

Stabilization by switching control methods

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Abstract. In this paper we consider some stabilization problems for the wave equation with switching. We prove exponential stability results for appropriate damping coefficients. The proof of the main results is based on D'Alembert formula and some energy estimates.

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1 Introduction

Our main goal is to study the pointwise or boundary stabilization of a switching delay wave equation in $(0, \ell)$. More precisely, we consider the systems given by :

$$u_{tt}(x, t) - u_{xx}(x, t) = 0, \quad \text{in } (0, \ell) \times (0, 2\ell), \quad (1.1)$$

$$u_{tt}(x, t) - u_{xx}(x, t) + a u_t(\xi, t - 2\ell) \delta_\xi = 0, \quad \text{in } (0, \ell) \times (2\ell, +\infty), \quad (1.2)$$

$$u(0, t) = 0, u_x(\ell, t) = 0, \quad \text{on } (0, +\infty), \quad (1.3)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } (0, \ell), \quad (1.4)$$

and

$$u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad \text{in } (0, \ell) \times (0, +\infty), \quad (1.5)$$

$$u(0, t) = 0 \quad \text{on } (0, +\infty), \quad (1.6)$$

$$u_x(\ell, t) = 0 \quad \text{on } (0, 2\ell), \quad (1.7)$$

$$u_x(\ell, t) = \mu_1 u_t(\ell, t) \quad \text{on } (2(2i+1)\ell, 2(2i+2)\ell), \forall i \in \mathbb{N}, \quad (1.8)$$

$$u_x(\ell, t) = \mu_2 u_t(\ell, t - 2\ell) \quad \text{on } (2(2i+2)\ell, 2(2i+3)\ell), \forall i \in \mathbb{N}, \quad (1.9)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } (0, \ell), \quad (1.10)$$

where $\ell > 0$, μ_1, μ_2, a and $\xi \in (0, \ell)$ are constants. Here and below we denote by \mathbb{N} the set of the natural numbers while $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

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Note that in both cases, the feedbacks are unbounded.

Delay effects arise in many applications and practical problems and it is well-known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in absence of delay (see e.g. [9, 10, 11], [15]). Nevertheless recent papers reveal that particular choice of delays may reconstitute exponential stability property, see [12, 13, 18].

We refer also to [1, 2, 15, 16] for stability results for systems with time delay due to the presence of “good” feedbacks compensating the destabilizing delay effect.

Note that the above systems are exponentially stable in absence of time delay, and if $\mu_1 = \mu_2 < 0$ (see e.g. [8]) for the second system and if $a > 0$, ξ admits a coprime factorization $\frac{p}{q}$ and p is odd (the best rate is obtained for $\xi = \frac{\ell}{2}$, see e.g. [4]) for the first system.

In this paper we propose a new approach that consists to stabilize the wave system by a control law that uses informations from the past (by switching or not). This means that the stabilization is obtained by a control method (that we propose to call switching control method) and not by a feedback law. For the first system this law is given by the term $a u_t(\xi, t - 2\ell) \delta_\xi$ in (1.2) for $t \geq 2\ell$, while for the second system it corresponds to the term $\mu_2 u_t(\ell, t - 2\ell)$ in a switched control form. Using D’Alembert formula and some energy estimates, we will show that for any $a \in (0, 2)$ and $\xi = \frac{\ell}{2}$, system (1.1)–(1.4) is exponentially stable. On the other hand we show that appropriate choices of μ_1 and μ_2 yield the exponential stability of (1.5)–(1.10).

The same approach is briefly treated in higher dimension, here our approach combines observability estimates and some energy estimates.

For the existence results, let us recall the following facts. Let $A = -\partial_x^2$ be the unbounded operator in $H = L^2(0, \ell)$ with domain

$$H_1 = \mathcal{D}(A) = \{u \in H^2(0, \ell); u(0) = 0, u_x(\ell) = 0\},$$

$$H_{\frac{1}{2}} = \mathcal{D}(A^{\frac{1}{2}}) = \{u \in H^1(0, \ell); u(0) = 0\}.$$

We define

$$B_1 \in \mathcal{L}(\mathbb{R}, H_{-\frac{1}{2}}), B_1 k = k \sqrt{a} \delta_\xi, \forall k \in \mathbb{R}, B_1^* u = \sqrt{a} u(\xi), \forall u \in H_{\frac{1}{2}},$$

and

$$B_2 \in \mathcal{L}(\mathbb{R}, H_{-\frac{1}{2}}), B_2 k = \sqrt{\mu_1} A_{-1} D k = k \sqrt{\mu_1} \delta_\ell, \forall k \in \mathbb{R}, B_2^* u = \sqrt{\mu_1} u(\ell), \forall u \in H_{\frac{1}{2}},$$

where A_{-1} is the extension of A to $H_{-1} = (\mathcal{D}(A))'$ and D is the Dirichlet map ($Dk = kx$ on $(0, \ell)$) and $H_{-\frac{1}{2}} = (H_{\frac{1}{2}})'$ (the duality is in the sense of H).

To study the well-posedness of the systems (1.1)–(1.4) and (1.5)–(1.10), we write them as an abstract Cauchy problem in a product space, and use the semigroup approach. For this purpose, take the Hilbert space $\mathcal{H} := H_{\frac{1}{2}} \times H$ and the unbounded linear operators

$$A : \mathcal{D}(A) = H_1 \times H_{\frac{1}{2}} \subset \mathcal{H} \longrightarrow \mathcal{H}, \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -A u_1 \end{pmatrix}, \quad (1.11)$$

and

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) = \{(u, v) \in [H^2(0, \ell) \times H^1(0, \ell)] \cap \mathcal{H} : u_x(\ell) = \mu_2 v(\ell)\} \subset \mathcal{H} \longrightarrow \mathcal{H},$$

$$\mathcal{A}_d \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -Au_1 \end{pmatrix}. \quad (1.12)$$

It is well known that the operators $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and $(\mathcal{A}_d, \mathcal{D}(\mathcal{A}_d))$ defined by (1.11) and (1.12), generate a strongly continuous semigroup of contractions on \mathcal{H} denoted respectively $(\mathcal{T}(t))_{t \geq 0}$ (we also denote $(\mathcal{T}_{-1}(t))_{t \geq 0}$ the extension of $(\mathcal{T}(t))_{t \geq 0}$ to H_{-1}), $(\mathcal{T}_d(t))_{t \geq 0}$.

Proposition 1.1. *1. The system (1.1)–(1.4) is well-posed. More precisely, for every $(u_0, u_1) \in \mathcal{H}$, the solution of (1.1)–(1.4) is given by*

$$\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} u^0(t) \\ u_t^0(t) \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, & 0 \leq t \leq 2\ell, \\ \begin{pmatrix} u^j(t) \\ u_t^j(t) \end{pmatrix} = \mathcal{T}(t - 2j\ell) \begin{pmatrix} u^{j-1}(2j\ell) \\ u_t^{j-1}(2j\ell) \end{pmatrix} + \\ \int_{2j\ell}^t \mathcal{T}_{-1}(t-s) \begin{pmatrix} 0 \\ -a u_t^{j-1}(s-2\ell)\delta_\xi \end{pmatrix} ds, & 2j\ell \leq t \leq 2(j+1)\ell, j \geq 1. \end{cases}$$

and satisfies $(u^j, u_t^j) \in C([2j\ell, 2(j+1)\ell], \mathcal{H})$, $j \in \mathbb{N}$.

2. The system (1.5)–(1.10) is well-posed. More precisely, for every $(u_0, u_1) \in \mathcal{H}$, the solution of (1.5)–(1.10) is given by

$$\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} u^0(t) \\ u_t^0(t) \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, & 0 \leq t \leq 2\ell, \\ \begin{pmatrix} u^{2j+1}(t) \\ u_t^{2j+1}(t) \end{pmatrix} = \mathcal{T}_d(t - 2(2j+1)\ell) \begin{pmatrix} u^{2j}(2(2j+1)\ell) \\ u_t^{2j}(2(2j+1)\ell) \end{pmatrix}, & 2(2j+1)\ell \leq t \leq 2(2j+2)\ell, j \in \mathbb{N}, \\ \begin{pmatrix} u^{2j+2}(t) \\ u_t^{2j+2}(t) \end{pmatrix} = \mathcal{T}(t - 2(2j+2)\ell) \begin{pmatrix} u^{2j+1}(2(2j+2)\ell) \\ u_t^{2j+1}(2(2j+2)\ell) \end{pmatrix} + \\ \int_{2(2j+2)\ell}^t \mathcal{T}_{-1}(t-s) \begin{pmatrix} 0 \\ -\mu_1 u_t^{2j+1}(s-2\ell)\delta_\ell \end{pmatrix} ds, & 2(2j+2)\ell \leq t \leq 2(2j+3)\ell, j \in \mathbb{N} \end{cases}$$

and satisfies

$$(u^0, u_t^0) \in C([0, 2\ell], \mathcal{H}), (u^{2j+1}, u_t^{2j+1}) \in C([2(2j+1)\ell, 2(2j+2)\ell], \mathcal{H}), j \in \mathbb{N}, \\ (u^{2j+2}, u_t^{2j+2}) \in C([2(2j+2)\ell, 2(2j+3)\ell], \mathcal{H}), j \in \mathbb{N}.$$

For any solution of problem (1.1)–(1.4) respectively of (1.5)–(1.10) we define the energy

$$E_p(t) = E_b(t) = \frac{1}{2} \int_0^\ell \{|u_x(x, t)|^2 + |u_t(x, t)|^2\} dx. \quad (1.13)$$

The main result of this paper is the following.

Theorem 1.2. 1. We suppose that $\xi = \frac{\ell}{2}$. Then for any $a \in (0, 2)$ there exist positive constants C_1, C_2 such that for all initial data in \mathcal{H} , the solution of problem (1.1)–(1.4) satisfies

$$E_p(t) \leq C_1 e^{-C_2 t}. \quad (1.14)$$

The constant C_1 depends on the initial data, on ℓ and on a , while C_2 depends only on ℓ and on a .

2. For any μ_1, μ_2 satisfying one of the following conditions

$$\begin{aligned} 1 < \mu_2 < \mu_1, \\ \mu_1 < \mu_2 < 1, \end{aligned} \quad (1.15)$$

there exist positive constants C_1, C_2 such that for all initial data in \mathcal{H} , the solution of problem (1.5)–(1.10) satisfies

$$E_b(t) \leq C_1 e^{-C_2 t}. \quad (1.16)$$

The constant C_1 depends on the initial data, on ℓ and on μ_1, μ_2 , while C_2 depends only on ℓ and on μ_1, μ_2 .

The paper is organized as follows. The second section deals with the well-posedness of the problem while, in the third section, we prove the exponential stability of the systems (1.1)–(1.4) and of (1.5)–(1.10) by using a suitable D’Alembert formula. In section 4 we give the same type of results for a multidimensional system. Some comments and related questions are given in the last section.

2 Proof of Proposition 1.1

Consider the evolution problems

$$\dot{y}^j(t) + Ay^j(t) = B_1 v^j(t), \quad \text{in } (2j\ell, 2(j+1)\ell), \quad j \in \mathbb{N}^*, \quad (2.1)$$

$$y^j(2j\ell) = \dot{y}^j(2j\ell) = 0, \quad j \in \mathbb{N}^*. \quad (2.2)$$

$$\ddot{\phi}(t) + A\phi(t) = 0, \quad \text{in } (0, +\infty), \quad (2.3)$$

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = \phi_1. \quad (2.4)$$

A natural question is the regularity of y^j when $v^j \in L^2(2\ell j, 2(j+1)\ell j)$, $j \in \mathbb{N}^*$. By applying standard energy estimates we can easily check that $y^j \in C([2j\ell, 2(j+1)\ell]; H) \cap C^1([2j\ell, 2(j+1)\ell]; H_{-\frac{1}{2}})$. However if B_1 satisfies a certain admissibility condition then y^j is more regular. More precisely the following result, which is a version of the general transposition method (see, for instance, Lions and Magenes [14]) holds true.

It is clear that the system (2.3)–(2.4) admits a unique solution ϕ having the regularity

$$\phi \in C([0, 2\ell]; H_{\frac{1}{2}}) \cap C^1([0, 2\ell]; H),$$

$$(\phi, \dot{\phi})(t) = \mathcal{T}(t) \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}, \quad 0 \leq t \leq 2\ell.$$

Moreover, $B_1^* \phi(\cdot) \in H^1(0, 2\ell)$, and for all $T \in (0, 2\ell)$ there exists a constant $C > 0$ such that

$$\|(B_1^* \phi)'(\cdot)\|_{L^2(0, T)} \leq C \|(\phi_0, \phi_1)\|_{H_{\frac{1}{2}} \times H}, \quad \forall (\phi_0, \phi_1) \in H_{\frac{1}{2}} \times H. \quad (2.5)$$

Lemma 2.1. *Suppose that $v^j \in L^2([2j\ell, 2(j+1)\ell])$, $j \in \mathbb{N}^*$. Then the problem (2.1)–(2.2) admits a unique solution having the regularity*

$$y^j \in C([2j\ell, 2(j+1)\ell]; H_{\frac{1}{2}}) \cap C^1([2\ell j, 2(j+1)\ell]; H), \quad j \in \mathbb{N}^*, \quad (2.6)$$

and

$$(y^j, \dot{y}^j)(t) = \int_{2j\ell}^t \mathcal{T}_{-1}(t - 2j\ell - s) \begin{pmatrix} 0 \\ B_1 v_j(s) \end{pmatrix} ds, \quad 2j\ell \leq t \leq 2(j+1)\ell, \quad j \geq 1.$$

Proof. If we set $Z(t) = \begin{pmatrix} y^j(t + 2j\ell) \\ \dot{y}^j(t + 2j\ell) \end{pmatrix}$ it is clear that (2.1)–(2.2) can be written as

$$\dot{Z}^j + \mathcal{A}Z^j(t) = \mathcal{B}_1 v^j(t + 2j\ell) \text{ on } (0, 2\ell), \quad Z^j(0) = 0,$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} : H_{\frac{1}{2}} \times H \rightarrow [\mathcal{D}(\mathcal{A})]',$$

$$\mathcal{B}_1 = \begin{pmatrix} 0 \\ B_1 \end{pmatrix} : \mathbb{R} \rightarrow [\mathcal{D}(\mathcal{A})]'$$

It is well known that \mathcal{A} is a skew adjoint operator so it generates a group of isometries in $[\mathcal{D}(\mathcal{A})]'$, denoted by $\mathcal{S}(t) (= \mathcal{T}_{-1}(t))$.

After simple calculations we get that the operator $\mathcal{B}_1^* : \mathcal{D}(\mathcal{A}) \rightarrow \mathbb{R}$ is given by

$$\mathcal{B}_1^* \begin{pmatrix} u^j \\ v^j \end{pmatrix} = B_1^* v^j, \quad \forall (u^j, v^j) \in \mathcal{D}(\mathcal{A}).$$

This implies that

$$\mathcal{B}_1^* \mathcal{S}^*(t) \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = B_1^* \dot{\phi}(t), \quad \forall (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A}),$$

with ϕ satisfying (2.3)–(2.4). From the inequality above and (2.5) we deduce that there exists a constant $C > 0$ such that for all $T \in (0, 2\ell)$

$$\int_0^T \left| \mathcal{B}_1^* \mathcal{S}^*(t) \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \right|^2 dt \leq C \|(\phi_0, \phi_1)\|_{H_{\frac{1}{2}} \times H}^2, \quad \forall (\phi_0, \phi_1) \in \mathcal{D}(\mathcal{A}).$$

According to Theorem 3.1 in [7, p.187] (see also [17]) the inequality above implies the interior regularity (2.6). \square

The existence result for problem (1.1)–(1.4) is now made by induction. First on $[0, 2\ell]$ (case $j = 0$), we take

$$\begin{pmatrix} u^0(t) \\ u_t^0(t) \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad \forall t \in [0, 2\ell].$$

That is clearly a solution of (1.1)–(1.4) on $(0, 2\ell)$ and that has the regularity $(u^0, u_t^0) \in C([0, 2\ell]; \mathcal{H})$. Now for $j \geq 1$, we take for all $t \in [2j\ell, 2(j+1)\ell]$,

$$\begin{aligned} \begin{pmatrix} u^j(t) \\ u_t^j(t) \end{pmatrix} &= \begin{pmatrix} \phi(t+2j\ell) \\ \dot{\phi}(t+2j\ell) \end{pmatrix} + \begin{pmatrix} y^j(t) \\ \dot{y}^j(t) \end{pmatrix} \\ &= \mathcal{T}(t+2j\ell) \begin{pmatrix} u^{j-1}(2j\ell) \\ \dot{u}^{j-1}(2j\ell) \end{pmatrix} + \int_{2j\ell}^t \mathcal{T}_{-1}(t-s) \begin{pmatrix} 0 \\ -a u_t^{j-1}(s-2\ell)\delta_\xi \end{pmatrix} ds, \end{aligned}$$

where y^j (resp. ϕ) is solution of (2.1)–(2.2) (resp. (2.3)–(2.4)) with $v^j(t) = -a u_t^{j-1}(t-2\ell)$ (that belongs to $L^2(2j\ell, 2(j+1)\ell)$) and $\phi_0 = u^{j-1}(2j\ell)$, $\phi_1 = \dot{u}^{j-1}(2j\ell)$. This solution has the announced regularity due to the above arguments.

By the same way we prove the second assertion of Proposition 1.1.

3 Proof of Theorem 1.2

We show that the system (1.1)–(1.4) can be reformulated as follows:

$$u_{tt}^0(x, t) - u_{xx}^0(x, t) = 0, \quad \text{in } (0, \ell) \times (0, 2\ell), \quad (3.1)$$

$$u_{tt}^{j-}(x, t) - u_{xx}^{j-}(x, t) = 0, \quad \text{in } (0, \xi) \times (2j\ell, 2(j+1)\ell), \quad j \in \mathbb{N}^*, \quad (3.2)$$

$$u_{tt}^{j+}(x, t) - u_{xx}^{j+}(x, t) = 0, \quad \text{in } (\xi, \ell) \times (2j\ell, 2(j+1)\ell), \quad j \in \mathbb{N}^*, \quad (3.3)$$

$$u^{j-}(\xi, t) = u^{j+}(\xi, t), \quad \text{on } (2j\ell, 2(j+1)\ell), \quad j \in \mathbb{N}^*, \quad (3.4)$$

$$-u_x^{j-}(\xi, t) + u_x^{j+}(\xi, t) = -a u_t^{j+}(\xi, t-2\ell) \quad \text{on } (2j\ell, 2(j+1)\ell), \quad j \in \mathbb{N}^*, \quad (3.5)$$

$$u^{j-}(0, t) = 0, \quad u_x^{j+}(\ell, t) = 0, \quad \text{on } (2j\ell, 2(j+1)\ell), \quad j \in \mathbb{N}^*, \quad (3.6)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } (0, \ell). \quad (3.7)$$

Note that if $u_t^{j+}(\xi, t-2\ell)$ is replaced by $u_t^{j+}(\xi, t)$, then the energy is decaying if $a > 0$ (and is exponentially decaying if ξ satisfies some conditions: ξ admits a coprime factorization $\frac{p}{q}$ and p is odd).

Hence we look for u solution of (1.1)–(1.4) in the form:

$$u(x, t) = \alpha_-(x+t) - \alpha_-(t-x), \quad \forall x \in (0, \xi), \quad t \geq 0, \quad (3.8)$$

and

$$u(x, t) = \alpha_+(x-\ell+t) + \alpha_+(t-x+\ell), \quad \forall x \in (\xi, \ell), \quad t \geq 0, \quad (3.9)$$

where α_- and α_+ have to be determined. From this expression we directly see that

$$u(0, t) = 0, \quad \text{and } u_x(\ell, t) = 0, \quad \forall t \geq 0,$$

in order words (1.3) holds. Hence it remains to impose the initial conditions at $t = 0$ and the transmission conditions at $x = \xi$.

In order to fulfil the initial conditions (1.4) for $x \leq \xi$, we take

$$\begin{aligned}\alpha_-(x) &= -\frac{1}{2}u_0(-x) + \frac{1}{2}\int_0^{-x} u_1(s)ds \quad \forall x \in (-\xi, 0), \\ \alpha_-(x) &= \frac{1}{2}u_0(x) + \frac{1}{2}\int_0^x u_1(s)ds \quad \forall x \in [0, \xi).\end{aligned}$$

In that way α_- is uniquely determined in $(-\xi, \xi)$.

In the same manner to fulfil the initial conditions (1.4) for $x \geq \xi$, we take

$$\begin{aligned}\alpha_+(y) &= \frac{1}{2}u_0(\ell + y) + \frac{1}{2}\int_0^{\ell+y} u_1(s)ds \quad \forall x \in (-(\ell - \xi), 0), \\ \alpha_+(y) &= \frac{1}{2}u_0(\ell - y) - \frac{1}{2}\int_0^{\ell-y} u_1(s)ds \quad \forall y \in [0, \ell - \xi).\end{aligned}$$

In that way α_+ is uniquely determined in $(-(\ell - \xi), \ell - \xi)$.

To check (3.1), we need the continuity of u and u_x at ξ , that is equivalent to

$$\begin{aligned}\alpha_-(\xi + t) - \alpha_-(t - \xi) &= \alpha_+(\xi - \ell + t) + \alpha_+(t - \xi + \ell), \forall t \in (0, 2\ell), \\ \alpha'_-(\xi + t) + \alpha'_-(t - \xi) &= \alpha'_+(\xi - \ell + t) - \alpha'_+(t - \xi + \ell), \forall t \in (0, 2\ell).\end{aligned}$$

By setting $y = \xi + t$, this is equivalent to

$$\begin{aligned}\alpha_-(y) - \alpha_+(y - 2\xi + \ell) &= \alpha_-(y - 2\xi) + \alpha_+(y - \ell), \forall y \in (\xi, \xi + 2\ell), \\ \alpha'_-(y) + \alpha'_+(y + \ell - 2\xi) &= -\alpha'_-(y - 2\xi) + \alpha'_+(y - \ell), \forall y \in (\xi, \xi + 2\ell).\end{aligned}$$

Differentiating the first identity in y , taking the sum and the difference of the two identities, we get

$$\alpha'_-(y) = \alpha'_+(y - \ell), \forall y \in (\xi, \xi + 2\ell), \quad (3.10)$$

$$\alpha'_+(y + \ell - 2\xi) = -\alpha'_-(y - 2\xi), \forall y \in (\xi, \xi + 2\ell). \quad (3.11)$$

By iteration this allows to find α_- (resp. α_+) up to $2\ell + \xi$ (resp. $3\ell - \xi$). Indeed fix $\varepsilon \leq 2 \min\{\xi, \ell - \xi\}$, then in a first step for $y \in (\xi, \xi + \varepsilon)$, we remark that $y - \ell$ belongs to $(\xi - \ell, \xi + \varepsilon - \ell)$ which is included in $(\xi - \ell, \ell - \xi)$ the set where α_+ is defined up to now. This allows to obtain $\alpha'_-(y)$ for all $y \in (\xi, \xi + \varepsilon)$. In the same manner $\alpha'_-(y - 2\xi)$ is well-defined and this allows then to obtain $\alpha'_+(y + \ell - 2\xi)$ for all $y \in (\xi, \xi + \varepsilon)$. We now iterate this argument, namely for $y \in (\xi + \varepsilon, \xi + 2\varepsilon)$, the right-hand sides of (3.10)–(3.11) are meaningful, and consequently we obtain $\alpha'_-(y)$ (resp. $\alpha'_+(y + \ell - 2\xi)$) for such y . We iterate this procedure up to $y \in (\xi + (k - 1)\varepsilon, \xi + k\varepsilon)$, with $k \in \mathbb{N}$ such that

$$\xi + k\varepsilon = \xi + 2\ell.$$

This proves the announced statement.

For $y > \xi + 2\ell$, we need to take into account (3.4) and (3.5), that take the form

$$\alpha_-(\xi + t) - \alpha_-(t - \xi) = \alpha_+(\xi - \ell + t) + \alpha_+(t - \xi + \ell), \forall t > 2\ell,$$

$$\alpha'_-(\xi + t) + \alpha'_-(t - \xi) = \alpha'_+(\xi - \ell + t) - \alpha'_+(t - \xi + \ell) + a(\alpha'_+(\xi + t - 3\ell) + \alpha'_+(t - \xi - \ell)), \forall t > 2\ell.$$

By setting $y = \xi + t$, this is equivalent to

$$\alpha_-(y) - \alpha_+(y - 2\xi + \ell) = \alpha_-(y - 2\xi) + \alpha_+(y - \ell), \forall y > \xi + 2\ell,$$

$$\alpha'_-(y) + \alpha'_+(y + \ell - 2\xi) = -\alpha'_-(y - 2\xi) + \alpha'_+(y - \ell) + a(\alpha'_+(y - 3\ell) + \alpha'_+(y - 2\xi - \ell)), \forall y > \xi + 2\ell.$$

As before differentiating the first equation in y and taking the sum and the difference, we arrive at (compare with (3.10)–(3.11))

$$\alpha'_-(y) = \alpha'_+(y - \ell) + \frac{a}{2}(\alpha'_+(y - 3\ell) + \alpha'_+(y - 2\xi - \ell)), \forall y > \xi + 2\ell, \quad (3.12)$$

$$\alpha'_+(y + \ell - 2\xi) = -\alpha'_-(y - 2\xi) + \frac{a}{2}(\alpha'_+(y - 3\ell) + \alpha'_+(y - 2\xi - \ell)), \forall y > \xi + 2\ell. \quad (3.13)$$

The same iterative argument allows to show that $\alpha_-(y)$ (resp. $\alpha_+(y)$) is uniquely defined for $y > 2\ell + \xi$ (resp. $y > 3\ell - \xi$). Note that this construction based on the D'Alembert formula re-proves the existence result from Proposition 1.1. This construction is only valid in one dimension and for a constant coefficients operator, while the semigroup approach of Proposition 1.1 is valid in a more general setting (see below).

The main point is this last iterative relation between $\alpha'_-(y)$, $\alpha'_+(y + \ell - 2\xi)$ and previous evaluations.

Let us now take $\xi = \frac{\ell}{2}$, then we can equivalently write (3.12)–(3.13) as the following system

$$\begin{pmatrix} \alpha'_-(y) \\ \alpha'_+(y) \\ \alpha'_+(y - \ell) \\ \alpha'_+(y - 2\ell) \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{a}{2} & \frac{a}{2} \\ -1 & 0 & \frac{a}{2} & \frac{a}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha'_-(y - \ell) \\ \alpha'_+(y - \ell) \\ \alpha'_+(y - 2\ell) \\ \alpha'_+(y - 3\ell) \end{pmatrix}. \quad (3.14)$$

As in [12, 13] we are reduced to calculate the eigenvalues of the matrix

$$M_a = \begin{pmatrix} 0 & 1 & \frac{a}{2} & \frac{a}{2} \\ -1 & 0 & \frac{a}{2} & \frac{a}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

whose characteristic polynomial is given by

$$p_a(\lambda) = \lambda^4 + (1 - \frac{a}{2})\lambda^2 + \frac{a}{2}.$$

The zeroes of p_a are given by

$$\lambda^2 = \frac{a - 2 \pm \sqrt{a^2 - 12a + 4}}{4}.$$

Consequently the eigenvalues of M_a are strictly less than 1 in modulus if and only if

$$|a - 2 \pm \sqrt{a^2 - 12a + 4}| < 4. \quad (3.15)$$

In the case $a^2 - 12a + 4 \geq 0$ we see that (3.15) holds if and only if

$$0 < a \leq 6 - 4\sqrt{2}. \quad (3.16)$$

On the contrary in the case $a^2 - 12a + 4 < 0$ we check that (3.15) holds if and only if

$$6 - 4\sqrt{2} \leq a < 2. \quad (3.17)$$

Hence we conclude that (3.15) holds if and only if $a \in (0, 2)$.

Since

$$p'_a(\lambda) = \lambda(4\lambda^2 + 2 - a),$$

we can conclude that for $a \in (0, 2)$, all eigenvalues of M_a are of modulus < 1 and simple. In that case, there exists a matrix V_a such that

$$M_a = V_a^{-1} D_a V_a,$$

where D_a is the diagonal matrix made of the eigenvalues of M_a .

Now coming back to (3.14) and using an inductive argument, we can deduce that for all $j \in \mathbb{N}$, and for all $y \in (\frac{5\ell}{2} + j\ell, \frac{5\ell}{2} + (j+1)\ell]$, we have

$$C(y) = M_a^j C(y - j\ell),$$

where for shortness we have written

$$C(y) := \begin{pmatrix} \alpha'_-(y) \\ \alpha'_+(y) \\ \alpha'_+(y - \ell) \\ \alpha'_+(y - 2\ell) \end{pmatrix}.$$

Therefore using the previous factorization of M_a , we get

$$C(y) = V_a^{-1} D_a^j V_a C(y - j\ell).$$

Finally we find a positive constant C_a (depending only on a) such that for all $j \in \mathbb{N}$, and all $y \in (\frac{5\ell}{2} + j\ell, \frac{5\ell}{2} + (j+1)\ell]$, we have

$$\|C(y)\|_2 \leq C_a \rho_a^j \|C(y - j\ell)\|_2, \quad (3.18)$$

where ρ_a is the spectral radius of D_a that is < 1 (if $a \in (0, 2)$).

By simple calculation we see that

$$E(t) = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} (\alpha'_-(x+t)^2 + \alpha'_+(x+t)^2) dx.$$

Now we closely follow the arguments of [12, 13] to conclude the exponential decay of the system. Namely for all $j \in \mathbb{N}$, and for all $t \in (2\ell + j\ell, 2\ell + (j+2)\ell]$, we can apply (3.18) with $y = x + t$ for any $x \in (-\frac{\ell}{2}, \frac{\ell}{2})$ and consequently

$$\begin{aligned} E(t) &\leq \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \|C(x+t)\|_2^2 dx \\ &\leq C_a^2 \rho_a^{2j} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \|C(x+t-j\ell)\|_2^2 dx. \end{aligned}$$

Finally as for $t \in (2\ell + j\ell, 2\ell + (j+2)\ell]$ and $x \in (-\frac{\ell}{2}, \frac{\ell}{2})$, $x+t-j\ell$ belongs to a compact set, the quantity

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \|C(x+t-j\ell)\|_2^2 dx$$

is bounded independently of j . This means that we have found a constant K_a such that for all $j \in \mathbb{N}$, and all $t \in (2\ell + j\ell, 2\ell + (j+2)\ell]$, one has

$$E(t) \leq K_a \rho_a^{2j}.$$

This leads to the conclusion because $\rho_a^{2j} = e^{2j \ln \rho_a} \leq e^{2t \frac{\ln \rho_a}{\ell}}$.

Now we study problem (1.5)–(1.10) and look for a solution u in the form:

$$u(x, t) = \alpha(x+t) - \alpha(t-x), \forall x \in (0, \ell), \quad t \geq 0. \quad (3.19)$$

Hence we see that (1.6) always holds. In order to fulfil the initial conditions (1.10), we take

$$\begin{aligned} \alpha(x) &= -\frac{1}{2}u_0(-x) + \frac{1}{2} \int_0^{-x} u_1(s) ds \quad \forall x \in (-\ell, 0), \\ \alpha(x) &= \frac{1}{2}u_0(x) + \frac{1}{2} \int_0^x u_1(s) ds \quad \forall x \in [0, \ell]. \end{aligned}$$

To check (1.7) we need that

$$\alpha'(\ell+t) + \alpha'(t-\ell) = 0, \quad \text{for } 0 < t < 2\ell,$$

or equivalently

$$\alpha'(y) = -\alpha'(y-2\ell) \quad \forall y \in (\ell, 3\ell).$$

Since the right-hand side is known we get the existence of α on $(\ell, 3\ell)$.

The condition (1.8) is satisfied if

$$\alpha'(\ell+t) + \alpha'(t-\ell) = \mu_1(\alpha'(\ell+t) - \alpha'(t-\ell)), \quad \text{for } t \in ((2i+1)2\ell, (2i+2)2\ell),$$

that is equivalent to

$$(1 - \mu_1)\alpha'(y) = -(1 + \mu_1)\alpha'(y-2\ell), \quad \forall y \in ((2i+1)2\ell + \ell, (2i+2)2\ell + \ell).$$

Hence for $\mu_1 \neq 1$, we find that

$$\alpha'(y) = \kappa \alpha'(y - 2\ell), \forall y \in ((2i + 1)2\ell + \ell, (2i + 2)2\ell + \ell), \quad (3.20)$$

where $\kappa = \frac{1+\mu_1}{\mu_1-1}$.

In the same manner to check (1.9) we require that

$$\alpha'(\ell + t) + \alpha'(t - \ell) = \mu_2(\alpha'(t - \ell) - \alpha'(t - 3\ell)), \text{ for } t \in ((2i + 2)2\ell, (2i + 3)2\ell),$$

or equivalently

$$\alpha'(y) = (\mu_2 - 1)\alpha'(y - 2\ell) - \mu_2\alpha'(y - 4\ell), \forall y \in ((2i + 2)2\ell + \ell, (2i + 3)2\ell + \ell). \quad (3.21)$$

By recurrence we can show that α is well-defined on the whole $(-\ell, \infty)$.

Now combining (3.20) and (3.21) we see that for $y \in ((2i + 1)2\ell + \ell, (2i + 2)2\ell + \ell)$ with $i \geq 1$ we obtain

$$\alpha'(y) = \kappa \alpha'(y - 2\ell) = \kappa((\mu_2 - 1)\alpha'(y - 4\ell) - \mu_2\alpha'(y - 6\ell)). \quad (3.22)$$

For $y > 7\ell$ we can define the vector

$$U(y) := \begin{pmatrix} \alpha'(y) \\ \alpha'(y - 2\ell) \end{pmatrix}$$

and then, from (3.21) and (3.22) we deduce

$$U(y) = MU(y - 4\ell),$$

where M is the matrix

$$M = \begin{pmatrix} \kappa(\mu_2 - 1) & -\kappa\mu_2 \\ (\mu_2 - 1) & -\mu_2 \end{pmatrix}.$$

The eigenvalues of M are $\lambda_1 = 0$ and $\lambda_2 = \kappa(\mu_2 - 1) - \mu_2$. Therefore, exponential stability holds if

$$|\kappa(\mu_2 - 1) - \mu_2| < 1. \quad (3.23)$$

Indeed the energy E_b of our system defined by

$$E_b(t) = \frac{1}{2} \int_0^\ell (u_t(x, t)^2 + u_x(x, t)^2) dx$$

is here equal to

$$E_b(t) = \int_{-\ell}^\ell \alpha'(x + t)^2 dx.$$

Hence, the previous arguments show that the energy is exponentially decaying if condition (3.23) is satisfied.

Finally by distinguishing the case $\mu_1 > 1$ to the case $\mu_1 < 1$, we easily check that (3.23) is equivalent to (1.15).

4 The multidimensional case

We study the following internal stabilization problem of a switching delay wave equation in $\Omega \subset \mathbb{R}^d$, $d \geq 1$. For given times $T^* > 0$ and $\tau \in (0, T^*]$, consider the problem

$$u_{tt}(x, t) - \Delta u(x, t) + b_1 u_t(x, t) = 0 \quad \text{in } \Omega \times (i(T^* + \tau), i(T^* + \tau) + T^*), \quad (4.1)$$

$$u_{tt}(x, t) - \Delta u(x, t) + b_2 u_t(x, t - \tau) = 0 \quad \text{in } \Omega \times (i(T^* + \tau) + T^*, (i+1)(T^* + \tau)), \quad (4.2)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty) \quad (4.3)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (4.4)$$

where $i \in \mathbb{N}$, $b_1 > 0$ and b_2 is a real number.

Note that in the interval $(0, T^*)$ the damping is a standard one, in the sense that it induces an exponential decay of the energy. Hence by standard technique (see e.g. [19, 3]), if T^* is fixed such that the observability estimate in Ω is valid, there exists $\alpha \in (0, 1)$ such that

$$E(T^*) \leq \alpha E(0), \quad (4.5)$$

where $E(t)$ is the standard energy, $E(t) := \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2) dx$.

Now for $t \in (T^*, T^* + \tau)$, by integration by parts, we see that

$$E'(t) = - \int_{\Omega} b_2 u_t(x, t) u_t(x, t - \tau) dx.$$

Hence by Cauchy-Schwarz's inequality we find that

$$E'(t) \leq 2|b_2| E(t)^{1/2} E(t - \tau)^{1/2}.$$

Since $t - \tau$ belongs to $(0, T^*)$ and since the energy is decaying on this interval, by (4.5), we find that

$$E'(t) \leq 2\sqrt{\alpha}|b_2| E(0)^{1/2} E(t)^{1/2}.$$

This can be equivalently written as

$$\frac{d}{dt} E(t)^{1/2} \leq \sqrt{\alpha}|b_2| E(0)^{1/2},$$

and integrating this estimate between T^* and $t \in (T^*, T^* + \tau)$, we obtain

$$E(t)^{1/2} - E(T^*)^{1/2} \leq \sqrt{\alpha}|b_2| E(0)^{1/2} (t - T^*) \leq \sqrt{\alpha}|b_2| E(0)^{1/2} \tau.$$

Using again (4.5), we arrive at

$$E(t)^{1/2} \leq \sqrt{\alpha}(1 + |b_2|\tau) E(0)^{1/2}.$$

As a consequence if the factor $\tilde{\alpha}^{1/2} := \sqrt{\alpha}(1 + |b_2|\tau)$ is strictly less than 1, then we will get a property like (4.5) but in the interval $(0, T^* + \tau)$, namely

$$E(t) \leq \tilde{\alpha} E(0), \quad \forall t \in (0, T^* + \tau). \quad (4.6)$$

Note that the condition $\tilde{\alpha}^{1/2} < 1$ is equivalent to

$$|b_2| < \frac{1 - \sqrt{\alpha}}{\sqrt{\alpha\tau}}, \quad (4.7)$$

that means that b_2 has to be small enough. Since our system is invariant by a translation of $T^* + \tau$, this argument may be repeated between $i(T^* + \tau)$ and $(i + 1)(T^* + \tau)$, and therefore we find that

$$E(t) \leq \tilde{\alpha}^{i+1} E(0), \forall t \in (i(T^* + \tau), (i + 1)(T^* + \tau)).$$

Writing $\tilde{\alpha}^{i+1} = e^{(i+1)(T^* + \tau) \frac{\log \tilde{\alpha}}{(T^* + \tau)}}$ and using the fact that $\frac{\log \tilde{\alpha}}{(T^* + \tau)} < 0$, we arrive at

$$E(t) \leq e^{t \frac{\log \tilde{\alpha}}{(T^* + \tau)}} E(0), \forall t \in (i(T^* + \tau), (i + 1)(T^* + \tau)), \quad (4.8)$$

which proves the exponential decay of the energy. In conclusion we have proved the next result.

Theorem 4.1. *Assume that T^* is the minimal time of observability for the wave equation with internal damping, that $\tau \in (0, T^*]$ and that (4.7) holds. Then the energy of the system (4.1) – (4.4) decays exponentially to zero.*

Remark 4.2. 1. Our arguments also hold if we replace the internal damping in $\Omega \times (2iT^*, (2i + 1)T^*)$ by a boundary damping. Similarly the global internal damping can be replaced by a local one, as far as the exponential decay is guaranteed. Obviously in both cases, the time T^* of observability has to be changed. The converse situation, namely keep internal damping in $\Omega \times (2iT^*, (2i + 1)T^*)$ and take a boundary damping with delay in $\Omega \times ((2i + 1)T^*, 2(i + 1)T^*)$ is more delicate because we are not able to prove that

$$\int_{\partial\Omega} b_2 u_t(x, t) u_t(x, t - \tau) dx \leq 2|b_2| E(t)^{1/2} E(t - \tau)^{1/2}.$$

Hence another argument should be found.

2. Instead of taking a constant coefficient b_2 we can also take $b_2 \in L^\infty(\Omega)$. In this case the condition (4.7) has to be replaced by

$$\sup_{\Omega} |b_2| < \frac{1 - \sqrt{\alpha}}{\sqrt{\alpha\tau}}.$$

5 Comments and related questions

1. The statement of Theorem 1.2 concerning problem (1.1)–(1.4) remains valid in the case $\frac{\xi}{\ell} = \frac{p}{q}$ with $p, q \in \mathbb{N}$ with p odd and q even. We did not give its proof since it is too technical and do not bring any new ideas. We have chosen $\xi = \frac{\ell}{2}$ because this is the best location for the decay rate in the absence of delay.

2. In the same manner we can obtain the same result as Theorem 1.2 for the following problem:

$$u_{tt}(x, t) - u_{xx}(x, t) = \mu_1 u_t(\xi, t) \delta_\xi, \quad \text{in } (0, \ell) \times (2i\ell, 2(i+1)\ell), \forall i \in \mathbb{N}, \quad (5.1)$$

$$u_{tt}(x, t) - u_{xx}(x, t) + a u_t(\xi, t - 2\ell) \delta_\xi = 0, \\ \text{in } (0, \ell) \times (2(i+1)\ell, 2(i+2)\ell), \forall i \in \mathbb{N}, \quad (5.2)$$

$$u(0, t) = 0, \quad u_x(\ell, t) = 0, \quad \text{on } (0, +\infty), \quad (5.3)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } (0, \ell). \quad (5.4)$$

3. Let H be a Hilbert space equipped with the norm $\|\cdot\|_H$, and let $A : \mathcal{D}(A) \rightarrow H$ be a self-adjoint, positive and invertible operator. We introduce the scale of Hilbert spaces H_α , $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A^\alpha)$ with the norm $\|z\|_\alpha = \|A^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. The operator A can be extended (or restricted) to each H_α such that it becomes a bounded operator

$$A : H_\alpha \rightarrow H_{\alpha-1} \quad \text{for } \alpha \in \mathbb{R}. \quad (5.5)$$

The second ingredient needed for our construction is a bounded linear operator $B : U \rightarrow H_{-\frac{1}{2}}$, where U is another Hilbert space identified with its dual. The operator B^* is bounded from $H_{\frac{1}{2}}$ to U .

The systems that we considered in this paper enter in one of the following abstract problems:

$$\ddot{w}(t) + Aw(t) = 0, \quad 0 \leq t \leq T_0, \quad (5.6)$$

$$\ddot{w}(t) + Aw(t) + \mu BB^* \dot{w}(t - T_0) = 0, \quad t \geq T_0, \quad (5.7)$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1, \quad (5.8)$$

or

$$\ddot{w}(t) + Aw(t) = 0, \quad 0 \leq t \leq T_0, \quad (5.9)$$

$$\ddot{w}(t) + Aw(t) + \mu_1 BB^* \dot{w}(t) = 0, \quad (2i+1)T_0 \leq t \leq (2i+2)T_0, \quad \forall i \in \mathbb{N}, \quad (5.10)$$

$$\ddot{w}(t) + Aw(t) + \mu_2 BB^* \dot{w}(t - T_0) = 0, \quad (2i+2)T_0 \leq t \leq (2i+3)T_0, \quad \forall i \in \mathbb{N}, \quad (5.11)$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1, \quad (5.12)$$

where $T_0 > 0$ is the time delay, μ, μ_1, μ_2 are real numbers and the initial datum (w_0, w_1) belongs to a suitable space.

Assume that there exist $T \geq T_0, C > 0$ such that

$$\int_0^T \|B^* \phi'(s)\|_U^2 ds \leq C \|(w_0, w_1)\|_{H_{\frac{1}{2}} \times H}^2 \quad (5.13)$$

for $(w_0, w_1) \in H_1 \times H_{\frac{1}{2}}$ and ϕ is the solution of the undamped evolution equation

$$\ddot{\phi}(t) + A\phi(t) = 0, \quad t \geq 0, \quad (5.14)$$

$$\phi(0) = w_0, \dot{\phi}(0) = w_1.$$

To study the well-posedness of the system (5.6)–(5.8), we write it as an abstract Cauchy problem in a product Banach space, and use the semigroup approach. For this take the Hilbert space $\mathcal{H} := H_{\frac{1}{2}} \times H$ and the unbounded linear operators

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) = H_1 \times H_{\frac{1}{2}} \subset \mathcal{H} \longrightarrow \mathcal{H}, \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -Au_1 \end{pmatrix} \quad (5.15)$$

and

$$\begin{aligned} \mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) = \left\{ (u, v) \in \mathcal{H}; v \in H_{\frac{1}{2}}, Au + \mu_1 BB^*v \in H \right\} \subset \mathcal{H} &\longrightarrow \mathcal{H}, \\ \mathcal{A}_d \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -Au_1 - \mu_1 BB^*u_2 \end{pmatrix}. & \quad (5.16) \end{aligned}$$

The operators $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and $(\mathcal{A}_d, \mathcal{D}(\mathcal{A}_d))$ defined by (5.16) generate a strongly continuous semigroup of contractions on \mathcal{H} denoted respectively by $(\mathcal{T}(t))_{t \geq 0}$ and $(\mathcal{T}_d(t))_{t \geq 0}$ (as before let $(\mathcal{T}_{-1}(t))_{t \geq 0}$ be the extension of $(\mathcal{T}(t))_{t \geq 0}$ to H_{-1}).

Proposition 5.1. (a) *Assume that the inequality (5.13) holds. Then the system (5.6)–(5.8) is well-posed. More precisely, for every $(u_0, u_1) \in \mathcal{H}$, the solution of (5.6)–(5.8) is given by*

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} u^0(t) \\ \dot{u}^0(t) \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, & 0 \leq t \leq T_0, \\ \begin{pmatrix} u^j(t) \\ \dot{u}^j(t) \end{pmatrix} = \mathcal{T}(t - jT_0) \begin{pmatrix} u^{j-1}(jT_0) \\ \dot{u}^{j-1}(jT_0) \end{pmatrix} + \\ \int_{jT_0}^t \mathcal{T}_{-1}(t-s) \begin{pmatrix} 0 \\ -\mu BB^* \dot{u}^{j-1}(s) \end{pmatrix} ds, & jT_0 \leq t \leq (j+1)T_0, j \geq 1. \end{cases}$$

and satisfies $(u^j, \dot{u}^j) \in C([jT_0, (j+1)T_0], \mathcal{H})$, $j \in \mathbb{N}$.

(b) *Assume that the inequality (5.13) holds. Then, the system (5.17)–(5.22) is well-posed. More precisely, for every $(u_0, u_1) \in \mathcal{H}$, the solution of (5.17)–(5.22) is given by*

$$\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} u^0(t) \\ u_t^0(t) \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, & 0 \leq t \leq T_0, \\ \begin{pmatrix} u^{2j+1}(t) \\ u_t^{2j+1}(t) \end{pmatrix} = \mathcal{T}_d(t - (2j+1)T_0) \begin{pmatrix} u^{2j}((2j+1)T_0) \\ u_t^{2j}((2j+1)T_0) \end{pmatrix}, & (2j+1)T_0 \leq t \leq (2j+2)T_0, j \in \mathbb{N}, \\ \begin{pmatrix} u^{2j+2}(t) \\ u_t^{2j+2}(t) \end{pmatrix} = \mathcal{T}(t - (2j+2)T_0) \begin{pmatrix} u^{2j+1}((2j+2)T_0) \\ u_t^{2j+1}((2j+2)T_0) \end{pmatrix} + \\ \int_{2(2j+2)T_0}^t \mathcal{T}_{-1}(t-s) \begin{pmatrix} 0 \\ -\mu_2 BB^* u_t^{2j+1}(s - 2\ell)\delta_\ell \end{pmatrix} ds, & (2j+2)T_0 \leq t \leq (2j+3)T_0, j \in \mathbb{N} \end{cases}$$

and satisfies

$$(u^0, u_t^0) \in C([0, 2\ell], \mathcal{H}), (u^{2j+1}, u_t^{2j+1}) \in C([2(2j+1)\ell, 2(2j+2)\ell], \mathcal{H}), j \in \mathbb{N},$$

$$(u^{2j+2}, u_t^{2j+2}) \in C([2(2j+2)\ell, 2(2j+3)\ell], \mathcal{H}), j \in \mathbb{N}.$$

We now give two multi-dimensional illustrations of this setting. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary Γ . We assume that Γ is divided into two parts Γ_0 and Γ_1 , i.e. $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and $meas \Gamma_1 \neq 0$ (and satisfied some Lions geometric condition or some geometric control condition, see [6] and [5] for more details). Note that the condition $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ is only made in order to simplify the presentation, hence our analysis can be performed without this assumption in a similar manner.

We further fix a time interval $T_0 > 0$ and a delay $\tau > 0$. In this domain Ω we consider the initial boundary value problems with switching boundary conditions:

$$u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (5.17)$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (5.18)$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T_0), \quad (5.19)$$

$$u(x, t) = \mu_1 \frac{\partial G(u_t)}{\partial n}(x, t) \quad \text{on } \Gamma_1 \times ((2i+1)T_0, (2i+2)T_0), \quad (5.20)$$

$$u(x, t) = \mu_2 \frac{\partial G(u_t)}{\partial n}(x, t - \tau) \quad \text{on } \Gamma_1 \times ((2i+2)T_0, (2i+3)T_0), \quad (5.21)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (5.22)$$

and

$$u_{tt} + \Delta^2 u = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (5.23)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (5.24)$$

$$\Delta u = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (5.25)$$

$$\Delta u = 0 \quad \text{on } \Gamma_1 \times (0, T_0), \quad (5.26)$$

$$\Delta u(x, t) = -\mu_1 \frac{\partial G(u_t)}{\partial n}(x, t) \quad \text{on } \Gamma_1 \times ((2i+1)T_0, (2i+2)T_0), \quad (5.27)$$

$$\Delta u(x, t) = -\mu_2 \frac{\partial G(u_t)}{\partial n}(x, t - \tau) \quad \text{on } \Gamma_1 \times ((2i+2)T_0, (2i+3)T_0), \quad (5.28)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (5.29)$$

where $G = (-\Delta)^{-1} : H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$, $i \in \mathbb{N}$, μ_1 and μ_2 are real parameters.

Note that the above systems are exponentially stable in absence of time delay, that is if $\tau = 0$ and if $\mu_1 = \mu_2 > 0$.

According to [6] and [5] the inequality (5.13) is satisfied for some $T_0 > 0$ and Proposition 5.1 implies that the wave system (5.17)–(5.22) and the plate system (5.23)–(5.29) admit a finite energy solution.

For $\tau = T_0$, where T_0 is fixed such that the observability estimate in Γ_1 is valid, by the same method used in Theorem 4.1, we can prove an exponential stability result for both systems.

References

- [1] E. M. AIT BEN HASSI, K. AMMARI, S. BOULITE AND L. MANIAR, *Feedback stabilization of a class of evolution equations with delay*, J. Evol. Equ., **1** (2009), 103-121.
- [2] K. AMMARI, S. NICAISE AND C. PIGNOTTI, *Feedback boundary stabilization of wave equations with interior delay*, Systems Control Lett., **59** (2010), 623–628.
- [3] K. AMMARI AND M. TUCSNAK, *Stabilization of second order evolution equations by a class of unbounded feedbacks*, ESAIM Control Optim. Calc. Var., **6** (2001), 361-386.
- [4] K. AMMARI, A. HENROT AND M. TUCSNAK, *Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string*, Asymptot. Anal., **28** (2001), 215-240.
- [5] K. AMMARI, G. TENENBAUM AND M. TUCSNAK, *A sharp geometric condition for the boundary exponential stabilizability of a square plate by moment feedbacks only*, International series of numerical Mathematics, Vol **155**, 1-11, Birkhäuser Verlag Basel/Switzerland, 2007.
- [6] K. AMMARI, *Dirichlet boundary stabilization of the wave equation*, Asymptot. Anal. **30** (2002), 117-130.
- [7] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR AND S. K. MITTER, *Representation and control of infinite Dimensional Systems*. Vol I, Birkhäuser, 1992.
- [8] G. CHEN, *Control and stabilization for the wave equation in a bounded domain I-II*, SIAM J. Control Optim., **17** (1979), 66–81; **19** (1981), 114–122.
- [9] R. DATKO, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks*, SIAM J. Control Optim., **26** (1988), 697-713.
- [10] R. DATKO, J. LAGNESE AND P. POLIS, *An exemple on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim., **24** (1985), 152-156.
- [11] R. DATKO, *Two examples of ill-posedness with respect to time delays revisited*, IEEE Trans. Automatic Control, **42** (1997), 511–515.
- [12] M. GUGAT, *Boundary feedback stabilization by time delay for one-dimensional wave equations*, IMA Journal of Mathematical Control and Information, **27** (2010), 189-203.
- [13] M. GUGAT AND M. TUCSNAK, *An example for the switching delay feedback stabilization of an infinite dimensional system: The boundary stabilization of a string*, System Control Lett., **60** (2011), 226-233.
- [14] J. L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications*. Vol 1, Dunod, Paris, 1968.

- [15] S. NICAISE AND C. PIGNOTTI, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim., **45** (2006), 1561–1585.
- [16] S. NICAISE AND J. VALEIN, *Stabilization of second order evolution equations with unbounded feedback with delay*, ESAIM Control Optim. Calc. Var., **16** (2010), 420–456.
- [17] M. TUCSNAK AND G. WEISS, *Observation and control for operator semigroups*. Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2009.
- [18] J. M. WANG, B. Z. GUO AND M. KRSTIC, *Wave equation stabilization by delays equal to even multiples of the wave propagation time*, SIAM J. Control Optim., **49** (2011), 517-554.
- [19] E. ZUAZUA, *Exponential decay for the semilinear wave equation with locally distributed damping*, Comm. Partial Differential Equations, **15** (1990), 205–235.