g \in L^p_m(I; L^p_{\vec{\mu}}(\Omega))$, there exists a unique strong solution $v \in L^p_m(I; L^p_{\vec{\mu}}(\Omega))$ of $(A + B_0)v = g$ i.e. there exists $(v_n)_n \subset D(A) \cap D(B_0)$ such that $v_n \to v$ and $Av_n + B_0 v_n \to g$.*

Moreover we have

$$v = \frac{1}{2\pi i} \int_\gamma (A + z\,I)^{-1}(z\,I - B_0)^{-1} g\,dz, \tag{3.4}$$

with $\gamma : \mathbb{R} \to \mathbb{C}$ defined for example by

$$\begin{aligned} \gamma(s) &= |s|\, e^{-i(\frac{\pi}{2}+\delta)}, \quad \textit{for } s \le 0, \\ &= |s|\, e^{i(\frac{\pi}{2}+\delta)}, \quad \textit{for } s > 0, \end{aligned}$$

with $\delta \in]0, \theta_A - \frac{\pi}{2}[$ and θ_A given by Theorem 2.3.

Proof. Observe that by Theorem 2.4, we have $D(A) \subset E$ and, for all $\lambda > 0$,

$$\|(A + \lambda I)^{-1}\| \le \frac{1}{\lambda}.$$

By [21, Thm I-4.2], this implies that $-A$ is dissipative. As E is reflexive and $R(I + A) = E$, we have by [21, Thm I-4.6] that $D(A)$ is dense in E. Hence by Lumer-Phillips and Hille-Yosida Theorems, A is closed. It is easy to observe also that $\sigma(-A) = \{-\nu_k \mid k \in \mathbb{N}\}$ where $(\nu_k)_k$ is the strictly increasing sequence of eigenvalues of $-\Delta$ in $H^1_0(\Omega)$. In particular $\nu_1 > 0$ and $\lim_{k\to\infty} \nu_k = +\infty$.

Concerning B_0 it is easy to observe that $D(B_0)$ is a dense subset of E and that B_0 is closed. Moreover a simple calculation proves that $\sigma(B_0) = i\mathbb{Z}^*$ and therefore $\rho(-B_0) \supset \{\lambda \in \mathbb{C} \mid \Re\lambda > 0\}$. Let us show that for all $\theta_B < \frac{\pi}{2}$, there exists $M \geq 0$ such that, for all $\mu \in S_{B_0} = \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta_B\}$, $\|(B_0 + \mu I)^{-1}\| \leq \frac{M}{|\mu|}$. To this aim, it is enough to prove that if u is a solution of

$$\begin{array}{ll} \partial_t u(x,t) + \mu\, u(x,t) = f(x,t), & \text{in } \Omega \times]-\pi,\pi[, \\ u(\cdot,-\pi) = u(\cdot,\pi), & \text{in } \Omega, \\ \displaystyle\int_{-\pi}^{\pi} u(x,t)\, dt = 0, & \end{array} \tag{3.5}$$

then

$$|\mu|\, \|u\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))} \lesssim \|f\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))}.$$

Multiplying the equation in (3.5) by $v := w^p |u|^{p-2}\bar{u}$ and integrating, we obtain

$$\int_\Omega \int_{-\pi}^{\pi} \partial_t u\, v\, dtdx + \mu \int_\Omega \int_{-\pi}^{\pi} w^p\, |u|^p\, dtdx = \int_\Omega \int_{-\pi}^{\pi} w^p f |u|^{p-2}\bar{u}\, dtdx. \tag{3.6}$$

Observe that, by periodicity,

$$\int_\Omega \int_{-\pi}^{\pi} \partial_t u\, v\, dtdx = -\int_\Omega \int_{-\pi}^{\pi} \partial_t v\, u\, dtdx.$$

Moreover by [2], we have

$$\begin{aligned} &\int_\Omega \int_{-\pi}^{\pi} \partial_t u\, v\, dtdx \\ &\quad= -\frac{p}{2}\int_\Omega \int_{-\pi}^{\pi} w^p |u|^{p-2} u\, \partial_t \bar{u}\, dtdx - \frac{p-2}{2}\int_\Omega \int_{-\pi}^{\pi} w^p |u|^{p-4} \bar{u}^2 u\, \partial_t u\, dtdx \\ &\quad= -\frac{p}{2}\overline{\int_\Omega \int_{-\pi}^{\pi} v\, \partial_t u\, dtdx} - \frac{p-2}{2}\int_\Omega \int_{-\pi}^{\pi} w^p |u|^{p-2}\bar{u}\, \partial_t u\, dtdx, \end{aligned}$$

i.e.

$$\frac{p}{2}\left(\int_\Omega \int_{-\pi}^{\pi} v\partial_t u\, dtdx + \overline{\int_\Omega \int_{-\pi}^{\pi} v\partial_t u\, dtdx}\right) = 0.$$

Hence taking the real part of (3.6) gives

$$\Re(\mu)\, \|u\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))} \leq \|f\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))}.$$

As $|\arg(\mu)| \leq \theta_B < \frac{\pi}{2}$, we have $|\Im(\mu)| \lesssim \Re(\mu)$ and hence

$$|\mu|\, \|u\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))} \lesssim \|f\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))}. \tag{3.7}$$

We conclude that (H_1) is satisfied with θ_A given by Theorem 2.3 and $\theta_B = \frac{\pi}{2} - \delta_B$ with $0 < \delta_B < \delta < \theta_A - \frac{\pi}{2}$.

It remains to verify (H_3). This can be easily deduced from the fact that the variables are separate in these two operators.

Hence we can apply Theorem 2.1 to conclude. □

Remark 3.1 Observe that, multiplying the equation

$$\partial_t u = f - \mu u,$$

by $|\partial_t u|^{p-2}\partial_t \overline{u}$, integrating and using the inequality (3.7), we obtain also

$$(1+|\mu|)\,\|u\|_{L^p(I;L^p_{\bar{\mu}}(\Omega))} \lesssim \|f\|_{L^p(I;L^p_{\bar{\mu}}(\Omega))}. \tag{3.8}$$

As it is clear that, for each t, we have

$$[(A+z\,I)^{-1}h](t) = (-\Delta + z\,I)^{-1}(h(t)),$$

we can use the decomposition (2.4) and rewrite (3.4) as

$$v = v_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N},\lambda'_j=k\lambda_j}} v_{\lambda'_j} \tag{3.9}$$

with

$$\begin{aligned} v_R(x,t) &= \frac{1}{2\pi i}\int_\gamma R(z)(z\,I-B_0)^{-1}g\,dz \\ v_{\lambda'_j}(x,t) &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z),(z\,I-B_0)^{-1}g\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z),(z\,I-B_0)^{-1}g\right\rangle P_{j,\lambda'_j}(r\sqrt{z})e^{-r\sqrt{z}}\,r^{\lambda'_j}\sin(\lambda'_j\theta)\,dz. \end{aligned} \tag{3.10}$$

Proposition 3.2. *Under the assumptions of Theorem 2.3, let us denote* $\sigma_j := 1-\frac{1}{p}-\frac{\mu_j+\lambda'_j}{2}$. *For all* $s\in]0,\min(1-\sigma_j,1/p)[$, *for all* $g\in W^{s,p}_m(I,L^p_\mu(\Omega))$, *there exist* $\tilde{q}_{\lambda'_j}\in W^{s+\sigma_j,p}_m(I)$ *and* $\tilde{E}_{\lambda'_j}$ *such that* $v_{\lambda'_j}$ *defined by* (3.10) *can be written as*

$$v_{\lambda'_j} = (\tilde{E}_{\lambda'_j} *_t \tilde{q}_{\lambda'_j})\,r^{\lambda'_j}\sin(\lambda'_j\theta) = \Big(\int_{-\pi}^{\pi}\tilde{E}_{\lambda'_j}(x,\tau)\,\tilde{q}_{\lambda'_j}(t-\tau)\,d\tau\Big)\,r^{\lambda'_j}\sin(\lambda'_j\theta).$$

Moreover we have

$$\begin{aligned} \tilde{q}_{\lambda'_j}(t) &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z),(z\,I-B_0)^{-1}g\right\rangle dz, \\ \tilde{E}_{\lambda'_j}(x,t) &= \sum_{k\in\mathbb{Z}^*} e^{ikt}P_{j,\lambda'_j}(r\sqrt{ik})\,e^{-r\sqrt{ik}}, \end{aligned}$$

where we used the notations introduced at the end of Section 2, and the operator

$$U_0 : W^{s,p}_m(I, L^p_\mu(\Omega)) \to W^{s+\sigma_j,p}_m(I) : g \mapsto \tilde{q}_{\lambda'_j}$$

is continuous.

Remark 3.2 Observe that, by the domain of summation in (3.9), we have $\sigma_j > 0$ and the condition $\mu_j > -\lambda_j$ implies that $\sigma_j < 1$.

Proof. First observe that $v_{\lambda'_j} \in L^p([-\pi,\pi]; L^p_{\vec{\mu}}(\Omega))$ and hence we can take its partial Fourier series in t.

Step 1: For all $f \in L^p_{\vec{\mu}}(\Omega)$, the application $\mathbb{C} \to \mathbb{C} : z \mapsto \left\langle T_{\lambda'_j}(z), f \right\rangle$ is holomorphic on $\mathcal{A} := \{z \in \mathbb{C} \mid |\arg(z)| < \theta_A\}$ and continuous on $\overline{\mathcal{A}}$. In fact the problem

$$\begin{cases} -\Delta u + zu = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.11}$$

is equivalent to

$$\begin{cases} -\Delta u = f - zu, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.12}$$

By [18] as in [8], we know that the solution u of this second problem admits the decomposition

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} d_{\lambda'_j}\, r^{\lambda'_j} \sin(\lambda'_j\theta)$$

with

$$d_{\lambda'_j} = d_{\lambda'_j}(f - zu) = \int_\Omega (f - zu) w_{\lambda'_j}$$

and $w_{\lambda'_j}$ independent of z. Moreover we have

$$\left\langle T_{\lambda'_j}(z), f \right\rangle = d_{\lambda'_j}(f - zu).$$

Hence it remains to consider the regularity of u with respect to z.

Let us consider the operator

$$S : \mathbb{C} \to \mathcal{L}(L^p_{\vec{\mu}}(\Omega), L^p_{\vec{\mu}}(\Omega)) : z \mapsto (-\Delta + z\,I)^{-1}.$$

Observe that

$$S(z)f - S(z+h)f = -h\, S(z)\, S(z+h)\, f$$

and hence

$$\begin{aligned}
&\frac{\langle T_{\lambda'_j}(z+h), f\rangle - \langle T_{\lambda'_j}(z), f\rangle}{h} \\
&\qquad = \frac{1}{h}\left(\int_\Omega w_{\lambda'_j}(f-(z+h)S(z+h)f) - \int_\Omega w_{\lambda'_j}(f - zS(z)f)\right) \\
&\qquad = \frac{1}{h}\left(z\int_\Omega w_{\lambda'_j}(S(z)f - S(z+h)f) - h\int_\Omega w_{\lambda'_j} S(z+h)f\right) \\
&\qquad = -z\int_\Omega w_{\lambda'_j}\, S(z)\, S(z+h)\, f - \int_\Omega w_{\lambda'_j} S(z+h) f.
\end{aligned}$$

By Theorem 2.4 we have $C>0$ such that, for all $f \in L^p_{\vec{\mu}}(\Omega)$, $z \in \mathcal{A}$ and for $h \in \mathbb{C}$ small enough,

$$\begin{aligned}
\|S(z+h)f - S(z)f\|_{L^p_{\vec{\mu}}(\Omega)} &= |h|\, \|S(z)\, S(z+h)\, f\|_{L^p_{\vec{\mu}}(\Omega)} \\
&\leq \frac{C\,|h|}{(1+|z|)\,(1+|z+h|)} \|f\|_{L^p_{\vec{\mu}}(\Omega)}.
\end{aligned}$$

Hence we conclude that, for $z \in \mathcal{A}$,

$$\lim_{h\to 0} \frac{\langle T_{\lambda'_j}(z+h), f\rangle - \langle T_{\lambda'_j}(z), f\rangle}{h} = -z\int_\Omega w_{\lambda'_j}\, S(z)^2 f - \int_\Omega w_{\lambda'_j} S(z) f,$$

and, for all $f \in L^p_{\vec{\mu}}(\Omega)$, the application $\mathbb{C} \to \mathbb{C} : z \mapsto \left\langle T_{\lambda'_j}(z), f\right\rangle$ is holomorphic on $\mathcal{A}$. In a similar way, we prove the continuity of this application on $\overline{\mathcal{A}}$.

Step 2: The Fourier series coefficient in t of $v_{\lambda'_j}(x,t)$ satisfies

$$\hat{v}_{\lambda'_j}(x,k) = -\left\langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\right\rangle \tilde{\psi}_{\lambda'_j,ik}(x).$$

By definition of the Fourier series in t and applying Fubini's theorem we obtain

$$\hat{v}_{\lambda'_j}(x,k) = \int_{-\pi}^{\pi} e^{-ikt} v_{\lambda'_j}(x,t)\, dt = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)\, dz.$$

For $R>k$, let us consider the domain D_R bounded by $\gamma_{1,R} : [-1,1] \to \mathbb{C}$ defined by $\gamma_{1,R}(s) = Re^{i(\frac{\pi}{2}+\delta)s}$ and $\gamma_{2,R} := -\gamma|_{[-R,R]}$ and, for $\epsilon > 0$ its subdomain $D_{R,\epsilon}$, bounded by $\gamma_{1,R}$ and $\gamma_{2,R,\epsilon} := -\gamma|_{[-R,R]} + \epsilon$.

As, for $\epsilon > 0$, the function

$$z \to \left\langle T_{\lambda'_j}(z), \hat{g}(\cdot,k)\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)$$

is holomorphic in an open domain containing $D_{R,\epsilon}$ then, by the Cauchy formula,

$$\frac{1}{2\pi i}\int_{\partial D_{R,\epsilon}}\left\langle T_{\lambda'_j}(z),\frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle\tilde{\psi}_{\lambda'_j,z}(x)\,dz=\left\langle T_{\lambda'_j}(ik),\hat{g}(\cdot,k)\right\rangle\tilde{\psi}_{\lambda'_j,ik}(x).$$

As this fonction of ϵ is continuous for $\epsilon\in[0,\epsilon_0[$ and constant, we deduce

$$\frac{1}{2\pi i}\left[\int_{\gamma_{1,R}}\left\langle T_{\lambda'_j}(z),\frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle\tilde{\psi}_{\lambda'_j,z}(x)\,dz+\int_{\gamma_{2,R}}\left\langle T_{\lambda'_j}(z),\frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle\tilde{\psi}_{\lambda'_j,z}(x)\,dz\right]\\=\left\langle T_{\lambda'_j}(ik),\hat{g}(\cdot,k)\right\rangle\tilde{\psi}_{\lambda'_j,ik}(x).$$

Let us prove that

$$\lim_{R\to+\infty}\int_{\gamma_{1,R}}\left\langle T_{\lambda'_j}(z),\frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle\tilde{\psi}_{\lambda'_j,z}(x)\,dz=0,$$

and

$$\int_{\gamma}\left\langle T_{\lambda'_j}(z),\frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle\tilde{\psi}_{\lambda'_j,z}(x)\,dz=-\lim_{R\to+\infty}\int_{\gamma_{2,R}}\left\langle T_{\lambda'_j}(z),\frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle\tilde{\psi}_{\lambda'_j,z}(x)\,dz.$$

Observe that by Hölder's inequality, we have

$$\begin{aligned}\|\hat{g}(\cdot,k)\|^p_{L^p_{\tilde{\mu}}(\Omega)}&=\int_\Omega w^p(x)\,|\int_{-\pi}^{\pi}e^{-ikt}g(x,t)\,dt|^p\,dx\\&\lesssim\int_\Omega w^p(x)\int_{-\pi}^{\pi}|g(x,t)|^p\,dt\,dx\lesssim\|g\|^p_{L^p(I;L^p_{\tilde{\mu}}(\Omega))}.\end{aligned}$$

Hence by (2.6) we have

$$\left|\frac{\left\langle T_{\lambda'_j}(z),\hat{g}(\cdot,k)\right\rangle}{z-ik}\tilde{\psi}_{\lambda'_j,z}(x)\right|\\\lesssim\frac{|z|^{\frac{\mu_j+\lambda'_j}{2}-(1-\frac{1}{p})}}{|z|-|k|}\,|P_{j,\lambda'_j}(r\sqrt{z})e^{-r\sqrt{z}}|\,|r^{\lambda'_j}\sin(\lambda'_j\theta)|\,\|g\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))},$$

with $\frac{\mu_j+\lambda'_j}{2}-(1-\frac{1}{p})=-\sigma_j<0$. This implies

$$\left|\int_{\gamma_{1,R}}\frac{\left\langle T_{\lambda'_j}(z),\hat{g}(\cdot,k)\right\rangle}{z-ik}\,\tilde{\psi}_{\lambda'_j,z}(x)\,dz\right|\\\lesssim\frac{1}{R^{\sigma_j}}\int_{-1}^{1}|P_{j,\lambda'_j}(r\sqrt{Re^{i(\frac{\pi}{2}+\delta)s}})\,e^{-r\sqrt{Re^{i(\frac{\pi}{2}+\delta)s}}}|\,|r^{\lambda'_j}\sin(\lambda'_j\theta)|\,ds\|g\|_{L^p(I;L^p_{\tilde{\mu}}(\Omega))},$$

from which we deduce that

$$\lim_{R\to+\infty}\int_{\gamma_{1,R}} \frac{\left\langle T_{\lambda'_j}(z), \hat{g}(\cdot,k)\right\rangle}{z-ik}\,\tilde{\psi}_{\lambda'_j,z}(x)\,dz = 0.$$

The argument to prove

$$\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)\,dz = -\lim_{R\to+\infty}\int_{\gamma_{2,R}} \left\langle T_{\lambda'_j}(z), \frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)\,dz,$$

is similar. Step 2 is then proved.

Step 3: The operator $U_0 : W^{s,p}_m(I; L^p_{\vec{\mu}}(\Omega)) \to W^{s+\sigma_j,p}_m(I) : g \mapsto \tilde{q}_{\lambda'_j}$ *with*

$$\tilde{q}_{\lambda'_j}(t) = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), (zI - B_0)^{-1} g\right\rangle\,dz$$

is continuous. By the results of [11], as $0 < s < 1/p$, we know that

$$W^{s,p}_m(I; L^p_{\vec{\mu}}(\Omega)) = \{g \in E \mid \int_0^\infty \rho^{sp}\|B_0(B_0 - \rho e^{\pm i(\frac{\pi}{2}+\delta)} I)^{-1} g\|^p_E \,\frac{d\rho}{\rho} < \infty\}.$$

We have a similar characterization of $W^{s+\sigma,p}_m(I)$ by considering the operator

$$N : D(N) \subset E_1 \to E_1 : u \mapsto \partial_t u$$

with

$$E_1 = L^p_m(I),$$
$$D(N) = \{u \in E_1 \mid \partial_t u \in L^p(I),\ u(-\pi) = u(\pi)\}.$$

Hence
if $s + \sigma_j < 1/p$ then

$$W^{s+\sigma_j,p}_m(I) = \{g \in E_1 \mid \int_0^\infty \tau^{(s+\sigma)p}\|N(N + \tau\, I)^{-1} g\|^p_{L^p(I)}\,\frac{d\tau}{\tau} < \infty\},$$

if $s + \sigma_j > 1/p$ then

$$W^{s+\sigma_j,p}_{2\pi,m}(I) = \{g \in E_1 \mid \int_0^\infty \tau^{(s+\sigma)p}\|N(N + \tau\, I)^{-1} g\|^p_{L^p(I)}\,\frac{d\tau}{\tau} < \infty\},$$

Claim 1: For $\tau \geq 0$, *we have*

$$N(N + \tau I)^{-1}\tilde{q}_{\lambda'_j} = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_0(zI - B_0)^{-1} g\right\rangle \frac{dz}{z+\tau}. \qquad (3.13)$$

First observe that

$$N(N+\tau I)^{-1}\tilde{q}_{\lambda'_j} = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_0(B_0+\tau I)^{-1}(zI-B_0)^{-1}g\right\rangle\, dz. \quad (3.14)$$

Let us take the Fourier coefficients in t of

$$\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_0(B_0+\tau I)^{-1}(zI-B_0)^{-1}g\right\rangle\, dz.$$

By Cauchy theorem, we obtain

$$\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{ik}{(ik+\tau)(z-ik)}\hat{g}(\cdot,k)\right\rangle\, dz = -\left\langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\right\rangle \frac{ik}{ik+\tau}.$$

In the same way, if we take the Fourier coefficients in t of

$$\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_0(zI-B_0)^{-1}g\right\rangle \frac{dz}{z+\tau},$$

by Cauchy theorem, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{ik}{z-ik}\hat{g}(\cdot,k)\right\rangle \frac{dz}{z+\tau} \\
&\qquad = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \Big(\frac{ik}{\tau+ik}\,\frac{1}{z-ik} - \frac{ik}{\tau+ik}\,\frac{1}{z+\tau}\Big)\hat{g}(\cdot,k)\right\rangle dz \\
&\qquad = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{ik}{\tau+ik}\,\frac{1}{z-ik}\,\hat{g}(\cdot,k)\right\rangle dz \\
&\qquad = -\left\langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\right\rangle \frac{ik}{\tau+ik}
\end{aligned}$$

as $\left\langle T_{\lambda'_j}(z), \frac{\hat{g}(\cdot,k)}{z+\tau}\right\rangle$ is holomorphic on the right of γ.

As the Fourier coefficients of the two functions coincide, the two functions are equal.

Claim 2: For $0 < s < \min(1-\sigma_j, 1/p)$, *the operator* $U_0 : W^{s,p}_m(I; L^p_{\bar{\mu}}(\Omega)) \to W^{s+\sigma_j,p}_m(I) : g \mapsto \tilde{q}_{\lambda'_j}$ *is continuous.* As $0 < s < 1/p$, for $g \in W^{s,p}_m(I; L^p_{\bar{\mu}}(\Omega))$ we have

$$\|B_0(B_0 - \rho e^{i(\frac{\pi}{2}+\delta)}I)^{-1}g\|_E = \eta(\rho),$$

with

$$\int_0^\infty \rho^{sp}|\eta(\rho)|^p\,\frac{d\rho}{\rho} < \infty.$$

By (3.13), denoting $\theta_0 = \frac{\pi}{2} + \delta$, we have

$$N(N+\tau I)^{-1}\tilde{q}_{\lambda'_j} = \frac{1}{2\pi i}\int_0^{+\infty} \Big\langle T_{\lambda'_j}(\rho e^{i\theta_0}), B_0(\rho e^{i\theta_0}I - B_0)^{-1}g\Big\rangle \frac{e^{i\theta_0}d\rho}{\rho e^{i\theta_0}+\tau} - \frac{1}{2\pi i}\int_0^{+\infty}\Big\langle T_{\lambda'_j}(\rho e^{-i\theta_0}), B_0(\rho e^{-i\theta_0}I - B_0)^{-1}g\Big\rangle \frac{e^{-i\theta_0}d\rho}{\rho e^{-i\theta_0}+\tau},$$

and hence

$$\begin{aligned}
&\tau^{(s+\sigma_j)p}\|N(N+\tau I)^{-1}\tilde{q}_{\lambda'_j}\|^p_{L^p(I)}\\
&= \tau^{(s+\sigma_j)p}\int_{-\pi}^{\pi}\Bigg|\frac{1}{2\pi i}\int_0^{+\infty}\Big\langle T_{\lambda'_j}(\rho e^{i\theta_0}), B_0(\rho e^{i\theta_0}I - B_0)^{-1}g\Big\rangle \frac{e^{i\theta_0}d\rho}{\rho e^{i\theta_0}+\tau}\\
&\qquad - \frac{1}{2\pi i}\int_0^{+\infty}\Big\langle T_{\lambda'_j}(\rho e^{-i\theta_0}), B_0(\rho e^{-i\theta_0}I - B_0)^{-1}g\Big\rangle \frac{e^{-i\theta_0}d\rho}{\rho e^{-i\theta_0}+\tau}\Bigg|^p dt.
\end{aligned}$$

Using the inequality $\| \int_0^{+\infty} f\|_{L^p(I)} \le \int_0^{+\infty} \|f\|_{L^p(I)}$, we have

$$\begin{aligned}
&\tau^{(s+\sigma_j)p}\|N(N+\tau I)^{-1}\tilde{q}_{\lambda'_j}\|^p_{L^p(I)}\\
&\lesssim \tau^{(s+\sigma_j)p}\left(\int_0^{+\infty}\left(\int_{-\pi}^{\pi}\left|\frac{\Big\langle T_{\lambda'_j}(\rho e^{i\theta_0}), B_0(\rho e^{i\theta_0}I - B_0)^{-1}g\Big\rangle}{|\rho e^{i\theta_0}+\tau|}\right|^p dt\right)^{1/p} d\rho\right)^p\\
&\quad + \tau^{(s+\sigma_j)p}\left(\int_0^{+\infty}\left(\int_{-\pi}^{\pi}\left|\frac{\Big\langle T_{\lambda'_j}(\rho e^{-i\theta_0}), B_0(\rho e^{-i\theta_0}I - B_0)^{-1}g\Big\rangle}{|\rho e^{-i\theta_0}+\tau|}\right|^p dt\right)^{1/p} d\rho\right)^p.
\end{aligned}$$

As the two terms behave in the same way, let us consider only the first term of the sum. We have, using (2.6),

$$\begin{aligned}
&\tau^{(s+\sigma_j)p}\left(\int_0^{+\infty}\left(\int_{-\pi}^{\pi}\left|\frac{\Big\langle T_{\lambda'_j}(\rho e^{i\theta_0}), B_0(\rho e^{i\theta_0}I - B_0)^{-1}g\Big\rangle}{|\rho e^{i\theta_0}+\tau|}\right|^p dt\right)^{1/p} d\rho\right)^p\\
&\lesssim \tau^{(s+\sigma_j)p}\left(\int_0^{+\infty}\left(\int_{-\pi}^{\pi}\left[\rho^{\frac{\mu_j+\lambda'_j}{2}-1+\frac{1}{p}}\|B_0(\rho e^{i\theta_0}I - B_0)^{-1}g\|_{L^p_{\vec{\mu}}(\Omega)}\right]^p dt\right)^{1/p}\frac{d\rho}{|\rho e^{i\theta_0}+\tau|}\right)^p\\
&\lesssim \tau^{(s+\sigma_j)p}\left(\int_0^{+\infty}\rho^{\frac{\mu_j+\lambda'_j}{2}-1+\frac{1}{p}}\|B_0(\rho e^{i\theta_0}I - B_0)^{-1}g\|_{L^p(I,L^p_{\vec{\mu}}(\Omega))}\frac{d\rho}{|\rho e^{i\theta_0}+\tau|}\right)^p\\
&\lesssim \tau^{(s+\sigma_j)p}\left(\int_0^{+\infty}\rho^{\frac{\mu_j+\lambda'_j}{2}-1+\frac{1}{p}}\eta(\rho)\frac{1}{|\rho e^{i\theta_0}+\tau|}d\rho\right)^p\\
&\lesssim \left(\int_0^{+\infty}\rho^s\eta(\rho)\frac{\tau^{s+\sigma_j}\rho^{\frac{\mu_j+\lambda'_j}{2}-1+\frac{1}{p}-s}}{|\rho e^{i\theta_0}+\tau|}d\rho\right)^p.
\end{aligned}$$

As $\sigma_j = -\frac{\mu_j+\lambda'_j}{2} + 1 - \frac{1}{p}$ we obtain

$$\begin{aligned}\tau^{(s+\sigma_j)p}\|N(N+\tau I)^{-1}\tilde{q}_{\lambda'_j}\|^p_{L^p(I)} &\lesssim \left(\int_0^{+\infty} \rho^s \eta(\rho) \frac{\tau^{s+\sigma_j}|\rho|^{-(s+\sigma_j)}}{|\rho e^{i\theta_0}+\tau|} d\rho\right)^p \\ &= \left(\int_0^{+\infty} \rho^s \eta(\rho) \frac{(\tau/\rho)^{s+\sigma_j}}{|e^{i\theta_0}+\frac{\tau}{\rho}|}\frac{d\rho}{\rho}\right)^p.\end{aligned}$$

This is a multiplicative convolution and we have, by Young inequality, that $\tilde{q}_{\lambda'_j} \in W^{s+\sigma_j,p}(I)$ if

$$\int_0^{+\infty} \frac{\xi^{s+\sigma_j}}{|e^{i\theta_0}+\xi|}\frac{d\xi}{\xi} < \infty,$$

which is true as $0 < s + \sigma_j < 1$.

Conclusion. By Step 2, we have

$$\begin{aligned}\hat{v}_{\lambda'_j}(x,k) &= -\left\langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\right\rangle \tilde{\psi}_{\lambda'_j,ik}(x) \\ &= -\left\langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\right\rangle P_{j,\lambda'_j}(r\sqrt{ik})e^{-r\sqrt{ik}}\, r^{\lambda'_j}\sin(\lambda'_j\theta).\end{aligned} \tag{3.15}$$

Let $\tilde{q}_{\lambda'_j}(t) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \left\langle T_{\lambda'_j}(z), (z\,I - B_0)^{-1}g\right\rangle dz$. Applying again the Cauchy theorem as above, we see that its Fourier series coefficient in t is given by

$$\hat{q}_{\lambda'_j}(k) = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{\hat{g}(\cdot,k)}{z-ik}\right\rangle dz = -\left\langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\right\rangle.$$

Moreover the function $\tilde{E}_{\lambda'_j}(x,t) = \sum_{k\in\mathbb{Z}^*} e^{ikt}P_{j,\lambda'_j}(r\sqrt{ik})\,e^{-r\sqrt{ik}}$ is such that, for all $r > 0$, $\tilde{E}_{\lambda'_j}(r,\cdot) \in L^2(-\pi,\pi)$ and even $\tilde{E}_{\lambda'_j}(r,\cdot) \in \mathcal{C}^\infty([-\pi,\pi])$. Hence we deduce from (3.15) that

$$v_{\lambda'_j}(x,t) = (\tilde{E}_{\lambda'_j} *_t \tilde{q}_{\lambda'_j})\, r^{\lambda'_j}\sin(\lambda'_j\theta),$$

which allows to conclude. □

Let us go back to the problem (1.1).

Proposition 3.3. *Under the assumptions of Theorem 2.3, let $\sigma_j = 1 - \frac{1}{p} - \frac{\mu_j+\lambda'_j}{2}$. Then, for all $s \in]0, \min(1-\sigma_j, 1/p)[$ and for all $h \in W^{s,p}(I, L^p_{\vec{\mu}}(\Omega))$, the problem* (1.1) *has a unique strong solution u with*

$$u = u_R + \sum_{j=1}^J \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} u_{\lambda'_j}$$

and, with the notations of Propositions 3.1, 3.2 and of (3.3),

$$u_R(x,t) = \frac{1}{2\pi i}\int_\gamma R(z)(z\,I - B_0)^{-1}(h - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\cdot,s)\,ds)\,dz + \bar{u}_R(x)$$
$$u_{\lambda'_j}(x,t) = (E_{\lambda'_j} *_t q_{\lambda'_j})\, r^{\lambda'_j}\sin(\lambda'_j\theta)$$

with $q_{\lambda'_j} \in W^{s+\sigma_j,p}(I)$ *and* $E_{\lambda'_j}$ *verifying*

$$\begin{aligned} q_{\lambda'_j}(t) &= \tilde{q}_{\lambda'_j}(t) + \bar{c}_{\lambda'_j} \\ &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), (z\,I - B_0)^{-1}(h - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\cdot,s)\,ds)\right\rangle dz + \bar{c}_{\lambda'_j}, \end{aligned}$$
$$E_{\lambda'_j}(x,t) = \tilde{E}_{\lambda'_j}(x,t) + \frac{1}{2\pi} = \sum_{k\in\mathbb{Z}^*} e^{ikt} P_{j,\lambda'_j}(r\sqrt{ik})\, e^{-r\sqrt{ik}} + \frac{1}{2\pi}.$$

Moreover, the operator

$$U : W^{s,p}(I, L^p_\mu(\Omega)) \to W^{s+\sigma_j,p}(I) : h \mapsto q_{\lambda'_j}$$

is continuous.

Proof. By the previous results it is enough to prove that

$$(E_{\lambda'_j} *_t q_{\lambda'_j})\, r^{\lambda'_j}\sin(\lambda'_j\theta) = v_{\lambda'_j}(x,t) + \bar{c}_{\lambda'_j}.$$

Observe that, as $\int_{-\pi}^{\pi} \tilde{q}_{\lambda'_j}(s)\,ds = 0$ and $\int_{-\pi}^{\pi} \tilde{E}_{\lambda'_j}(r,t-s)\,ds = 0$, we have

$$\begin{aligned} (E_{\lambda'_j} *_t q_{\lambda'_j})(x,t) &= \int_{-\pi}^{\pi} E_{\lambda'_j}(r,t-s)\, q_{\lambda'_j}(s)\,ds \\ &= \int_{-\pi}^{\pi} \tilde{E}_{\lambda'_j}(r,t-s)\, q_{\lambda'_j}(s)\,ds + \frac{1}{2\pi}\int_{-\pi}^{\pi} q_{\lambda'_j}(s)\,ds \\ &= \int_{-\pi}^{\pi} \tilde{E}_{\lambda'_j}(r,t-s)\, q_{\lambda'_j}(s)\,ds + \bar{c}_{\lambda'_j} \\ &= \int_{-\pi}^{\pi} \tilde{E}_{\lambda'_j}(r,t-s)\, \tilde{q}_{\lambda'_j}(s)\,ds + \bar{c}_{\lambda'_j} \\ &= (\tilde{E}_{\lambda'_j} *_t q_{\lambda'_j})(x,t) + \bar{c}_{\lambda'_j} \\ &= v_{\lambda'_j}(x,t) + \bar{c}_{\lambda'_j}. \end{aligned}$$

The result follows. □

In the next result we extend Proposition 3.3 to $h \in L^p(I, L^p_{\vec{\mu}}(\Omega))$.

Theorem 3.4. *Under the assumptions of Theorem 2.3, let* $\sigma_j = 1-\frac{1}{p}-\frac{\mu_j+\lambda'_j}{2}$. *Then, for all* $h \in L^p(I, L^p_{\vec{\mu}}(\Omega))$, *the problem* (1.1) *has a unique strong solution* u *with*

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} u_{\lambda'_j}$$

and, with the notations of Propositions 3.1, 3.2 and of (3.3),

$$u_R(x,t) = \frac{1}{2\pi i}\int_\gamma R(z)(z\,I - B_0)^{-1}(h - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\cdot,s)\,ds)\,dz + \bar{u}_R(x)$$
$$u_{\lambda'_j}(x,t) = (E_{\lambda'_j} *_t q_{\lambda'_j})\, r^{\lambda'_j}\sin(\lambda'_j\theta)$$

with $q_{\lambda'_j} \in W^{\sigma_j,p}(I)$ *and* $E_{\lambda'_j}$ *verifying*

$$\begin{aligned} q_{\lambda'_j}(t) &= \tilde{q}_{\lambda'_j}(t) + \bar{c}_{\lambda'_j} \\ &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), (z\,I - B_0)^{-1}(h - \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\cdot,s)\,ds)\right\rangle dz + \bar{c}_{\lambda'_j}, \end{aligned}$$
$$E_{\lambda'_j}(x,t) = \tilde{E}_{\lambda'_j}(x,t) + \frac{1}{2\pi} = \sum_{k\in\mathbb{Z}^*} e^{ikt} P_{j,\lambda'_j}(r\sqrt{ik})\, e^{-r\sqrt{ik}} + \frac{1}{2\pi}.$$

Moreover the mapping $L^p(I, L^p_{\vec{\mu}}(\Omega)) \to W^{\sigma_j,p}(I) : h \mapsto q_{\lambda'_j}$ *is continuous.*

Proof. We already know by Proposition 3.3 that, for $0 < s < \min(1-\sigma_j, 1/p)$, the operator U which maps h to $q_{\lambda'_j}$ is continuous from $W^{s,p}(I; L^p_{\vec{\mu}}(\Omega))$ to $W^{s+\sigma_j,p}(I)$. We prove that U is also continuous from $W^{s-1,p}(I; L^p_{\vec{\mu}}(\Omega))$ to $W^{s-1+\sigma_j,p}(I)$, which allows to conclude by interpolation.

Claim 1: For $0 < s < \min(1-\sigma_j, 1/p)$, *the operator* $U : W^{s-1,p}(I; L^p_{\vec{\mu}}(\Omega)) \to W^{s-1+\sigma_j,p}(I) : h \mapsto q_{\lambda'_j}$ *is continuous.* We have

$$h \in W^{s-1,p}(I; L^p_{\vec{\mu}}(\Omega)) \Leftrightarrow \exists h_0, h_1 \in W^{s,p}(I; L^p_{\vec{\mu}}(\Omega)),\ h = h_0 + \frac{\partial}{\partial t}h_1.$$

Hence we define

$$U : W^{s-1,p}(I; L^p_{\vec{\mu}}(\Omega)) \to W^{s-1+\sigma_j,p}(I)$$

by

$$Uh = Uh_0 + \frac{\partial}{\partial t}Uh_1,$$

where $h = h_0 + \frac{\partial}{\partial t}h_1$ with h_0, $h_1 \in W^{s,p}(I; L^p_{\vec{\mu}}(\Omega))$.

Let us show that U is well defined. Assume that h admits a second decomposition $h = \tilde{h}_0 + \frac{\partial}{\partial t}\tilde{h}_1$ with $\tilde{h}_0,\, \tilde{h}_1 \in W^{s,p}(I; L^p_{\vec{\mu}}(\Omega))$. In that case we have

$$h_0 - \tilde{h}_0 = \frac{\partial}{\partial t}(\tilde{h}_1 - h_1),$$

and hence $w := \tilde{h}_1 - h_1 \in W^{s+1,p}(I; L^p_{\vec{\mu}}(\Omega))$. As $w \in W^{s+1,p}(I; L^p_{\vec{\mu}}(\Omega))$ we have that $D_\tau w = \frac{w(t+\tau)-w(t)}{\tau} \to w'$ in $W^{s,p}(I; L^p_{\vec{\mu}}(\Omega))$ as $\tau \to 0$. As $U : W^{s,p}(I; L^p_{\vec{\mu}}(\Omega)) \to W^{s+\sigma_j,p}(I)$ is linear and continuous we have

$$D_\tau U w = U D_\tau w \to U w' \text{ as } \tau \to 0,$$

from which we deduce that Uw is derivable and

$$\frac{\partial}{\partial t} U w = U \frac{\partial}{\partial t} w$$

and hence, if $h_0 + \frac{\partial}{\partial t} h_1 = \tilde{h}_0 + \frac{\partial}{\partial t}\tilde{h}_1$, we have

$$U(h_0 - \tilde{h}_0) = U(\frac{\partial}{\partial t}(\tilde{h}_1 - h_1)) = \frac{\partial}{\partial t} U(\tilde{h}_1 - h_1)$$

which means

$$U h_0 + \frac{\partial}{\partial t} U h_1 = U\tilde{h}_0 + \frac{\partial}{\partial t} U\tilde{h}_1$$

i.e. U is well defined.

It remains to prove that U is continuous. We have for $h = h_0 + \frac{\partial}{\partial t} h_1$ with $h_0,\, h_1 \in W^{s,p}(I; L^p_{\vec{\mu}}(\Omega))$,

$$\begin{aligned} \|Uh\|_{W^{s-1+\sigma_j,p}(I)} &\lesssim \|Uh_0\|_{W^{s+\sigma_j,p}(I)} + \|Uh_1\|_{W^{s+\sigma_j,p}(I)} \\ &\lesssim \|h_0\|_{W^{s,p}(I;L^p_{\vec{\mu}}(\Omega))} + \|h_1\|_{W^{s,p}(I;L^p_{\vec{\mu}}(\Omega))} \\ &\lesssim \|h\|_{W^{s-1,p}(I;L^p_{\vec{\mu}}(\Omega))}, \end{aligned}$$

which proves the Claim.

Claim 2: $L^p(I; L^p_{\vec{\mu}}(\Omega)) \hookrightarrow (W^{s,p}(I; L^p_{\vec{\mu}}(\Omega)), W^{s-1,p}(I; L^p_{\vec{\mu}}(\Omega)))_{s,p}$. By Fubini's theorem, we have

$$W^{s,p}(I, L^p_{\vec{\mu}}(\Omega)) = L^p_{\vec{\mu}}(\Omega, W^{s,p}(I)),$$

as well as

$$W^{1-s,p}_0(I, L^p_{\vec{\mu}}(\Omega)) = L^p_{\vec{\mu}}(\Omega, W^{1-s,p}_0(I)),$$

and then by duality

$$W^{s-1,p}(I,L^p_{\vec{\mu}}(\Omega))=L^p_{\vec{\mu}}(\Omega,W^{s-1,p}(I)).$$

Hence we may write

$$(W^{s,p}(I,L^p_{\vec{\mu}}(\Omega)),W^{s-1,p}(I,L^p_{\vec{\mu}}(\Omega)))_{s,p}=(L^p_{\vec{\mu}}(\Omega,W^{s,p}(I)),L^p_{\vec{\mu}}(\Omega,W^{s-1,p}(I)))_{s,p},$$

and applying [25, Theorem 1.18.4, p.128], we deduce that

$$(W^{s,p}(I,L^p_{\vec{\mu}}(\Omega)),W^{s-1,p}(I,L^p_{\vec{\mu}}(\Omega)))_{s,p}=L^p_{\vec{\mu}}(\Omega,(W^{s,p}(I),W^{s-1,p}(I))_{s,p}).$$

Hence by [10, Thm 6.2], we obtain

$$(W^{s,p}(I,L^p_{\vec{\mu}}(\Omega)),W^{s-1,p}(I,L^p_{\vec{\mu}}(\Omega)))_{s,p}=L^p_{\vec{\mu}}(\Omega,B^{0,p}(I)).$$

As Taibleson's results [24] yield $L^p(I)\hookrightarrow B^{0,p}(I)$, we have shown that

$$L^p_{\vec{\mu}}(\Omega,L^p(I))\hookrightarrow L^p_{\vec{\mu}}(\Omega,B^{0,p}(I))=(W^{s,p}(I,L^p_{\vec{\mu}}(\Omega)),W^{s-1,p}(I,L^p_{\vec{\mu}}(\Omega)))_{s,p}.$$

We conclude by Fubini's theorem that

$$L^p(I,L^p_{\vec{\mu}}(\Omega))=L^p_{\vec{\mu}}(\Omega,L^p(I)).$$

Conclusion. By interpolation the application

$$\begin{aligned}U:(W^{s,p}(I;L^p_{\vec{\mu}}(\Omega)),W^{s-1,p}(I;L^p_{\vec{\mu}}(\Omega)))_{s,p}&\\ \to(W^{s+\sigma_j,p}(I),W^{s-1+\sigma_j,p}(I))_{s,p}:h\mapsto q_{\lambda'_j}&\end{aligned}$$

is continuous.

As, for $\sigma_j\notin\mathbb{N}$,

$$(W^{s+\sigma_j,p}(I),W^{s-1+\sigma_j,p}(I))_{s,p}=B^{\sigma_j,p}(I)=W^{\sigma_j,p}(I),$$

and, by Remark 3.2, $\sigma_j\in]0,1[$, we have a continuous operator

$$U:L^p(I;L^p_{\vec{\mu}}(\Omega))\to W^{\sigma_j,p}(I):h\mapsto q_{\lambda'_j},$$

and the result follows. □

4 Regularity of $q_{\lambda'_j} \to (\frac{\partial}{\partial t} - \Delta)(\eta_j u_{\lambda'_j})$

In order to consider the regularity of u_R we observe that u_R satisfies

$$\partial_t u_R + u_R - \Delta u_R = h - \sum_{j=1}^{J} \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} (\partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j})) + u_R$$

Hence we need informations on the regularity of $\partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j})$. This is the aim of this section.

Lemma 4.1. *The kernel H defined on $\mathbb{R}^+ \times [-\pi, \pi]$ by*

$$H(r,t) = \sum_{k\in\mathbb{Z}} \sqrt{ik}\, e^{-r\sqrt{ik}}\, e^{ikt} = \sum_{k\in\mathbb{Z}} H_k(r)\, e^{ikt} \tag{4.1}$$

admits the decomposition

$$H(r,t) = H_1(r,t) + H_2(r,t), \tag{4.2}$$

with

$$|H_1(r,t)| \lesssim 1, \tag{4.3}$$

and

$$|H_2(r,t)| \lesssim \frac{1}{(r^2 + |t|)^{3/2}}. \tag{4.4}$$

Moreover, for $l \in \mathbb{N}$,

$$\frac{\partial^{1+l}}{\partial r^{1+l}} H(r,t) = H_1^{(l)}(r,t) + H_2^{(l)}(r,t),$$

with $H_1^{(l)}$ and $H_2^{(l)}$ satisfying

$$|H_1^{(l)}(r,t)| \lesssim 1 \tag{4.5}$$

and

$$|H_2^{(l)}(r,t)| \lesssim (|t| + r^2)^{-(2+l/2)}. \tag{4.6}$$

Proof. Let us define the function

$$E_p^{(0)}(r,t) = \sum_{k\neq 0} \frac{e^{-|r|\sqrt{ik}}}{\sqrt{ik}} e^{ikt}. \tag{4.7}$$

We verify that $E_p^{(0)} \in L^2(\mathbb{R}\times[-\pi,\pi])$ as, using Parseval identity,

$$\begin{aligned}\int_{-\infty}^{+\infty}\sum_{k\neq 0}\frac{|e^{-|r|\sqrt{ik}}|^2}{|k|}\,dr &= 4\int_0^{+\infty}\sum_{k>0}\frac{e^{-2r\sqrt{k}\cos(\pi/4)}}{|k|}\,dr\\ &= 2\int_0^{+\infty}\sum_{k>0}\frac{e^{-s\cos(\pi/4)}}{|k|^{3/2}}\,ds\\ &= 2\left(\sum_{k>0}\frac{1}{|k|^{3/2}}\right)\int_0^{+\infty} e^{-s\cos(\pi/4)}\,ds<+\infty.\end{aligned}$$

Considering the finite sum for $|k|\le K$ and passing to the limit as $K\to+\infty$, we can take the Fourier transform in r of $E_p^{(0)}$ using the fact that the Fourier transform is an isometry and $E_p^{(0)}\in L^2(\mathbb{R}\times[-\pi,\pi])$. This gives

$$\widehat{E}_p^{(0)}(\xi,t)=\sum_{k\neq 0}\frac{1}{\sqrt{ik}}\widehat{e^{-|r|\sqrt{ik}}}(\xi)e^{ikt}.$$

As, for $k\neq 0$,

$$\begin{aligned}\widehat{e^{-|r|\sqrt{ik}}}(\xi) &= \int_{-\infty}^{+\infty} e^{-|r|\sqrt{ik}}e^{-i\xi r}\,dr=\int_0^{+\infty} e^{-r\sqrt{ik}}e^{-i\xi r}\,dr+\int_{-\infty}^0 e^{r\sqrt{ik}}e^{-i\xi r}\,dr\\ &= \int_0^{+\infty} e^{-r\sqrt{ik}}(e^{-i\xi r}+e^{i\xi r})\,dr=\frac{2\sqrt{ik}}{\xi^2+ik},\end{aligned}$$

we obtain

$$\widehat{E}_p^{(0)}(\xi,t)=2\sum_{k\neq 0}\frac{1}{\xi^2+ik}e^{ikt}. \tag{4.8}$$

Let E be the elementary solution of the heat equation in $\mathbb{R}^2$ i.e.

$$E(r,t)=\frac{M(t)}{\sqrt{4\pi t}}e^{-\frac{r^2}{4t}}, \tag{4.9}$$

where

$$\begin{aligned}M(t) &= 1,\ \text{ if } t>0,\\ &= 0,\ \text{ if } t<0.\end{aligned}$$

Then we have that the Fourier transform of E in r is given by

$$\widehat{E}(\xi,t)=\frac{M(t)}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{r^2}{4t}}e^{-i\xi r}\,dr=\frac{M(t)}{\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{r^2}{4t}}\cos(\xi r)\,dr=M(t)e^{-\xi^2 t}.$$

Consider now the function $R_0(r,t)$ which has as Fourier transform in r the function

$$\widehat{R}_0(\xi,t) = -\frac{e^{-\xi^2(t+\pi)}}{2\sinh(\xi^2\pi)} + \frac{1}{2\xi^2\pi}.$$

With these notations we have

$$(\widehat{E} - \widehat{R}_0)(\xi,t) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \beta_k e^{ikt},$$

with

$$\beta_k = \frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{e^{-\xi^2(t+\pi)}}{\sinh(\xi^2\pi)} + 2M(t)e^{-\xi^2 t}\right)e^{-ikt}\,dt - \frac{1}{2\xi^2\pi}\int_{-\pi}^{\pi} e^{-ikt}\,dt.$$

Hence we have, for $k \neq 0$,

$$\begin{aligned}
\beta_k &= \frac{1}{2}\left\{\frac{e^{-\xi^2\pi}}{\sinh(\xi^2\pi)}\int_{-\pi}^{\pi} e^{-(\xi^2+ik)t}\,dt + 2\int_0^{\pi} e^{-(ik+\xi^2)t}\,dt\right\}\\
&= \frac{-1}{2(\xi^2+ik)}\left\{\frac{e^{-\xi^2\pi}}{\sinh(\xi^2\pi)}(e^{-(\xi^2+ik)\pi} - e^{(\xi^2+ik)\pi}) + 2(e^{-(\xi^2+ik)\pi} - 1)\right\}\\
&= \frac{1}{\xi^2+ik},
\end{aligned}$$

and, in the same way,

$$\beta_0 = 0.$$

Hence, we obtain

$$(\widehat{E} - \widehat{R}_0)(\xi,t) = \frac{1}{2\pi}\sum_{k\neq 0}\frac{1}{\xi^2+ik}e^{ikt}. \tag{4.10}$$

By (4.8) and (4.10), we deduce that

$$\widehat{E}_p^{(0)}(\xi,t) = 4\pi(\widehat{E}(\xi,t) - \widehat{R}_0(\xi,t))$$

and hence

$$E_p^{(0)}(r,t) = 4\pi(E(r,t) - R_0(r,t)).$$

Now observe that

$$\begin{aligned}
\frac{\partial^2}{\partial r^2}E_p^{(0)}(r,t) &= \sum_{k\neq 0}\sqrt{ik}e^{-|r|\sqrt{ik}}e^{ikt} - 2\delta_0(r)\sum_{k\neq 0} e^{ikt}\\
&= H(|r|,t) - 2\delta_0(r)(\sum_{k\in\mathbb{Z}} e^{ikt} - 1)\\
&= H(|r|,t) - 2\delta_0(r)(2\pi\delta_0(t) - 1)\\
&= H(|r|,t) - 4\pi\delta_0(r)\delta_0(t) + 2\delta_0(r).
\end{aligned}$$

In the same way, we have

$$\frac{\partial^2}{\partial r^2}(E-R_0)(r,t) = \frac{\partial^2}{\partial r^2}E(r,t) - \frac{\partial^2}{\partial r^2}R_0(r,t),$$

and, denoting $\widehat{R}(\xi,t) = -\frac{e^{-\xi^2(t+\pi)}}{2\sinh(\xi^2\pi)}$, we obtain

$$\begin{aligned}
\widehat{\frac{\partial^2}{\partial r^2}R_0}(\xi,t) &= -\xi^2\widehat{R}_0(\xi,t) = -\xi^2(\widehat{R}(\xi,t) + \frac{1}{2\xi^2\pi}) \\
&= -\xi^2\widehat{R}(\xi,t) - \frac{1}{2\pi} = -\xi^2\widehat{R}(\xi,t) - \frac{1}{2\pi}\widehat{\delta}_0(\xi) \\
&= \widehat{\frac{\partial^2 R}{\partial r^2}}(\xi,t) - \frac{1}{2\pi}\widehat{\delta}_0(\xi).
\end{aligned}$$

Hence we deduce

$$\begin{aligned}
H(|r|,t) - 4\pi\delta_0(r)\delta_0(t) &= \frac{\partial^2}{\partial r^2}E_p^{(0)}(r,t) - 2\delta_0(r) \\
&= 4\pi\frac{\partial^2 E}{\partial r^2}(r,t) - 4\pi(\frac{\partial^2 R}{\partial r^2}(r,t) - \frac{1}{2\pi}\delta_0(r)) - 2\delta_0(r)
\end{aligned}$$

and therefore, for $r>0$,

$$H(r,t) = 4\pi\frac{\partial^2 E}{\partial r^2}(r,t) - 4\pi\frac{\partial^2 R}{\partial r^2}(r,t).$$

This suggests to decompose $H(r,t)$ for $r>0$ in the following way

$$H = H_1 + H_2 \quad \text{with} \quad H_1 = -4\pi\frac{\partial^2 R}{\partial r^2} \quad \text{and} \quad H_2 = 4\pi\frac{\partial^2 E}{\partial r^2}. \tag{4.11}$$

For H_1 we have

$$\widehat{H_1}(\xi,t) = 4\pi\xi^2\widehat{R}(\xi,t) = -2\pi\frac{\xi^2 e^{-\xi^2(t+\pi)}}{\sinh(\xi^2\pi)}.$$

Hence

$$\begin{aligned}
H_1(r,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ir\xi}\widehat{H_1}(\xi,t)\,d\xi \\
&= -\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{ir\xi}\frac{\xi^2\pi}{\sinh(\xi^2\pi)}e^{-\xi^2(t+\pi)}\,d\xi
\end{aligned}$$

As $t+\pi \geq 0$ we have $e^{-\xi^2(t+\pi)} \leq 1$ and hence

$$|H_1(r,t)| \leq \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\xi^2\pi}{\sinh(\xi^2\pi)}\,d\xi \leq \frac{1}{\pi\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{x^2}{\sinh(x^2)}\,dx,$$

which implies (4.3) as $\int_{-\infty}^{+\infty}\frac{x^2}{\sinh(x^2)}\,dx < \infty$.

Now let us consider the estimate (4.4). We have chosen

$$H_2 = 4\pi \frac{\partial^2 E}{\partial r^2},$$

and hence we have,

$$H_2(r,t) = -2\pi \frac{M(t)}{\sqrt{\pi}}\, t^{-3/2}\, (\frac{1}{2} - \frac{r^2}{4t})\, e^{-\frac{r^2}{4t}}.$$

This implies

$$|H_2(r,t)| \leq 2\pi \frac{t^{-3/2}}{\sqrt{\pi}} \left| \frac{1}{2} - \frac{r^2}{4t} \right| e^{-\frac{r^2}{4t}}.$$

Recall that for all $x \geq 0$ we have

$$|1-x|e^{-x} \lesssim (1+x)^{-3/2}$$

and hence

$$|H_2(r,t)| \lesssim t^{-3/2}(1+\frac{r^2}{t})^{-3/2} \lesssim (t+r^2)^{-3/2}.$$

The result concerning the derivatives can be deduced easily by similar considerations. For what concerns $H_1^{(l)}$, we just have to observe that

$$H_1^{(l)}(r,t) = \frac{\partial^{1+l} H_1(r,t)}{\partial r^{1+l}} = \int_{-\infty}^{+\infty} e^{ir\xi} \frac{(i\xi)^{3+l}}{\sinh(\xi^2 \pi)} e^{-\xi^2(t+\pi)}\, d\xi$$

and hence, as above,

$$|H_1^{(l)}(r,t)| \lesssim 1.$$

On the other hand, by recurrence, we can prove that, for some $a_i \in \mathbb{R}$,

$$\begin{aligned} H_2^{(l)}(r,t) &= \frac{\partial^{1+l} H_2(r,t)}{\partial r^{1+l}} = 4\pi \frac{\partial^{3+l} E(r,t)}{\partial r^{3+l}} \\ &= 2\pi \frac{M(t)}{\sqrt{\pi}} \frac{1}{\sqrt{t}} e^{-\frac{r^2}{4t}} \left[\sum_{i=0}^{[\frac{3+l}{2}]} a_i \frac{r^{3+l-2i}}{t^{3+l-i}} \right] \\ &= 2\pi \frac{M(t)}{\sqrt{\pi}} \frac{1}{t^{\frac{4+l}{2}}} e^{-\frac{r^2}{4t}} \left[\sum_{i=0}^{[\frac{3+l}{2}]} a_i \left(\frac{r^2}{t} \right)^{\frac{3+l}{2}-i} \right], \end{aligned}$$

and we conclude as above. □

Theorem 4.2. *Under the assumptions of Theorem 3.4 and recalling that* $\sigma_j = 1 - \frac{1}{p} - \frac{\mu_j+\lambda'_j}{2}$*, the mapping* $q_{\lambda'_j} \to (\frac{\partial}{\partial t} - \Delta)(\eta_j u_{\lambda'_j})$ *is continuous from* $W^{\sigma_j,p}(I)$ *into* $L^p(I; L^p_{\vec{\mu}}(\Omega))$.

Proof. Recall that, by Remark 3.2, $0 < \sigma_j < 1$.

Case 1: $P_{j,\lambda'_j} \equiv 1$ *i.e.* $\lambda'_j + \mu_j - 1 + \frac{2}{p} > 0$. Let us take the Fourier series in t of $f = \eta_j(\frac{\partial}{\partial t} - \Delta)u_{\lambda'_j}$. We obtain

$$\begin{aligned}\hat{f}_k &= \eta_j(r)\,(\widehat{(\tfrac{\partial}{\partial t} - \Delta)u_{\lambda'_j}})_k = \eta_j(r)\,(ik\,I - \Delta)\,\hat{u}_{\lambda'_j,k} \\ &= -c_{\lambda'_j}(ik)\,(2\lambda'_j + 1)\,\sqrt{ik}\,e^{-r\sqrt{ik}}\,r^{\lambda'_j-1}\sin(\lambda'_j\theta)\,\eta_j(r),\end{aligned}$$

with $c_{\lambda'_j}(ik) = \langle T_{\lambda'_j}(ik), \hat{g}(\cdot,k)\rangle = -\hat{q}_{\lambda'_j}(k)$.

Let us consider the kernel H given by Lemma 4.1 i.e.

$$H(r,t) = \sum_{k\in\mathbb{Z}} \sqrt{ik}\,e^{-r\sqrt{ik}}\,e^{ikt} = \sum_{k\in\mathbb{Z}} H_k(r)\,e^{ikt}.$$

Hence, we have

$$\begin{aligned} f &= (H *_t q_{\lambda'_j})\,(2\lambda'_j + 1)\,r^{\lambda'_j-1}\sin(\lambda'_j\theta)\,\eta_j(r) \\ &= (2\lambda'_j + 1)\,r^{\lambda'_j-1}\sin(\lambda'_j\theta)\,\eta_j(r)\int_{-\pi}^{\pi} H(r,s)\,q_{\lambda'_j}(t-s)\,ds.\end{aligned}$$

As

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(r,s)\,ds = H_0(r) = 0,$$

we have

$$f = (2\lambda'_j + 1)\,r^{\lambda'_j-1}\sin(\lambda'_j\theta)\,\eta_j(r)\int_{-\pi}^{\pi} H(r,s)\,[q_{\lambda'_j}(t-s) - q_{\lambda'_j}(t)]\,ds. \quad (4.12)$$

By (4.2) we define $f = f_1 + f_2$ with

$$f_i = (2\lambda'_j + 1)\,r^{\lambda'_j-1}\sin(\lambda'_j\theta)\,\eta_j(r)\int_{-\pi}^{\pi} H_i(r,s)\,[q_{\lambda'_j}(t-s) - q_{\lambda'_j}(t)]\,ds.$$

Step 1: Study of f_1. By (4.3) we have

$$\begin{aligned} |f_1(x,t)| &\lesssim r^{\lambda'_j-1}\eta_j(r)\int_{-\pi}^{\pi} |q_{\lambda'_j}(t-s) - q_{\lambda'_j}(t)|\,ds \\ &\lesssim r^{\lambda'_j-1}\eta_j(r)\,\{\int_{-\pi}^{\pi} |q_{\lambda'_j}(s)|\,ds + 2\pi\,|q_{\lambda'_j}(t)|\} \\ &\lesssim r^{\lambda'_j-1}\eta_j(r)\,\{\|q_{\lambda'_j}\|_{W^{\sigma_j,p}(I)} + |q_{\lambda'_j}(t)|\}.\end{aligned}$$

This implies that

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{\Omega}|f_1(x,t)|^p r^{\mu_j p} r\,drdt &\lesssim \left(\int_0^1 r^{(\lambda'_j-1)p+\mu_j p+1}dr\right)\\ &\qquad\left[2\pi\|q_{\lambda'_j}\|^p_{W^{\sigma_j,p}(I)}+\int_{-\pi}^{\pi}|q_{\lambda'_j}(t)|^p dt\right]\\ &\lesssim \|q_{\lambda'_j}\|^p_{W^{\sigma_j,p}(I)}\int_0^1 r^{(\lambda'_j-1)p+\mu_j p+1}dr.\end{aligned}$$

The integral $\int_0^1 r^{(\lambda'_j-1)p+\mu_j p+1}dr$ converges if $(\lambda'_j-1)+\mu_j+\frac{2}{p}>0$ which is the condition to have $P_{j,\lambda'_j}\equiv 1$.

Step 2: Study of f_2. By (4.4), we have

$$|f_2(x,t)|\lesssim r^{\lambda'_j-1}\eta_j(r)\int_{-\pi}^{\pi}\frac{|q_{\lambda'_j}(t-s)-q_{\lambda'_j}(t)|}{(r^2+|s|)^{3/2}}\,ds$$

from which we deduce

$$\begin{aligned}&\left(\int_{-\pi}^{\pi}|f_2(x,t)|^p\,dt\right)^{1/p} \\ &\quad\lesssim r^{\lambda'_j-1}\eta_j(r)\int_{-\pi}^{\pi}(r^2+|s|)^{-\frac{3}{2}}\left(\int_{-\pi}^{\pi}|q_{\lambda'_j}(t-s)-q_{\lambda'_j}(t)|^p\,dt\right)^{\frac{1}{p}}\,ds.\end{aligned} \tag{4.13}$$

By the assumption $q_{\lambda'_j}\in W^{\sigma_j,p}(I)$ we have

$$\int_{I^2}\frac{|q_{\lambda'_j}(x)-q_{\lambda'_j}(y)|^p}{|x-y|^{1+\sigma_j p}}\,dxdy\le\|q_{\lambda'_j}\|^p_{W^{\sigma_j,p}(I)}<\infty. \tag{4.14}$$

Making the change of variables $(x,y)=(t-s,t)$, (4.14) becomes

$$\int_{I^2}\frac{|q_{\lambda'_j}(t-s)-q_{\lambda'_j}(t)|^p}{|s|^{1+\sigma_j p}}\,dsdt\le\|q_{\lambda'_j}\|^p_{W^{\sigma_j,p}(I)}.$$

Denoting

$$\kappa(s)=\left(\int_{-\pi}^{\pi}\frac{|q_{\lambda'_j}(t-s)-q_{\lambda'_j}(t)|^p}{|s|^{1+\sigma_j p}}\,dt\right)^{1/p},$$

this estimates is equivalent to

$$\int_{-\pi}^{\pi}|\kappa(s)|^p ds\le\|q_{\lambda'_j}\|^p_{W^{\sigma_j,p}(I)}. \tag{4.15}$$

Going back to (4.13), we obtain

$$\|f_2(x,\cdot)\|_{L^p(I)} \lesssim r^{\lambda_j'-1}\eta_j(r)\int_{-\pi}^{\pi}(r^2+|s|)^{-\frac{3}{2}}|s|^{\frac{1}{p}+\sigma_j}\kappa(s)\,ds,$$

which implies

$$\begin{aligned}\int_{-\pi}^{\pi}\int_\Omega |f_2(x,t)|^p r^{\mu_j p}\, rdrd\theta dt \\ \leq \int_0^1 r^{(\lambda_j'-1+\mu_j)p+2}\left(\int_{-\pi}^{\pi}(r^2+|s|)^{-\frac{3}{2}}|s|^{\frac{1}{p}+\sigma_j+1}\kappa(s)\,\frac{ds}{|s|}\right)^p\frac{dr}{r}.\end{aligned} \tag{4.16}$$

Observe that, by definition of σ_j,

$$\begin{aligned} r^{(\lambda_j'-1+\mu_j)+\frac{2}{p}}(r^2+|s|)^{-\frac{3}{2}}|s|^{\sigma_j+1} &= r^{(\lambda_j'-1+\mu_j)+\frac{2}{p}}|s|^{\sigma_j+1-\frac{3}{2}}(\tfrac{r^2}{|s|}+1)^{-\frac{3}{2}} \\ &= r^{(\lambda_j'-1+\mu_j)+\frac{2}{p}}|s|^{2-\frac{1}{p}-\frac{\mu_j+\lambda_j'}{2}-\frac{3}{2}}(\tfrac{r^2}{|s|}+1)^{-\frac{3}{2}} \\ &= (\tfrac{r^2}{|s|})^{\frac{(\lambda_j'-1+\mu_j)+\frac{2}{p}}{2}}(\tfrac{r^2}{|s|}+1)^{-\frac{3}{2}} \\ &= k(\tfrac{r^2}{|s|}) \end{aligned}$$

where

$$k(\tau)=\tau^{\frac{(\lambda_j'-1+\mu_j)+\frac{2}{p}}{2}}(\tau+1)^{-\frac{3}{2}}.$$

With these notations, (4.16) becomes

$$\|f_2\|^p_{L^p(I;L^p_{\vec\mu}(\Omega))} \leq \int_0^1\Big(\int_{-\pi}^{\pi}k(\frac{r^2}{|s|})\,|s|^{\frac{1}{p}}\,\kappa(s)\,\frac{ds}{|s|}\Big)^p\frac{dr}{r}. \tag{4.17}$$

Making the change of variables $r'=r^2$, (4.17) becomes

$$\begin{aligned}\|f_2\|^p_{L^p(I;L^p_{\vec\mu}(\Omega))} &\lesssim \int_0^1\Big(\int_{-\pi}^{\pi}k(\frac{r'}{|s|})\,|s|^{\frac{1}{p}}\,\kappa(s)\,\frac{ds}{|s|}\Big)^p\frac{dr'}{r'} \\ &\lesssim \int_0^1\Big(\int_0^{\pi}k(\frac{r'}{|s|})\,|s|^{\frac{1}{p}}\,\kappa(s)\,\frac{ds}{|s|}\Big)^p\frac{dr'}{r'}.\end{aligned}$$

Let us define

$$\begin{aligned}\tilde\kappa(s) &= \kappa(s), && \text{if } s\in\,]0,\pi[, \\ &= 0, && \text{otherwise.}\end{aligned}$$

In that case

$$\int_0^{\pi}k(\frac{r'}{s})\,s^{\frac{1}{p}}\,\kappa(s)\,\frac{ds}{s}=\int_0^{\infty}k(\frac{r'}{s})\,s^{\frac{1}{p}}\,\tilde\kappa(s)\,\frac{ds}{s}=(k*_m(s^{\frac{1}{p}}\tilde\kappa))(r')$$

where $*_m$ denotes the multiplicative convolution. Now observe that, by (4.15), we have

$$\int_0^\infty |s^{\frac{1}{p}}\tilde{\kappa}(s)|^p \frac{ds}{s} = \int_0^\infty |\tilde{\kappa}(s)|^p\, ds = \int_0^\pi |\kappa(s)|^p\, ds \le \|q_{\lambda'_j}\|^p_{W^{\sigma_j,p}(I)}.$$

It remains to verify that $k \in L^1(0,\infty;\frac{dr}{r})$ i.e.

$$\int_0^\infty r^{\frac{\mu_j+\lambda'_j-1+\frac{2}{p}}{2}}\,(r+1)^{-\frac{3}{2}}\,\frac{dr}{r} < \infty.$$

This is true as, by the summation domain $\lambda'_j + \mu_j + \frac{2}{p} < 2$ and as $P_{j,\lambda'_j} \equiv 1$, $\lambda'_j + \mu_j - 1 + \frac{2}{p} > 0$.

Case 2: $\deg(P_{j,\lambda'_j}) = l_{j,\lambda'_j} - 1 \ge 1$. In that case, we have

$$\hat{f}_k = \eta_j(r)\,(ikI-\Delta)\hat{u}_{\lambda'_j,k} = c_{\lambda'}(ik)\,ik\,e^{-r\sqrt{ik}}\,r^{\lambda'_j}\sin(\lambda'_j\theta)\,\eta_j(r)\tilde{P}(r\sqrt{ik}) \quad (4.18)$$

with $\tilde{P}$ a polynomial function of degree $l_{j,\lambda'_j} - 2$ and l_{j,λ'_j} such that

$$l_{j,\lambda'_j} + \lambda'_j + \mu_j - 2 + \frac{2}{p} > 0. \quad (4.19)$$

Let us introduce the kernel

$$H^{(l)}(r,t) = \sum_{k\in\mathbb{Z}} (ik)^{1+l/2}\, e^{-r\sqrt{ik}}\, e^{ikt},$$

where $l = 0, 1, ...l_{j,\lambda'_j} - 2$. We see that

$$|H^{(l)}| = \left|\frac{\partial^{1+l}}{\partial r^{1+l}} H\right|.$$

By Lemma 4.1, we have

$$H^{(l)} = H_1^{(l)} + H_2^{(l)}, \quad (4.20)$$

with $H_1^{(l)}$ and $H_2^{(l)}$ satisfying (4.5) and (4.6).

In the same way as before, we define

$$f_i^{(l)} = r^{\lambda'_j+l}\sin(\lambda'_j\theta)\,\eta_j(r)\int_{-\pi}^{\pi} H_i^{(l)}(r,s)\,[q_{\lambda'_j}(t-s) - q_{\lambda'_j}(t)]\,ds.$$

and we have

$$|f| \lesssim \sum_{l=0}^{l_{j,\lambda'_j}-2} (|f_2^{(l)}| + |f_1^{(l)}|), \tag{4.21}$$

with

$$|f_i^{(l)}(x,t)| = r^{\lambda'_j+l}\eta_j(r) \int_{-\pi}^{\pi} |H_i^{(l)}(r,s)|\, |q_{\lambda'_j}(t-s) - q_{\lambda'_j}(t)|\, ds. \tag{4.22}$$

Hence we have, as in the first case,

$$\int_{-\pi}^{\pi}\int_{\Omega} |f_1^{(l)}(r,t)|^p r^{\mu_j p}\, r dr dt \lesssim \|q_{\lambda'_j}\|_{W^{\sigma_j,p}(I)} \int_0^1 r^{(\lambda'_j+\mu_j+l)p+1}\, dr,$$

where the last integral converges as $\lambda'_j+\mu_j+l+\frac{2}{p} > 0$ which is true as $\sigma_j < 1$ and $l \geq 0$.

In the same way as before, we have

$$\begin{aligned}\|f_2^{(l)}\|^p_{L^p(I;L^p_{\vec\mu}(\Omega))} &\lesssim \int_0^1 r^{(\lambda'_j+l+\mu_j)p+2} \left(\int_{-\pi}^{\pi} (r^2+|s|)^{-(2+\frac{l}{2})} |s|^{\frac{1}{p}+\sigma_j+1} \kappa(s)\, \frac{ds}{|s|}\right)^p \frac{dr}{r} \\ &\lesssim \int_0^1 |k^{(l)} *_m (s^{1/p}\tilde\kappa))(r)|^p\, \frac{dr}{r}\end{aligned}$$

with

$$k^{(l)}(u) = u^{\frac{\lambda'_j+l+\mu_j}{2}+\frac{1}{p}} (u+1)^{-(2+l/2)}.$$

We conclude observing that, as $0 < \sigma_j < 1$, we have

$$\int_0^\infty k^{(l)}(u)\, \frac{du}{u} < \infty.$$

Conclusions. Now observe that

$$(\frac{\partial}{\partial t} - \Delta)(\eta_j u_{\lambda'_j}) = \eta_j(\frac{\partial}{\partial t} - \Delta)u_{\lambda'_j} - 2\frac{\partial \eta_j}{\partial r}\,\frac{\partial u_{\lambda'_j}}{\partial r} - u_{\lambda'_j}\,\Delta\eta_j$$

with $\frac{\partial\eta_j}{\partial r}$ and $\Delta\eta_j$ equals to zero on $D_j(1/2) \cup (\Omega \setminus D_j(1))$. Hence it is easy to deduce that

$$\eta_j(\frac{\partial}{\partial t} - \Delta)u_{\lambda'_j} - 2\frac{\partial \eta_j}{\partial r}\,\frac{\partial u_{\lambda'_j}}{\partial r} - u_{\lambda'_j}\,\Delta\eta_j \in L^p(I, L^p_{\vec\mu}(\Omega)),$$

which concludes the proof. □

5 Application of the second strategy

Now we are able to consider the regularity of u_R and to prove our main result.

Theorem 5.1. *Let $p \geq 2$, Ω be a bounded polygonal domain of $\mathbb{R}^2$ and denote $\vec{\lambda} = (\lambda_j)_{1\leq j\leq J}$.*

Let $\vec{\mu}$ satisfies, for all $j = 1, \ldots, J$,

$$-\lambda_j < \mu_j < \tfrac{2p-2}{p},$$
$$4(p-1)\lambda_j^2 - \mu_j^2 p^2 > 0$$

and, for all $k \in \mathbb{Z}^*$ and all $j \in \{1, 2, \cdots, J\}$, $2-\frac{2}{p}-\mu_j \neq k\lambda_j$ and $\mu_j+k\lambda_j \neq 1$. Let $\sigma_j = -\frac{\mu_j+\lambda'_j}{2} + 1 - \frac{1}{p}$, then, for every $h \in L^p(I; L^p_{\vec{\mu}}(\Omega))$, there exists a unique solution $u \in L^p(I; L^p_{\vec{\mu}}(\Omega))$ of

$$\begin{aligned} \partial_t u - \Delta u = h(x,t), &\quad \text{in } \Omega \times]-\pi, \pi[,\\ u = 0, &\quad \text{on } \partial\Omega \times [-\pi, \pi],\\ u(\cdot, -\pi) = u(\cdot, \pi), &\quad \text{in } \Omega. \end{aligned}$$

Moreover u admits the decomposition

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j\\ \exists k\in\mathbb{N}, \lambda'_j = k\lambda_j}} u_{\lambda'_j},$$

with

$$u_R \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega))$$

and

$$u_{\lambda'_j} = (E_{\lambda'_j} *_t q_{\lambda'_j}) r^{\lambda'_j} \sin(\lambda'_j \theta)$$

where

$$q_{\lambda'_j} \in W^{\sigma_j, p}(I)$$

and

$$E_{\lambda'_j}(x,t) = \sum_{k\in\mathbb{Z}^*} e^{ikt} P_{j,\lambda'_j}(r\sqrt{ik})\, e^{-r\sqrt{ik}} + \frac{1}{2\pi}.$$

Proof. Recall that, in the notations of the end of Section 2, we define

$$R(z) : L^p_{\vec{\mu}}(\Omega) \to V^{2,p}_{\vec{\mu}}(\Omega) : g \mapsto u_R$$

where u_R is the regular part of the solution of

$$\left\{ \begin{array}{lll} -\Delta u + zu = g, & \text{in} & \Omega, \\ u = 0, & \text{on} & \partial\Omega, \end{array} \right.$$

and, for all $z \in \pi^+ \cup S_A$,

$$\|R(z)\|_{L^p_{\vec{\mu}}(\Omega)\to V^{2,p}_{\vec{\mu}}(\Omega)} + (1+|z|)\,\|R(z)\|_{L^p_{\vec{\mu}}(\Omega)\to L^p_{\vec{\mu}}(\Omega)} \lesssim 1.$$

Hence by interpolation we have (see for example [1, Thm 7.22]), for all $\theta \in [0,1[$,

$$\|R(z)\|_{L^p_{\vec{\mu}}(\Omega)\to (L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta} \lesssim \frac{1}{(1+|z|)^{1-\theta}}.$$

In that case, by Theorem 3.4 and Remark 3.1

$$\begin{aligned}
&\|u_R\|_{L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta)} \\
&\quad \le \Big\|\frac{1}{2\pi i}\int_\gamma R(z)\,(zI-B_0)^{-1}(h-\frac{1}{2\pi}\int_{-\pi}^{\pi} h(s,\cdot)\,ds)dz + \bar{u}_R\Big\|_{L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta)} \\
&\quad \le \frac{1}{2\pi}\int_\gamma \|R(z)\,(z\,I-B_0)^{-1}(h-\frac{1}{2\pi}\int_{-\pi}^{\pi} h(s,\cdot)\,ds)\|_{L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta)}\,dz \\
&\qquad\qquad + \|\bar{u}_R\|_{L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta)} \\
&\quad \lesssim \frac{1}{2\pi}\int_\gamma \frac{1}{(1+|z|)^{1-\theta}}\|(z\,I-B_0)^{-1}(h-\frac{1}{2\pi}\int_{-\pi}^{\pi} h(s,\cdot)\,ds)\|_{L^p(I;L^p_{\vec{\mu}}(\Omega))}\,dz \\
&\qquad\qquad + \|\bar{u}_R\|_{L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta)} \\
&\quad \lesssim \frac{1}{2\pi}\int_\gamma \frac{1}{(1+|z|)^{2-\theta}}\|h\|_{L^p(I;L^p_{\vec{\mu}}(\Omega))}\,dz + K\,\|h\|_{L^p(I;L^p_{\vec{\mu}}(\Omega))}
\end{aligned}$$

Hence, for all $\theta \in [0,1[$,

$$u_R \in L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta),$$

with the estimate

$$\|u_R\|_{L^p(I;(L^p_{\vec{\mu}}(\Omega),V^{2,p}_{\vec{\mu}}(\Omega))_\theta)} \le K(\theta)\|h\|_{L^p(I;L^p_{\vec{\mu}}(\Omega))}, \tag{5.1}$$

for some positive constant $K(\theta)$ that may depend on θ but not on h.

Let us show that

$$u_R \in L^p(I;V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}_{2\pi}(I;L^p_{\vec{\mu}}(\Omega)).$$

First observe that u_R is a strong solution of

$$\partial_t u_R + u_R - \Delta u_R = h - \sum_{j=1}^{J} \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k\in\mathbb{N}, \lambda'_j=k\lambda_j}} (\partial_t - \Delta)(\eta_j u_{\lambda'_j}) + u_R =: h_R$$

with, by the previous results, $h_R \in L^p(I; L^p_{\vec{\mu}}(\Omega))$.

Then we apply the second strategy with

$$E = L^p(I; L^p_{\vec{\mu}}(\Omega)),$$

and

$$A : D(A) \subset E \to E : u \mapsto -\Delta u, \quad \text{with} \quad D(A) = L^p(I; D(\Delta_{p,\vec{\mu}})),$$

$$B : D(B) \subset E \to E : u \mapsto \partial_t u + u, \quad \text{with} \quad D(B) = W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)).$$

The assumptions (H_3) and (H_5) can be verified as previously. The assumption (H_4) is satisfied by all $L^p_{\vec{\mu}}(\Omega)$ spaces (see for example [3]). It remains to verify (H_6). To this aim we will apply the following result of Coifman - Weiss (see [5] or for example [3]).

If $-A$ is the infinitesimal generator of a strongly continuous contraction semigroup in E which preserves the positivity then there exists $K > 0$ such that, for all $s \in \mathbb{R}$,

$$\|A^{is}\| \leq K(1 + |s|)\, e^{\frac{\pi}{2}|s|}.$$

For what concerns the operator A, we already know (see [8, proof of Corollary 2.14]) that $-A$ generates a C_0 semigroup of contractions $T(t)$. It remains to prove that $T(t)$ preserves the positivity.

Let $f \in L^p_{\vec{\mu}}(\Omega)$ with $f \geq 0$, $\lambda \in [0, +\infty[$ and $u \in H^1_0(\Omega)$ be the solution of

$$\forall w \in H^1_0(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla \overline{w} + \lambda \int_\Omega u\, \overline{w} = \int_\Omega f\, \overline{w}. \tag{5.2}$$

As f is real, the solution u is real. Let us decompose $u = u^+ - u^-$ with $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. Hence we have u^+, $u^- \in H^1_0(\Omega)$ and by (5.2) applied with $w = u^-$, we obtain

$$\int_\Omega |\nabla u^-|^2 + \lambda \int_\Omega |u^-|^2 = -\int_\Omega f\, u^- \leq 0.$$

Hence $u^- \equiv 0$ and $u \geq 0$.

By [21, Cor I-3.5] we have that if $-A$ is the generator of $T(t)$ then, for all $f \in E$,

$$\begin{aligned} T(t)f &= \lim_{\lambda\to\infty} e^{t(\lambda^2(\lambda I+A)^{-1}-\lambda I)} f \\ &= \lim_{\lambda\to\infty} e^{-\lambda t} e^{t\lambda^2(\lambda I+A)^{-1}} f \\ &= \lim_{\lambda\to\infty} e^{-t\lambda} \sum_{k\geq 0} \frac{(t\lambda^2)^k}{k!} (\lambda I + A)^{-k} f. \end{aligned}$$

By what we prove just before we have $(\lambda I + A)^{-1} f \geq 0$ and hence $T(t)f \geq 0$. We then deduce that the semi-group preserves the positivity. Hence, there exists $K > 0$ such that, for all $s \in \mathbb{R}$,

$$\|A^{is}\|_{L^p(I;L^p_{\vec{\mu}}(\Omega))} \leq K(1 + |s|)\, e^{\frac{\pi}{2}|s|}.$$

As $-A$ is symmetric on the Hilbert space $L^2(I, L^2(\Omega))$, we have also (see for example [3, p. 164])

$$\|A^{is}\|_{L^2(I,L^2(\Omega))} \leq 1.$$

Hence under the assumptions of Theorem 2.3, for $\theta \in]0, 1[$ close enough to 1 in such a way that $\vec{\nu} = \frac{\vec{\mu}}{\theta}$ and $q = \frac{2p\theta}{2-p(1-\theta)}$ satisfy the assumptions of Theorem 2.3, we have, by [25, 1.18.7/Th 4],

$$\begin{aligned} \|A^{is}\|_{L^p(I,L^p_{\vec{\mu}}(\Omega))} &\leq \|A^{is}\|^{1-\theta}_{L^2(I,L^2(\Omega))}\, \|A^{is}\|^{\theta}_{L^q(I,L^q_{\vec{\nu}}(\Omega))} \\ &\leq K^\theta\, (1 + |s|)^\theta\, e^{\theta \frac{\pi}{2}|s|}. \end{aligned}$$

Hence, for all such θ, for all $\epsilon > 0$ there exists $K(\epsilon, \theta) > 0$ such that

$$\|A^{is}\|_{L^p(I,L^p_{\vec{\mu}}(\Omega))} \leq K(\epsilon, \theta)\, e^{(\theta\frac{\pi}{2}+\epsilon)|s|},$$

from which we deduce the existence of $\tau_A < \frac{\pi}{2}$ such that

$$\|A^{is}\|_{L^p(I,L^p_{\vec{\mu}}(\Omega))} = 0(e^{\tau_A|s|}).$$

For what concerns B, observe that $\sigma(-B) = \{-(ki + 1) \mid k \in \mathbb{Z}\}$ and hence $\sigma(-B) \cap [0, +\infty[= \emptyset$. Moreover we have seen that $\mathbb{R}^+ \subset \rho(-B)$ and, for all $\lambda \in \mathbb{R}^+$,

$$\|(\lambda I + B)^{-1}\| \leq \frac{1}{\lambda + 1},$$

and hence, as in [8, proof of Corollary 2.14], we see that $-B$ is the generator of a C_0 semigroup $S(t)$ of contraction.

Let us show that $S(t)$ preserves the positivity. Consider the solution $u \in D(B)$ of

$$\partial_t u + u + \lambda u = f \geq 0,$$
$$u(-\pi) = u(\pi),$$

then

$$\begin{aligned} u(x,t) &= (B + \lambda I)^{-1} f \\ &= \int_{-\pi}^{\pi} \frac{e^{-(1+\lambda)(\pi - s)}}{e^{(1+\lambda)\pi} - e^{-(1+\lambda)\pi}} f(x,s)\, ds\; e^{-(1+\lambda)t} \\ &\qquad + \int_{-\pi}^{t} e^{-(1+\lambda)(t-s)}\, f(x,s)\, ds \geq 0. \end{aligned}$$

As above we deduce that $S(t)$ preserves the positivity. By the previous result of Coifman-Weiss, there exists $K > 0$ such that, for all $s \in \mathbb{R}$,

$$\|B^{is}\| \leq K\,(1 + |s|)\, e^{\frac{\pi}{2}|s|}.$$

Hence we obtain $\tau_A < \pi/2$ and $\tau_B \in]\pi/2, \pi - \tau_A[$ such that

$$\|A^{is}\| = 0(e^{\tau_A |s|}),$$
$$\|B^{is}\| = 0(e^{\tau_B |s|}).$$

As all the assumptions of the second strategy are satisfied, we have the existence of $w_R \in W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) \cap L^p(I; D(\Delta_{p,\vec{\mu}}))$ solution of

$$\begin{aligned} \partial_t w + w - \Delta w = h_R, &\quad \text{in } \Omega \times]-\pi, \pi[, \\ w = 0, &\quad \text{on } \partial\Omega \times [-\pi, \pi], \\ w(\cdot, -\pi) = w(\cdot, \pi), &\quad \text{in } \Omega. \end{aligned}$$

Claim: $w_R = u_R$ *and hence*

$$u_R \in W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) \cap L^p(I; D(\Delta_{p,\vec{\mu}})) \cap L^p(I; (L^p_{\vec{\mu}}(\Omega), V^{2,p}_{\vec{\mu}}(\Omega))_\theta).$$

It is easy to observe that u_R is a strong solution of

$$\begin{aligned} \partial_t u_R + u_R - \Delta u_R = h & \\ - \sum_{j=1}^{J} \sum_{\substack{0 < \lambda'_j < 2 - \frac{2}{p} - \mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} (\partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j})) + u_R, &\quad \text{in } \Omega \times I, \\ u_R = 0, &\quad \text{on } \partial\Omega \times \bar{I}, \\ u_R(\cdot, -\pi) = u_R(\cdot, \pi), &\quad \text{in } \Omega. \end{aligned} \tag{5.3}$$

In fact, by Theorem 3.4, we know that u is a strong solution of

$$\begin{aligned} \partial_t u - \Delta u = h, &\quad \text{in } \Omega \times I, \\ u = 0, &\quad \text{on } \partial\Omega \times \bar{I}, \\ u(\cdot, -\pi) = u(\cdot, \pi), &\quad \text{in } \Omega, \end{aligned}$$

i.e., for every $n \in \mathbb{N}$, there exist $u_n \in D(A) \cap D(B)$ and $h_n \in E$ such that $(A+B)u_n = h_n$, $u_n \to u$ and $h_n \to h$ in E. Moreover, as in Section 3, for every n, we have the decomposition

$$u_n = u_{n,R} + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} u_{n,\lambda'_j}.$$

By Theorems 3.4 and 4.2, we have

$$\partial_t(\eta_j u_{n,\lambda'_j}) - \Delta(\eta_j u_{n,\lambda'_j}) \to \partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j}).$$

By the estimate (5.1) we see that $u_{n,R} \to u_R$ in E.

Now observe that w_R is a strong solution of (5.3) as $w_{n,R} = w_R$ are such that $w_{n,R} \in D(A) \cap D(B)$ and $w_{n,R} \to w_R$.

Hence applying the first strategy to (5.3) with

$$\begin{aligned} E &= L^p(I, L^p_\mu(\Omega)) \\ A : D(A) &:= L^p(I; D(\Delta_{p,\vec{\mu}})) \subset E \to E : u \mapsto -\Delta u, \\ B : D(B) &:= W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) \subset E \to E : u \mapsto \partial_t u + u, \end{aligned}$$

we have by uniqueness of the strong solution that $w_R = u_R$ and hence

$$u_R \in W^{1,p}_{2\pi}(I; L^p_{\vec{\mu}}(\Omega)) \cap L^p(I; D(\Delta_{p,\vec{\mu}})) \cap L^p(I; (L^p_{\vec{\mu}}(\Omega), V^{2,p}_{\vec{\mu}}(\Omega))_\theta).$$

Claim: $D(\Delta_{p,\mu}) \cap (L^p_{\vec{\mu}}(\Omega), V^{2,p}_{\vec{\mu}}(\Omega))_\theta \subset V^{2,p}_{\vec{\mu}}(\Omega)$.

If $u \in D(\Delta_{p,\mu})$ then, by [18] as in [8], u admits the decomposition

$$u = u_1 + \sum_{j=1}^{J} \eta_j \sum_{\substack{0<\lambda'_j<2-\frac{2}{p}-\mu_j \\ \exists k \in \mathbb{N}, \lambda'_j = k\lambda_j}} c_{\lambda'_j}\, r^{\lambda'_j} \sin(\lambda'_j \theta).$$

Recall that by [7] we have that

$$V^{2,p}_{\vec{\mu}}(\Omega) \to W^{2,p}(\Omega) : u \mapsto w\, u$$

as well as

$$L^p_{\vec{\mu}}(\Omega) \to L^p(\Omega) : u \mapsto w\, u$$

are continuous. Hence, if $u \in (L^p_{\vec{\mu}}(\Omega), V^{2,p}_{\vec{\mu}}(\Omega))_\theta$ we have $w\, u \in (L^p(\Omega), W^{2,p}(\Omega))_\theta$. By [1] we know $(L^p(\Omega), W^{2,p}(\Omega))_\theta = W^{2\theta,p}(\Omega)$ and in particular $r^{\mu_j} u \in W^{2\theta,p}(D_j)$.

By [12] we have
if $\mu_j + \lambda'_j > 2\theta - \frac{2}{p}$, then $r^{\mu_j+\lambda'_j} \sin(\lambda'_j\theta) \in W^{2\theta,p}(D_j)$,
if $\mu_j + \lambda'_j \leq 2\theta - \frac{2}{p}$ and $\mu_j + \lambda'_j \notin \mathbb{N}$, then $r^{\mu_j+\lambda'_j} \sin(\lambda'_j\theta) \notin W^{2\theta,p}(D_j)$.

As $\mu_j + \lambda'_j < 2 - \frac{2}{p}$, for θ close to 1, we have $\mu_j + \lambda'_j \leq 2\theta - \frac{2}{p}$ and hence $u = u_1 \in V^{2,p}_{\vec{\mu}}(\Omega)$. □

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