Exponential stability of the wave equation with boundary time-varying delay

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Abstract

We consider the wave equation with a time-varying delay term in the boundary condition in a bounded and smooth domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions, we prove exponential stability of the solution. These results are obtained by introducing suitable energies and suitable Lyapunov functionals. Such analysis is also extended to a nonlinear version of the model.

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1 Introduction

We are interested in the effect of a time-varying delay in boundary stabilization of the wave equation in domains of $\mathbb{R}^n$. Delay effects arise in many practical problems and it is well known that they can induce some unstabilities, see [5, 6, 7, 25, 30].

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a boundary $\Gamma$ of class $C^2$. We assume that $\Gamma$ is divided into two parts $\Gamma_D$ and $\Gamma_N$, i.e. $\Gamma = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\Gamma_D \neq \emptyset$.

In this domain $\Omega$, we consider the initial boundary value problem

\begin{align*}
    u_{tt}(x, t) - \Delta u(x, t) &= 0 \quad \text{in } \Omega \times (0, +\infty) \tag{1.1} \\
    u(x, t) &= 0 \quad \text{on } \Gamma_D \times (0, +\infty) \tag{1.2} \\
    \frac{\partial u}{\partial \nu}(x, t) &= -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau(t)) \quad \text{on } \Gamma_N \times (0, +\infty) \tag{1.3} \\
    u(x, 0) &= u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega \tag{1.4} \\
    u_t(x, t - \tau(0)) &= f_0(x, t - \tau(0)) \quad \text{in } \Gamma_N \times (0, \tau(0)), \tag{1.5}
\end{align*}

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau(t) > 0$ is the time-varying delay, $\mu_1$ and $\mu_2$ are positive real numbers and the initial datum $(u_0, u_1, f_0)$ belongs to a suitable space.

On the function $\tau$ we assume that there exists a positive constant $\tau$ such that

$$0 \leq \tau(t) \leq \tau, \quad \forall \ t > 0.$$

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Moreover, we assume
\[ \tau'(t) < 1 \quad \forall \ t > 0, \] (1.7)
and
\[ \tau \in W^{2,\infty}([0, T]), \quad \forall \ T > 0. \] (1.8)

We are interested in giving an exponential stability result for such a problem.

Let us denote by \( \langle v, w \rangle \) or, equivalently, by \( v \cdot w \) the euclidean inner product between two vectors \( v, w \in \mathbb{R}^n \).

We assume that there exists \( x_0 \in \mathbb{R}^n \) such that denoting by \( m \) the standard multiplier
\[ m(x) := x - x_0, \]
we have
\[ m(x) \cdot \nu(x) \leq 0 \quad \text{on} \quad \Gamma_D \]
(1.9)
and, for some positive constant \( \delta \),
\[ m(x) \cdot \nu(x) \geq \delta \quad \text{on} \quad \Gamma_N. \] (1.10)

It is well–known that if \( \mu_2 = 0 \), that is in absence of delay, the energy of problem (1.1) – (1.5) is exponentially decaying to zero. See for instance Chen [3], Lagnese [16, 17], Lasiecka and Triggiani [18], Komornik and Zuazua [15], Komornik [13, 14]. On the contrary, if \( \mu_1 = 0 \), that is if we have only the delay part in the boundary condition on \( \Gamma_N \), system (1.1) – (1.5) becomes unstable. See, for instance Datko, Lagnese and Polis [7].

The above problem, with both \( \mu_1, \mu_2 > 0 \) and a constant delay \( \tau \), has been studied in one space dimension by Xu, Yung and Li [30] and on networks by Nicaise and Valein [26] and in higher space dimension by Nicaise and Pignotti [25]. Assuming that
\[ \mu_2 < \mu_1 \] (1.11)
in [25], a stabilization result in general space dimension is given, by using a suitable observability estimate. This is done by applying inequalities obtained from Carleman estimates for the wave equation by Lasiecka, Triggiani and Yao in [19] and by using compactness-uniqueness arguments.

The case of time–varying delay has been studied by Nicaise, Valein and Fridman [27] in one space dimension. In [27] an exponential stability result is given, under the condition
\[ \mu_2 < \sqrt{1 - d} \mu_1 \] (1.12)
where \( d \) is a constant such that
\[ \tau'(t) \leq d < 1, \quad \forall \ t > 0. \] (1.13)

Here, we extend this result to general space dimension. Moreover, we remove the hypothesis
\[ \tau(t) \geq \tau_0 > 0, \quad \forall \ t > 0, \] (1.14)
assumed in [27], that is the delay may degenerate.

We will study also a nonlinear version of the above model. Consider the system
\[ u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \] (1.15)
\[ u(x, t) = 0 \quad \text{on} \quad \Gamma_D \times (0, +\infty) \] (1.16)
\[ \frac{\partial u}{\partial \nu}(x, t) = -\beta_1(u_t(x, t)) - \beta_2(u_t(x, t - \tau(t))) \quad \text{on} \quad \Gamma_N \times (0, +\infty) \] (1.17)
\[ u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \] (1.18)
\[ u_t(x, t - \tau(0)) = g_0(x, t - \tau(0)) \quad \text{in} \quad \Gamma_N \times (0, \tau(0)), \] (1.19)
where \( \beta_j : \mathbb{R} \to \mathbb{R}, j = 1, 2 \), satisfy suitable growth assumptions. In particular we assume
\[ |\beta_j(s)| \leq c_j |s|, \quad \forall s \in \mathbb{R}, \quad j = 1, 2, \] (1.20)
for some positive constants $c_1, c_2$ and
\[ \beta_2(s) \cdot s \geq 0, \quad \forall s \in \mathbb{R}. \] (1.21)

Moreover we assume
\[ \exists \gamma_1 > 0, \forall x, y \in \mathbb{R}, (\beta_1(x) - \beta_1(y))(x - y) \geq \gamma_1(x - y)^2, \] (1.22)
and
\[ \exists \gamma_2 > 0, \forall x, y \in \mathbb{R}, |\beta_2(x) - \beta_2(y)| \leq \gamma_2|x - y|. \] (1.23)

Note that (1.20) and (1.22) imply \( c_2 \leq \gamma_2 \) and from (1.20) and (1.22) we deduce
\[ \beta_1(s) \cdot s \geq \gamma_1 s^2, \quad \forall s \in \mathbb{R}. \] (1.24)

Under a suitable relation between the above coefficients we can give a well–posedness result and an exponential stability estimate for problem (1.15) – (1.19). To prove the well–posedness of the nonlinear model we need to assume (1.14). In our opinion, this is only a technical assumption but at the moment we are not able to remove it.

The paper is organized as follows. Well–posedness of the problems is analysed in section 2 using semigroup theory. In subsection 2.1 we study the well-posedness of problem (1.1) – (1.5), while in subsection 2.2 we concentrate on problem (1.15) – (1.19). In section 3 and section 4 we prove the exponential stability of the linear and nonlinear problems respectively.

2 Well-posedness of the problems

Using semigroup theory we can give the well–posedness of problem (1.1) – (1.5) and problem (1.15) – (1.19).

2.1 Linear problem

Let us set
\[ z(x, \rho, t) = u_t(x, t - \tau(t) \rho), \quad x \in \Gamma_N, \; \rho \in (0, 1), \; t > 0. \] (2.1)
Then, problem (1.1) – (1.5) is equivalent to
\[ u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \] (2.2)
\[ \tau(t) z_t(x, \rho, t) + (1 - \tau'(t) \rho) z_p(x, \rho, t) = 0 \quad \text{in} \quad \Gamma_N \times (0, 1) \times (0, +\infty) \] (2.3)
\[ u(x, t) = 0 \quad \text{on} \quad \Gamma_D \times (0, +\infty) \] (2.4)
\[ \frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_1(x, t) - \mu_2 z(x, 1, t) \quad \text{on} \quad \Gamma_N \times (0, +\infty) \] (2.5)
\[ z(x, 0, t) = u_0(x, t) \quad \text{on} \quad \Gamma_N \times (0, \infty) \] (2.6)
\[ u(x, 0) = u_0(x, t) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \] (2.7)
\[ z(x, \rho, 0) = f_0(x, -\rho \tau(0)) \quad \text{in} \quad \Gamma_N \times (0, 1). \] (2.8)

To prove the well-posedness of (2.2) – (2.8) we have to distinguish two cases. First, we assume also (1.14), i.e. we assume
\[ 0 < \tau_0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t > 0. \] (2.9)
In the second case we assume only (1.6).
2.1.1 First case
Assume for the moment that (2.9) holds.

If we denote by

$$U := (u, u_t, z)^T,$$

then

$$U' = (u_t, u_{tt}, z_t)^T = \left( u_t, \Delta u, \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \right)^T.$$ 

Therefore, problem (2.2) - (2.8) can be rewritten as

$$\begin{cases}
U' = \mathcal{A}(t)U \\
U(0) = (u_0, u_1, f_0(\cdot, - \cdot \tau(0)))^T
\end{cases}$$

(2.10)

where the operator \( \mathcal{A}(t) \) is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} \frac{\Delta u}{\tau'(t)\rho - 1} z_\rho \\ \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \left\{ (u, v, z)^T \in (E(\Delta, \mathcal{L}^2(\Omega)) \cap V) \times V \times \mathcal{L}^2(\Gamma_N; \mathcal{H}^1(0, 1)) : \right.$$  

$$\left. \frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1) \text{ on } \Gamma_N; \ v = z(\cdot, 0) \text{ on } \Gamma_N \right\},$$

(2.11)

where, as usual,

$$V = \mathcal{H}^1_\Gamma(\Omega) = \{ u \in \mathcal{H}^1(\Omega) : u = 0 \ \text{ on } \Gamma_D \},$$

and

$$E(\Delta, \mathcal{L}^2(\Omega)) = \{ u \in \mathcal{H}^1(\Omega) : \Delta u \in \mathcal{L}^2(\Omega). \}.$$ 

Notice that the domain of the operator \( \mathcal{A}(t) \) is independent of the time \( t \), i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \ \forall t > 0.$$ 

(2.12)

Recall that for a function \( u \in E(\Delta, \mathcal{L}^2(\Omega)) \), then \( \frac{\partial u}{\partial \nu} \) belongs to \( H^{-1/2}(\Gamma_N) \) and the next Green formula is valid (see section 1.5 of [9])

$$\int_\Omega \nabla u \nabla w dx = -\int_\Omega \Delta uw dx + \langle \frac{\partial u}{\partial \nu}; w \rangle_{\Gamma_N}, \forall w \in \mathcal{H}^1_\Gamma(\Omega),$$

(2.13)

where \( \langle \cdot ; \cdot \rangle_{\Gamma_N} \) means the duality pairing between \( H^{-1/2}(\Gamma_N) \) and \( H^{1/2}(\Gamma_N). \)

Note further that for \( (u, v, z)^T \in \mathcal{D}(\mathcal{A}(t)), \ \frac{\partial u}{\partial \nu} \) belongs to \( \mathcal{L}^2(\Gamma_N) \), since \( z(\cdot, 1) \) is in \( \mathcal{L}^2(\Gamma_N). \)

Denote by \( \mathcal{H} \) the Hilbert space

$$\mathcal{H} := V \times \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Gamma_N \times (0, 1))$$

(2.14)

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_\mathcal{H} = \int_\Omega \{ \nabla u(x) \nabla \tilde{u}(x) + v(x) \tilde{v}(x) \} dx + \int_{\Gamma_N} \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho d\Gamma.$$ 

(2.15)

A general theory for equations of type (2.10) has been developed using semigroup theory [11, 12, 28]. The simplest way to prove existence and uniqueness results is to show that the triplet \( \{ \mathcal{A}, \mathcal{H}, Y \} \), with \( \mathcal{A} = \{ \mathcal{A}(t) : t \in [0, T] \} \) for some fixed \( T > 0 \) and \( Y = \mathcal{D}(\mathcal{A}(0)) \), forms a CD-system (or constant domain system, see [11, 12]). More precisely, the following theorem gives some existence and uniqueness results and is proved in Theorem 1.9 of [11] (see also Theorem 2.13 of [12] or [1]).
Theorem 2.1  Assume that

(i) \( Y = D(A(0)) \) is a dense subset of \( \mathcal{H} \),
(ii) (2.12) holds,
(iii) for all \( t \in [0, T] \), \( A(t) \) generates a strongly continuous semigroup on \( \mathcal{H} \) and the family \( \mathcal{A} = \{ A(t) : t \in [0, T] \} \) is stable with stability constants \( C \) and \( m \) independent of \( t \) (i.e., the semigroup \( (S_t(s))_{s \geq 0} \) generated by \( A(t) \) satisfies \( \| S_t(s)u \|_{\mathcal{H}} \leq Ce^{m|s|} \| u \|_{\mathcal{H}} \) for all \( u \in \mathcal{H} \) and \( s \geq 0 \)),
(iv) \( \partial \mathcal{A} \) belongs to \( L^\infty([0, T], B(Y, \mathcal{H})) \), the space of equivalent classes of essentially bounded, strongly measurable functions from \([0, T]\) into the set \( B(Y, \mathcal{H}) \) of bounded operators from \( Y \) into \( \mathcal{H} \).

Then, problem (2.10) has a unique solution \( U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H}) \) for any initial datum in \( Y \).

Our goal is then to check the above assumptions for problem (2.10).

Lemma 2.2  \( D(A(0)) \) is dense in \( \mathcal{H} \).

Proof. The proof is the same as the one of Lemma 2.1 of [27], we give it for the sake of completeness. Let \((f, g, h) \in \mathcal{H}\) be orthogonal to all elements of \( D(A(0)) \), namely

\[
0 = \left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega \{\nabla u(x)\nabla f(x) + v(x)g(x)\} dx + \int_{\Gamma_N} \int_0^1 z(x, \rho)h(x, \rho)d\rho d\Gamma,
\]

for all \((u, v, z) \in D(A(0))\).

We first take \( u = 0 \) and \( v = 0 \) and \( z \in D(\Gamma_N \times (0, 1)) \). As \((0, 0, 0) \in D(A(0))\), we get

\[
\int_{\Gamma_N} \int_0^1 z(x, \rho)h(x, \rho)d\rho d\Gamma = 0.
\]

Since \( D(\Gamma_N \times (0, 1)) \) is dense in \( L^2(\Gamma_N \times (0, 1)) \), we deduce that \( h = 0 \).

In the same manner, by taking \( u = 0 \), \( z = 0 \) and \( v \in D(\Omega) \) we see that \( g = 0 \).

The above orthogonality condition is then reduced to

\[
0 = \int_\Omega \nabla u \nabla f dx, \forall (u, v, z) \in D(A(0)).
\]

By restricting ourselves to \( v = 0 \) and \( z = 0 \), we obtain

\[
\int_\Omega \nabla u(x)\nabla f(x) dx = 0, \forall (u, 0, 0) \in D(A(0)).
\]

But we easily check that \((u, 0, 0) \in D(A(0))\) if and only if \( u \in D(\Delta) \) \( = \{ v \in E(\Delta, L^2(\Omega)) \cap V : \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N \} \), the domain of the Laplace operator with mixed boundary conditions. Since it is well known that \( D(\Delta) \) is dense in \( V \) (equipped with the inner product \( \langle ., . \rangle_V \)), we conclude that \( f = 0 \). ■

Assuming

\[
\mu_2 \leq \sqrt{1 - d} \mu_1,
\]

we will show that \( \mathcal{A}(t) \) generates a \( C_0 \) semigroup on \( \mathcal{H} \) and using the variable norm technique of Kato from [11] and Theorem 2.1, that problem (2.10) has a unique solution.

Let \( \xi \) be a positive real number such that

\[
\frac{\mu_2}{\sqrt{1 - d}} \leq \xi \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1 - d}}.
\]
Let us define on the Hilbert space $\mathcal{H}$ the following time-dependent inner product
\[
\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t := \int_\Omega \{ \nabla u(x) \nabla \tilde{u}(x) + v(x) \tilde{v}(x) \} dx + \xi \tau(t) \int_{\Gamma_N} \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho d\Gamma. \tag{2.18}
\]

Using this time-dependent inner product and Theorem 2.1 we obtain the following existence and uniqueness result:

**Theorem 2.3** For any initial datum $U_0 \in \mathcal{D}(A(0))$ there exists a unique solution
\[
U \in C([0, +\infty), \mathcal{D}(A(0))) \cap C^1([0, +\infty), \mathcal{H})
\]
of system (2.10).

**Proof.** We first notice that
\[
\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2} |t-s|}, \quad \forall t, s \in [0, T],
\tag{2.19}
\]
where $\phi = (u, v, z)^T$ and $c$ is a positive constant. Indeed, for all $s, t \in [0, T]$, we have
\[
\|\phi\|^2_t - \|\phi\|^2_s \leq \int_\Omega (|\nabla u(x)|^2 + v^2) dx + \xi \int_{\Gamma_N} \int_0^1 z(x, \rho)^2 d\rho d\Gamma.
\]

We notice that $1 - e^{\frac{c}{2} |t-s|} \leq 0$. Moreover $\tau(t) - \tau(s)e^{\frac{c}{2} |t-s|} \leq 0$ for some $c > 0$. Indeed,
\[
\tau(t) = \tau(s) + \tau'(a)(t-s), \quad \text{where } a \in (s, t),
\]
and thus,
\[
\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0} |t-s| \leq e^{\frac{c}{\tau_0} |t-s|},
\]
by (2.9), which proves (2.19).

Now we calculate $\langle A(t)U, U \rangle_t$ for a fixed $t$. Take $U = (u, v, z)^T \in \mathcal{D}(A(t))$. Then,
\[
\langle A(t)U, U \rangle_t = \left\langle \begin{pmatrix} v \\ \nabla \cdot (u \tau(t) + 1) z_p \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_t
\]
\[
= \int_\Omega \{ \nabla v(x) \nabla u(x) + v(x) \Delta u(x) \} dx - \xi \int_{\Gamma_N} \int_0^1 (1 - \tau'(t)\rho) z_p(x, \rho) z(x, \rho) d\rho d\Gamma.
\]

So, by Green’s formula,
\[
\langle A(t)U, U \rangle_t = \int_{\Gamma_N} \frac{\partial u}{\partial v}(x) v(x) d\Gamma - \xi \int_{\Gamma_N} \int_0^1 (1 - \tau'(t)\rho) z_p(x, \rho) z(x, \rho) d\rho d\Gamma. \tag{2.20}
\]

Integrating by parts in $\rho$, we get
\[
\int_{\Gamma_N} \int_0^1 z_p(x, \rho) z(x, \rho) d\rho d\Gamma = \int_{\Gamma_N} \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} z^2(x, \rho) d\rho d\Gamma
\]
\[
= \frac{\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 z^2(x, \rho) d\rho d\Gamma + \frac{1}{2} \int_{\Gamma_N} \left\{ z^2(x, 1)(1 - \tau'(t)) - z^2(x, 0) \right\} d\Gamma. \tag{2.21}
\]

\[\]
Therefore, from (2.20) and (2.21),
\[
\langle A(t)U, U \rangle_t = \int_{\Gamma_N} \frac{\partial u}{\partial \nu}(x,v(x))d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} z^2(x,1)\{1 - \tau'(t)\} - z^2(x,0)d\Gamma \\
- \frac{\xi^2}{2} \int_{\Gamma_N} \int_0^1 z^2(x,\rho)d\rho d\Gamma \\
= - \int_{\Gamma_N} (\mu_1 v(x) + \mu_2 z(x,1))v(x)d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} z^2(x,1)\{1 - \tau'(t)\} - z^2(x,0)d\Gamma \\
- \frac{\xi^2}{2} \int_{\Gamma_N} \int_0^1 z^2(x,\rho)d\rho d\Gamma \\
= -\mu_1 \int_{\Gamma_N} v^2(x)d\Gamma - \mu_2 \int_{\Gamma_N} z(x,1)v(x)d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} z^2(x,1)\{1 - \tau'(t)\}d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} v^2(x)d\Gamma \\
- \frac{\xi^2}{2} \int_{\Gamma_N} \int_0^1 z^2(x,\rho)d\rho d\Gamma,
\]
from which follows, using Cauchy-Schwarz's inequality and (1.13),
\[
\langle A(t)U, U \rangle_t \leq \left( -\mu_1 + \frac{\mu_2}{2\sqrt{1 - d}} + \frac{\xi}{2} \right) \int_{\Gamma_N} v^2(x)d\Gamma \\
+ \left( \frac{\mu_2\sqrt{1 - d}}{2} - \frac{\xi(1 - d)}{2} \right) \int_{\Gamma_N} z^2(x,1)d\Gamma + \kappa(t) \langle U, U \rangle_t,
\]
where
\[
\kappa(t) = \frac{(\tau'(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)}.
\]
Now, observe that from (2.17),
\[
-\mu_1 + \frac{\mu_2}{2\sqrt{1 - d}} + \frac{\xi}{2} \leq 0, \quad \frac{\mu_2\sqrt{1 - d}}{2} - \frac{\xi(1 - d)}{2} \leq 0.
\]
Then,
\[
\langle A(t)U, U \rangle_t - \kappa(t) \langle U, U \rangle_t \leq 0,
\]
which means that the operator \( \tilde{A}(t) = A(t) - \kappa(t)I \) is dissipative.

Moreover \( \kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2 + 1)^{\frac{1}{2}}} \) is bounded on \([0, T] \) for all \( T > 0 \) (by (1.8) and (2.9)) and we have
\[
\frac{d}{dt} \tilde{A}(t)U = \left( \begin{array}{cc} 0 & 0 \\ \tau'(t)z & \tau'(t)z \end{array} \right) \\
\frac{d}{dt} \tilde{A}(t)U = \left( \begin{array}{cc} 0 & 0 \\ \tau'(t)z & \tau'(t)z \end{array} \right) \tilde{A}(t)U
\]
with \( \frac{\tau''(t)\tau(t) - \tau'(t)(\tau'(t)^2 + 1)^{\frac{1}{2}}}{\tau(t)^2} \) bounded on \([0, T] \) by (1.8) and (2.9). Thus
\[
\frac{d}{dt} \tilde{A}(t) \in L^\infty([0, T], B(D(A(0)), \mathcal{H})),
\]
the space of equivalence classes of essentially bounded, strongly measurable functions from \([0, T] \) into \( B(D(A(0)), \mathcal{H}) \).

Now, we will show that \( \lambda I - A(t) \) is surjective for fixed \( t > 0 \) and \( \lambda > 0 \). Given \( (f, g, h)^T \in \mathcal{H} \), we seek \( U = (u, v, z)^T \in D(A(t)) \) solution of
\[
(\lambda I - A(t)) \left( \begin{array}{c} u \\ v \\ z \end{array} \right) = \left( \begin{array}{c} f \\ g \\ h \end{array} \right).
\]
that is verifying
\[
\begin{aligned}
\begin{cases}
\lambda u - v &= f \\
\lambda v - \Delta u &= g \\
\lambda z + \frac{1-r'(t)\rho}{\tau(t)} z_\rho &= h.
\end{cases}
\end{aligned}
\tag{2.26}
\]
Suppose that we have found \( u \) with the appropriated regularity. Then,
\[
v := \lambda u - f \in V
\tag{2.27}
\]
and we can determine \( z \). Indeed, by (2.11),
\[
z(x, 0) = v(x), \quad \text{for } x \in \Gamma_N,
\tag{2.28}
\]
and, from (2.26),
\[
\lambda z(x, \rho) + \frac{1 - \tau'(t)\rho}{\tau(t)} z_\rho(x, \rho) = h(x, \rho), \quad \text{for } x \in \Gamma_N, \rho \in (0, 1).
\tag{2.29}
\]
Then, by (2.28) and (2.29), we obtain
\[
z(x, \rho) = v(x)e^{-\lambda \rho \tau(t)} + \tau(t)e^{-\lambda \rho \tau(t)} \int_0^\rho h(x, \sigma)e^{\lambda \sigma \tau(t)}d\sigma,
\]
if \( \tau'(t) = 0 \), and
\[
z(x, \rho) = \lambda u(x)e^{-\lambda \rho \tau(t)} - f(x)e^{-\lambda \rho \tau(t)} + \tau(t)e^{-\lambda \rho \tau(t)} \int_0^\rho h(x, \sigma)e^{\lambda \sigma \tau(t)}d\sigma,
\]
on \( \Gamma_N \times (0, 1) \),
\tag{2.30}
\]
if \( \tau'(t) = 0 \), and
\[
z(x, \rho) = \lambda u(x)e^{\lambda \tau(t)} + f(x)e^{\lambda \tau(t)} \int_0^\rho h(x, \sigma)e^{\lambda \sigma \tau(t)}d\sigma,
\tag{2.31}
\]
on \( \Gamma_N \times (0, 1) \),
\]
In particular, if \( \tau'(t) = 0 \)
\[
z(x, 1) = \lambda u(x)e^{-\lambda \tau(t)} + z_0(x), \quad x \in \Gamma_N .
\tag{2.32}
\]
with \( z_0 \in L^2(\Gamma_N) \) defined by
\[
z_0(x) = - f(x)e^{-\lambda \tau(t)} + \tau(t)e^{-\lambda \tau(t)} \int_0^1 h(x, \sigma)e^{\lambda \sigma \tau(t)}d\sigma, \quad x \in \Gamma_N,
\tag{2.33}
\]
and, if \( \tau'(t) \neq 0 \)
\[
z(x, 1) = \lambda u(x)e^{\lambda \tau(t)} + z_0(x), \quad x \in \Gamma_N,
\tag{2.34}
\]
with \( z_0 \in L^2(\Gamma_N) \) defined by
\[
z_0(x) = - f(x)e^{\lambda \tau(t)} \int_0^{\tau(t)} h(x, \sigma)e^{-\lambda \sigma \tau(t)}d\sigma + \tau(t)e^{-\lambda \tau(t)} \int_0^1 h(x, \sigma)e^{\lambda \sigma \tau(t)}d\sigma e^{-\lambda \tau(t)\rho}, \quad x \in \Gamma_N .
\tag{2.35}
\]
It remains to find $u$. By (2.27) and (2.26), the function $u$ satisfies

$$\lambda(\lambda u - f) - \Delta u = g,$$

that is

$$\lambda^2 u - \Delta u = g + \lambda f.$$  \hspace{1cm} (2.36)

Problem (2.36) can be reformulated as

$$\int_{\Omega} (\lambda^2 u - \Delta u)wdx = \int_{\Omega} (g + \lambda f)wdx, \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$  \hspace{1cm} (2.37)

Integrating by parts,

$$\int_{\Omega} (\lambda^2 u - \Delta u)wdx = \int_{\Omega} (\lambda^2 uw + \nabla u \nabla w)dx - \int_{\Gamma_N} \frac{\partial u}{\partial \nu} w d\Gamma = \int_{\Gamma_N} (\mu_1 v + \mu_2 z(x, 1))wd\Gamma.$$

If $\tau'(t) = 0$, by (2.27) and (2.32), we have

$$\int_{\Omega} (\lambda^2 u - \Delta u)wdx = \int_{\Omega} (\lambda^2 uw + \nabla u \nabla w)dx + \int_{\Gamma_N} \{\mu_1(\lambda u - f)w + \mu_2(\lambda u e^{-\lambda\tau(t)} + z_0)wd\Gamma, \quad \forall w \in H^1_{\Gamma_D}(\Omega),$$

and if $\tau'(t) \neq 0$, by (2.27) and (2.34),

$$\int_{\Omega} (\lambda^2 u - \Delta u)wdx = \int_{\Omega} (\lambda^2 uw + \nabla u \nabla w)dx + \int_{\Gamma_N} \{\mu_1(\lambda u - f)w + \mu_2(\lambda u e^{\lambda \frac{\omega}{\omega_0} ln(1-\tau'(t))} + z_0)wd\Gamma, \quad \forall w \in H^1_{\Gamma_D}(\Omega),$$

Therefore, (2.37) can be rewritten as

$$\int_{\Omega} (\lambda^2 uw + \nabla u \nabla w)dx + \int_{\Gamma_N} (\mu_1 + \mu_2 e^{-\lambda\tau(t)})\lambda uw d\Gamma = \int_{\Omega} (g + \lambda f)wdx + \mu_1 \int_{\Gamma_N} fw d\Gamma - \mu_2 \int_{\Gamma_N} z_0wd\Gamma, \quad \forall w \in H^1_{\Gamma_D}(\Omega),$$  \hspace{1cm} (2.38)

if $\tau'(t) = 0$, and

$$\int_{\Omega} (\lambda^2 uw + \nabla u \nabla w)dx + \int_{\Gamma_N} (\mu_1 + \mu_2 e^{\lambda \frac{\omega}{\omega_0} ln(1-\tau'(t))})\lambda uw d\Gamma = \int_{\Omega} (g + \lambda f)wdx + \mu_1 \int_{\Gamma_N} fw d\Gamma - \mu_2 \int_{\Gamma_N} z_0wd\Gamma, \quad \forall w \in H^1_{\Gamma_D}(\Omega),$$  \hspace{1cm} (2.39)

otherwise. As the left-hand side of (2.38) or (2.39) is coercive on $H^1_{\Gamma_D}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $u \in H^1_{\Gamma_D}(\Omega)$ of (2.38) or (2.39).

If we consider $w \in D(\Omega)$ in (2.38) or (2.39), $u$ solves in $D'(\Omega)$

$$\lambda^2 u - \Delta u = g + \lambda f,$$  \hspace{1cm} (2.40)

and thus $u \in E(\Delta, L^2(\Omega)).$

Using Green's formula (2.13) in (2.38) and using (2.40), we obtain, if $\tau'(t) = 0$

$$\int_{\Gamma_N} (\mu_1 + \mu_2 e^{-\lambda\tau(t)})\lambda uw d\Gamma + (\frac{\partial u}{\partial \nu}, w)_{\Gamma_N} = \mu_1 \int_{\Gamma_N} fw d\Gamma - \mu_2 \int_{\Gamma_N} z_0wd\Gamma,$$

from which follows

$$\frac{\partial u}{\partial \nu} + (\mu_1 + \mu_2 e^{-\lambda\tau(t)})\lambda u = \mu_1 f - \mu_2 z_0 \quad \text{on} \quad \Gamma_N.$$  \hspace{1cm} (2.41)
Therefore, from (2.41),
\[
\frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1) \quad \text{on} \quad \Gamma_N,
\]
where we have used (2.27) and (2.32).

We find the same result if \( \tau'(t) \neq 0 \).

So, we have found \((u, v, z)^T \in \mathcal{D}(A(t))\) which verifies (2.26), and thus \( \lambda I - A(t) \) is surjective for some \( \lambda > 0 \) and \( t > 0 \). Again as \( \kappa(t) > 0 \), this proves that
\[
\lambda I - \tilde{A}(t) = (\lambda + \kappa(t))I - A(t) \text{ is surjective} \quad (2.42)
\]
for any \( \lambda > 0 \) and \( t > 0 \).

Then, (2.19), (2.24) and (2.42) imply that the family \( \tilde{A} = \{ \tilde{A}(t) : t \in [0, T]\} \) is a stable family of generators in \( \mathcal{H} \) with stability constants independent of \( t \), by Proposition 1.1 from [11]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are verified by (2.12), (2.19), (2.24), (2.25), (2.42) and Lemma 2.2, and thus, the problem
\[
\begin{cases}
\tilde{U}' = \tilde{A}(t)\tilde{U} \\
\tilde{U}(0) = U_0
\end{cases}
\]
has a unique solution \( \tilde{U} \in C([0, +\infty), \mathcal{D}(A(0))) \cap C^1([0, +\infty), \mathcal{H}) \) for \( U_0 \in \mathcal{D}(A(0)) \). The requested solution of (2.10) is then given by
\[
U(t) = e^{\beta(t)}\tilde{U}(t)
\]
with \( \beta(t) = \int_0^t \kappa(s)ds \), because
\[
U'(t) = \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}'(t) = \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{A}(t)\tilde{U}(t) = e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{A}(t)\tilde{U}(t)) = e^{\beta(t)}\tilde{A}(t)\tilde{U}(t) = \tilde{A}(t)e^{\beta(t)}\tilde{U}(t) = \tilde{A}(t)U(t),
\]
which concludes the proof. \( \blacksquare \)

### 2.1.2 The general case

In this subsection (1.6) only holds, so \( \tau \) may be also degenerate, i.e. \( \tau(t) = 0 \) for some times \( t \). Taking
\[
\tau_{\epsilon}(t) = \tau(t) + \epsilon, \quad \forall 0 < \epsilon < \epsilon_0
\]
then
\[
0 < \epsilon \leq \tau_{\epsilon}(t) \leq \bar{\tau} + \epsilon, \quad (2.43)
\]
i.e. \( \tau_{\epsilon} \) satisfies (2.9). Therefore, by Theorem 2.3, there exists a unique solution
\[
U_{\epsilon} = (u^\epsilon, v^\epsilon, z^\epsilon)^T \in C([0, +\infty), \mathcal{D}(A_\epsilon(t))) \cap C^1([0, +\infty), \mathcal{H})
\]
for \( U_{\epsilon,0} \in \mathcal{D}(A_\epsilon(0)) \), of problem
\[
\begin{cases}
U_{\epsilon}' = A_\epsilon(t)U_{\epsilon} \\
U_{\epsilon}(0) = (u_0, u_1, f_0(\cdot, -\tau_{\epsilon}(0)))^T = U_{\epsilon,0}
\end{cases} \quad (2.44)
\]
where the operator \( A_\epsilon(t) \) is defined by
\[
A_\epsilon(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'_{\epsilon}(t)p - 1}{\tau_{\epsilon}(t)^p} z^p \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'_{\epsilon}(t)p - 1}{\tau_{\epsilon}(t)^p} z^p \end{pmatrix},
\]

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with domain
\[ \mathcal{D}(\mathcal{A}_t) = \mathcal{D}(\mathcal{A}(t)). \]

The aim is then to take the limit of \((u_\epsilon)_{\epsilon<\epsilon_0}\) when \(\epsilon\) tends to 0.

To pass at the limit, we need to have more regularity on the solution and, for that purpose, we use Theorem 2.13 of [11] (see also Theorem 3.2.3 of [1]).

We now fix \(0 < \epsilon < \epsilon_0\). We consider the family of Hilbert spaces
\[ X = X_0 = \mathcal{H}, \quad X_1 = \left(V \cap H^{3/2}(\Omega)\right) \times V \times L^2(\Gamma_N; H^1(0,1)), \]
\[ X_2 = \left(V \cap H^{3/2}(\Omega)\right) \times \left(V \cap H^{3/2}(\Omega)\right) \times L^2(\Gamma_N; H^2(0,1)), \]
with the usual norms
\[ \|\cdot\|_0 = \|\cdot\|_{\mathcal{H}}, \quad \|\cdot\|_1 = \|\cdot\|_{X_1}, \quad \|\cdot\|_2 = \|\cdot\|_{X_2}. \]
We can easily check that
\[ X_2 \hookrightarrow X_1 \hookrightarrow X_0 = X \]
and
\[ \|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2. \]
Let \(Y = \mathcal{D}(\mathcal{A}_t)\). \(Y\) is a dense subset of \(X = X_0 = \mathcal{H}\) and a subset of \(X_1\). Indeed, by a result of Lions and Magenes [22], if \(u \in H^1_{\Gamma_\rho}(\Omega)\), \(\Delta u \in L^2(\Omega)\) and \(\partial u / \partial \nu \in L^2(\Gamma_N)\), then \(u \in H^{3/2}(\Omega)\). Consequently
\[ \mathcal{D}(\mathcal{A}_t) \cap X_1 = \mathcal{D}(\mathcal{A}_t) = Y, \quad \forall t \in [0, T]. \]

The family of operators \(\mathcal{A}_t = \{\mathcal{A}_t(t) : t \in [0, T]\}\) is a stable family of generators in \(X = \mathcal{H}\) with stability constants independent of \(t\) (see the previous subsection).

We have that
\[ \frac{d}{dt} \mathcal{A}_t(t) \in L^\infty([0, T], B(D(\mathcal{A}_t(0)), X)) \cap L^\infty([0, T], B(D(\mathcal{A}_t(0)) \cap X_2, X_1)), \]
\[ \frac{d^2}{dt^2} \mathcal{A}_t(t) \in L^\infty([0, T], B(D(\mathcal{A}_t(0)) \cap X_2, X)), \]
because
\[ \frac{d}{dt} \mathcal{A}_t(t) U = \left( \begin{array}{c} 0 \\ \rho \frac{\tau'(t)\tau'(t)\rho - \tau'(t)\rho - 1}{(\tau'(t)+\tau)^2} \end{array} \right), \]
and
\[ \frac{d^2}{dt^2} \mathcal{A}_t(t) U = \left( \begin{array}{ccc} 0 & 0 & \rho \frac{\tau''(t)\rho - \tau''(t)\rho - 1}{(\tau'(t)+\tau)^2} \\ 0 & 0 & 2\tau'(t)\rho - \tau'(t)\rho - 1 \end{array} \right), \]
and by (1.6) and (1.8).

Finally, again with a result of [22] and as \(\frac{\tau'(t)\rho - 1}{(\tau(t)+\tau)^2}\) is bounded on \([0, T]\) by (1.6) and (1.8), if \(\phi \in \mathcal{D}(\mathcal{A}_t(t))\) and \(\mathcal{A}_t(t)\phi \in X\), then \(\phi \in X_1\) with
\[ \|\phi\|_1 \leq \nu (\|\mathcal{A}_t(t)\phi\|_0 + \|\phi\|_0), \]
and, if \(\phi \in \mathcal{D}(\mathcal{A}_t(t))\) and \(\mathcal{A}_t(t)\phi \in X_1\), then \(\phi \in X_2\) with
\[ \|\phi\|_2 \leq \nu (\|\mathcal{A}_t(t)\phi\|_1 + \|\phi\|_0). \]

Introduce now the space \(D^2(0)\) defined by
\[ D^2(0) = \{ \phi \in \mathcal{D}(A(0)) \cap X_2 : -A(0)\phi \in \mathcal{D}(A(0)) \}. \]

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Therefore, by the result of [11] (see also [1]), for all initial data \( U_{\epsilon,0} \in D^2(0) \), there exists a unique solution \( U_{\epsilon} \in C^1([0,T],\mathcal{H}) \cap C([0,T],D(A_{\epsilon}(0))) \) of (2.44) which satisfies, moreover,

\[
\frac{d^2}{dt^2} U_{\epsilon} \in C([0,T],\mathcal{H}).
\]

We then have more regularity of the solution with more regular initial data. Therefore, we can give a sense to the derivative of the stronger energy \( \tilde{E}_{\epsilon} \) defined as follows:

\[
\tilde{E}_{\epsilon}(t) = \frac{1}{2} \int_{\Omega} \left( (\Delta u_{\epsilon})^2 + 2 \nabla u_{\epsilon,t} \nabla u_{\epsilon,t} \right) dx + \frac{q\tau_{\epsilon}(t)}{2} \int_{\Gamma_N} \int_0^1 u_{\epsilon,t}^2(x,t-t_{\epsilon}(t)\rho) d\rho d\Gamma,
\]

for \((u_0, u_1, f_{0}(\cdot, \cdot, \tau_{\epsilon}(0)))^T \in D^2(0)\), where \( q \) is a suitable positive constant. Then the derivative of \( \tilde{E}_{\epsilon} \) gives

\[
\tilde{E}_{\epsilon}'(t) = \int_{\Omega} (u_{\epsilon,ttt_\epsilon} u_{\epsilon,t} + \nabla u_{\epsilon,t} \nabla u_{\epsilon,t}) dx + \frac{q\tau_{\epsilon}'(t)}{2} \int_{\Gamma_N} \int_0^1 u_{\epsilon,t}^2(x,t-t_{\epsilon}(t)\rho) d\rho d\Gamma
\]

\[
+ q\tau_{\epsilon}(t) \int_{\Gamma_N} \int_0^1 u_{\epsilon,tt}(x,t-t_{\epsilon}(t)\rho) u_{\epsilon,ttt_\epsilon}(x,t-t_{\epsilon}(t)\rho) (1 - \tau_{\epsilon}'(t)\rho) d\rho d\Gamma.
\]

By Green’s formula and integrating by parts in \( \rho \), we obtain

\[
\tilde{E}_{\epsilon}'(t) = \int_{\Gamma_N} \frac{\partial u_{\epsilon,t}}{\partial \nu} u_{\epsilon,tt} d\Gamma - \frac{q}{2} \int_{\Gamma_N} u_{\epsilon,tt}^2(x,t-t_{\epsilon}(t))(1 - \tau_{\epsilon}'(t)) d\Gamma + \frac{q}{2} \int_{\Gamma_N} u_{\epsilon,ttt_\epsilon}^2(x,t) d\Gamma.
\]

Since \( u_{\epsilon} \) satisfies (2.44),

\[
\frac{\partial u_{\epsilon,t}}{\partial \nu} = -\mu_1 u_{\epsilon,tt}(t) - \mu_2 u_{\epsilon,ttt}(t-t_{\epsilon}(t))(1 - \tau_{\epsilon}'(t)),
\]

and we obtain

\[
\tilde{E}_{\epsilon}'(t) = \left( \frac{q}{2} - \mu_1 \right) \int_{\Gamma_N} u_{\epsilon,tt}^2(t) d\Gamma - \mu_2 (1 - \tau_{\epsilon}'(t)) \int_{\Gamma_N} u_{\epsilon,tt}(x,t-t_{\epsilon}(t)) u_{\epsilon,ttt}(x,t) d\Gamma
\]

\[
- \frac{q}{2} \int_{\Gamma_N} u_{\epsilon,tt}^2(x,t-t_{\epsilon}(t))(1 - \tau_{\epsilon}'(t)) d\Gamma.
\]

By Cauchy-Schwarz’s inequality, we get, for \( \alpha > 0 \),

\[
\tilde{E}_{\epsilon}'(t) \leq \left( \frac{q}{2} - \mu_1 + \frac{\alpha \mu_2 (1 - \tau_{\epsilon}'(t))}{2} \right) \int_{\Gamma_N} u_{\epsilon,tt}^2(t) d\Gamma
\]

\[
+ \left( \frac{\mu_2 (1 - \tau_{\epsilon}'(t))}{2\alpha} - \frac{q(1 - \tau_{\epsilon}'(t))}{2} \right) \int_{\Gamma_N} u_{\epsilon,tt}^2(x,t-t_{\epsilon}(t)) d\Gamma.
\]

Let \( \tau_{\min}^\prime = \min_{t \in [0,T]} \tau_{\epsilon}'(t) \) and assume that

\[
(1 - d)(1 - \tau_{\min}^\prime) \leq 2.
\]

By (2.16) and (2.46), we have

\[
\mu_2 \leq \frac{\sqrt{2}}{\sqrt{1 - \tau_{\min}^\prime}} \mu_1,
\]

which implies

\[
\frac{\mu_2}{2\alpha} \leq 2\mu_1 + \mu_2 \alpha (\tau_{\min}^\prime - 1),
\]

with \( \alpha = \frac{1}{\sqrt{2(1 - \tau_{\min}^\prime)}} \). Consequently we can choose \( q > 0 \) such that

\[
\frac{\mu_2}{2\alpha} \leq q \leq 2\mu_1 + \mu_2 \alpha (\tau_{\min}^\prime - 1),
\]

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Under the assumption (2.46), we have
\[ \tilde{E}_\epsilon(t) \leq \tilde{E}_\epsilon(0), \quad \forall t > 0, \]
i.e., for all \( 0 < \epsilon < \epsilon_0 \) and \( t > 0 \),
\[
\begin{align*}
\int_{\Omega} \left( (u_{\epsilon,t})^2 + (\nabla u_{\epsilon,t})^2 \right) \, dx + q\tau_\epsilon(t) \int_{\Gamma_N} \int_0^1 u_{\epsilon,t}^2(x, t - \tau_\epsilon(t) \rho) \, d\rho \, d\Gamma \\
\leq \int_{\Omega} \left( (\Delta u_0)^2 + (\nabla u_1)^2 \right) \, dx + q(\tau(0) + \epsilon) \int_{\Gamma_N} \int_0^1 f_{\epsilon,t}^2(x, -(\tau(0) + \epsilon) \rho) \, d\rho \, d\Gamma.
\end{align*}
\] (2.47)
Therefore, assuming that \( (f_{0,t}(x, -(\tau(0) + \epsilon) \rho)_{0 < \epsilon < \epsilon_0} \) is bounded on \( L^2(\Gamma_N \times (0, 1)) \), the sequence \((u_\epsilon)_\epsilon\) is bounded on \( H^1((0, T); V) \cap H^2((0, T); L^2(\Omega)) \), and thus, there exists \( u \in H^1((0, T); V) \cap H^2((0, T); L^2(\Omega)) \) such that, up to a subsequence,
\[ u_\epsilon \rightharpoonup u \quad \text{in} \quad H^1((0, T); V) \cap H^2((0, T); L^2(\Omega)). \]
The limit \( u \) then satisfies (1.1) in \( D'(\Omega \times (0, T)) \) and (1.2), (1.5). Moreover \( u \) satisfies (1.3) since \( u_{\epsilon,t}|_{\Gamma_N} \rightharpoonup u_{t}|_{\Gamma_N} \) in \( L^2((0, T) \times \Gamma_N) \) and by using Lebesgue’s convergence theorem. In the same manner, we find that \( u \) verifies (1.5), since, by change of variable and by (2.47) we have
\[
\int_{\Gamma_N} \int_{-(\tau(t)+\epsilon)}^t u_{\epsilon,t}^2(x, t) \, dt \, d\Gamma \leq C, \quad \forall t \in [0, T],
\]
and thus
\[
\int_{\Gamma_N} \int_0^{-(\tau(0))} u_{\epsilon,t}^2(x, t) \, dt \, d\Gamma \leq C, \quad \forall t \in [0, T].
\]
In conclusion we have proved the next existence result.

**Theorem 2.4** Assume (2.46) and let \((f_{0,t}(x, -(\tau(0) + \epsilon) \rho)_{0 < \epsilon < \epsilon_0} \) be bounded on \( L^2(\Gamma_N \times (0, 1)) \). Then, for all initial data \( U_0 \in D^2(0) \), there exists a unique solution \( u \in H^1((0, T); V) \cap H^2((0, T); L^2(\Omega)) \) of (2.44).

### 2.2 Nonlinear problem

Here we restrict ourselves to the case where (2.9) holds.

As previously, if we set \( z(x, \rho, t) \) as in (2.1), problem (1.15) – (1.19) is equivalent to
\[
\begin{align*}
& u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \quad (2.48) \\
& \tau(t)z_{t}(x, \rho, t) + (1 - \tau'(t) \rho)z_{\rho}(x, \rho, t) = 0 \quad \text{in} \quad \Gamma_N \times (0, 1) \times (0, +\infty) \quad (2.49) \\
& u(x, t) = 0 \quad \text{on} \quad \Gamma_D \times (0, +\infty) \quad (2.50) \\
& \frac{\partial u}{\partial \nu}(x, t) = -\beta_1(u_t(x, t)) - \beta_2(z(x, 1, t)) \quad \text{on} \quad \Gamma_N \times (0, +\infty) \quad (2.51) \\
& z(x, 0, t) = u_t(x, t) \quad \text{on} \quad \Gamma_N \times (0, \infty) \quad (2.52) \\
& u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \quad (2.53) \\
& z(x, \rho, 0) = g_0(x, -\rho \tau(0)) \quad \text{in} \quad \Gamma_N \times (0, 1). \quad (2.54)
\end{align*}
\]

Then problem (2.48) – (2.54) can be rewritten as
\[
\begin{cases}
U' = A(t)U \\
U(0) = (u_0, u_1, g_0(\cdot, - \cdot \tau(0)))^T
\end{cases}
\] (2.55)
where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \frac{\Delta u}{\sigma(t)\rho - 1} \sigma \rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \left\{ (u, v, z)^T \in (E(\Delta, L^2(\Omega)) \cap V) \times V \times L^2(\Gamma_N; H^1(0, 1)) : \right\}.$$

Notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time $t$, i.e. (2.12) holds. Note further that for $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$, $\partial u/\partial \nu$ belongs to $L^2(\Gamma_N)$, by (1.20) and since $z(\cdot, 1)$ is in $L^2(\Gamma_N)$.

We observe that the operator $\mathcal{A}(t)$ defined before is nonlinear (due to the domain (2.56) of the operator $\mathcal{A}(t)$) and therefore the technique developed in Section 2 can not be applied here. For nonlinear operators $\mathcal{A}(t)$ similar results exist (see [4, 8, 10, 20]) but for maximal operators $\mathcal{A}(t)$ with one inner product independent of $t$. For our system we need a variant of such results for maximal monotone operators $\mathcal{A}(t)$ for a time-dependent inner product depending “smoothly” on $t$.

We have the following result from [10] (see also [24]):

**Theorem 2.5** Let $X$ be a real separable Hilbert space. For a fixed $T > 0$ and any time $t \in [0, T]$ we assume that there exists an inner product $\langle \cdot, \cdot \rangle$ on $X$ depending “smoothly” on $t$ in the following sense: there exists $c > 0$ such that

$$\frac{\|u\|}{\|u\|_t} \leq e^{c|t-s|}, \quad \forall u \in X, \forall t, s \in [0, T].$$

Assume furthermore that:

(i) for all $t \in [0, T]$, $\mathcal{A}(t)$ is a maximal monotone operator for the inner product $\langle \cdot, \cdot \rangle_t$;

(ii) the domain $\mathcal{D}(\mathcal{A}(t))$ of $\mathcal{A}(t)$ is independent of $t$, for all $t \in [0, T]$;

(iii) there exists a positive constant $K$ such that

$$\|\mathcal{A}(t)u - A(s)\|_0 \leq K|t - s|(1 + \|u\|_0 + \|A(s)u\|_0), \quad \forall u \in \mathcal{D}(\mathcal{A}(t)), \forall s, t \in [0, T],$$

where here $\|\cdot\|_0 = \|\cdot\|_{t=0}$. Then for all $v \in \mathcal{D}(\mathcal{A}(t))$ the evolution equation

$$\begin{cases} u' + A(t)u = 0 & \text{for } 0 \leq t \leq T \\ u(0) = v \end{cases}$$

has a unique solution $u \in C([0, T]; X)$ such that $u(t)$ belongs to $\mathcal{D}(\mathcal{A}(t))$ for all $t \in [0, T]$, its strong derivative $u'(t) = -A(t)u(t)$ exists and is continuous except at a countable numbers of values $t$.

Therefore to prove the existence and uniqueness of the solution of (2.55), we define an inner product depending “smoothly” on $t$.

For that, we assume that

$$\gamma_2 \leq \gamma_1 \sqrt{1 - d}$$

holds, where $\gamma_1$, $\gamma_2$ is defined by (1.22) and (1.23).

Let $\xi$ be a positive real number such that

$$\frac{\gamma_2}{\sqrt{1 - d}} \leq \xi \leq 2\gamma_1 - \frac{\gamma_2}{\sqrt{1 - d}}.$$ 

Note that, from (2.60), such a constant $\xi$ exists.

Let us define on the Hilbert space $\mathcal{H}$ the following time-dependent inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t := \int_0^1 \int_{\Gamma_N} \left\{ \nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{u}(x) \right\} dx + \xi\tau(t) \int_0^1 \int_{\Gamma_N} z(x, \rho)\tilde{z}(x, \rho) d\rho d\Gamma.$$ 

(2.62)
where $\xi$ is defined by (2.61).

Note that if $\beta_1$ and $\beta_2$ are linear, i.e. $\beta_i(s) = \mu_i s$ with $\mu_i > 0$, the assumptions (2.16) and (2.60), and (2.17) and (2.61) are exactly the same.

The aim of this section is then to prove the following theorem:

**Theorem 2.6** Assume (1.13), (1.20), (1.21), (1.22), (1.23), (2.9), (2.60) hold. Moreover assume that $\beta_2$ is nondecreasing. For any initial datum $U_0 \in D(A(0))$, then there exists a unique solution

$$U \in C([0, +\infty), D(A(t))) \cap C^1([0, +\infty), \mathcal{H})$$

of problem (2.55).

To prove Theorem 2.6, we thus check that (2.57) holds and that

$$\bar{A}_-(t) = \bar{A}(t) = -A(t) + \kappa(t)I$$

(2.63)

satisfies the assumptions (i) to (iii) of Theorem 2.5, where $\kappa$ is defined by (2.23).

The proof of (2.57) is the same that in Theorem 2.3, so we omit it.

We clearly have (ii) for $\bar{A}_-(t)$ since $D(A(t)) = D(A_-(t))$.

Therefore, it remains to show (i) and (iii), which is the aim of the three following lemmas.

**Lemma 2.7** Assume (1.22), (1.23), (2.60) and (2.61) hold. Then $\bar{A}_-(t)$ is a monotone operator in $\mathcal{H}$ for the inner product $(.,.)_\ell$ for any fixed $t \geq 0$, i.e.:

$$\langle \bar{A}_-(t)\phi_1 - \bar{A}_-(t)\phi_2, \phi_1 - \phi_2 \rangle \geq 0, \quad \forall \phi_1, \phi_2 \in D(\bar{A}_-(t)).$$

(2.64)

**Proof.** First, from the definition of $A(t)$, for $\phi = (u, v, z)^T \in D(A(t))$,

$$\langle A(t)\phi_1 - A(t)\phi_2, \phi_1 - \phi_2 \rangle = \int_{\Gamma_N} \{(\nabla v_1 - \nabla v_2)(\nabla u_1 - \nabla u_2) + (\Delta u_1 - \Delta u_2)(v_1 - v_2)\} dx + \xi \int_{\Gamma_N} \int_0^1 \left( \frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho} \right) (z_1 - z_2) (\tau'(t)\rho - 1) d\rho d\Gamma.$$

So, by Green’s formula,

$$\langle A(t)\phi_1 - A(t)\phi_2, \phi_1 - \phi_2 \rangle = \int_{\Gamma_N} (v_1 - v_2) \frac{\partial}{\partial \nu} (u_1 - u_2) d\Gamma + \xi \int_{\Gamma_N} \int_0^1 \left( \frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho} \right) (z_1 - z_2) (\tau'(t)\rho - 1) d\rho d\Gamma.$$

Integrating by parts in $\rho$, we get

$$\int_{\Gamma_N} \int_0^1 \left( \frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho} \right) (z_1 - z_2) (\tau'(t)\rho - 1) d\rho d\Gamma = \frac{1}{2} \int_{\Gamma_N} \int_0^1 (\tau'(t)\rho - 1) \frac{\partial}{\partial \rho} (z_1 - z_2)^2 d\rho d\Gamma$$

$$= -\frac{\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma + \frac{\tau'(t)}{2} \int_{\Gamma_N} (z_1(x, 1) - z_2(x, 1))^2 d\Gamma$$

$$+ \frac{1}{2} \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0))^2 d\Gamma.$$

Therefore

$$\langle A(t)\phi_1 - A(t)\phi_2, \phi_1 - \phi_2 \rangle = \int_{\Gamma_N} (v_1 - v_2) \frac{\partial}{\partial \nu} (u_1 - u_2) d\Gamma - \xi \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma$$

$$- \frac{\xi(1 - \tau'(t))}{2} \int_{\Gamma_N} (z_1(x, 1) - z_2(x, 1))^2 d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0))^2 d\Gamma.$$
As \( \phi_i \in \mathcal{D}(A(t)) \) for \( i = 1, 2 \), we obtain

\[
\langle A(t)\phi_1 - A(t)\phi_2, \phi_1 - \phi_2 \rangle_t = -\int_{\Gamma_N} (z_1(x,0) - z_2(x,0))(\beta_1(z_1(x,0)) - \beta_1(z_2(x,0)))d\Gamma
\]

\[
- \int_{\Gamma_N} (z_1(x,0) - z_2(x,0))(\beta_2(z_2(x,1)) - \beta_2(z_2(x,1)))d\Gamma - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N}\int_0^1 (z_1 - z_2)^2d\rho d\Gamma
\]

From (1.22), (1.23) and Cauchy-Schwarz's inequality

\[
\langle A(t)\phi_1 - A(t)\phi_2, \phi_1 - \phi_2 \rangle_t \leq \left( \frac{\xi}{2} - \gamma_1 + \frac{\gamma_2}{2\sqrt{1-d}} \right) \int_{\Gamma_N} (z_1(x,0) - z_2(x,0))^2d\Gamma
\]

\[+ \left( \frac{\gamma_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2} \right) \int_{\Gamma_N} (z_1(x,1) - z_2(x,1))^2d\Gamma
\]

\[\quad - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N}\int_0^1 (z_1 - z_2)^2d\rho d\Gamma.
\]

By (2.61) and the definition (2.23) of \( \kappa \), we get

\[
\langle A(t)\phi_1 - A(t)\phi_2, \phi_1 - \phi_2 \rangle_t \leq \kappa(t)\xi\tau(t) \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2d\rho d\Gamma \leq \kappa(t) \langle \phi_1 - \phi_2, \phi_1 - \phi_2 \rangle_t.
\]

By the definition (2.63) of \( \tilde{A}_-(t) \), we obtain (2.64). □

**Lemma 2.8** Assume that (1.8), (1.20), (1.21), (1.22) and (2.9) hold. Moreover assume that \( \beta_2 \) is nondecreasing. Then \( \tilde{A}_-(t) \) is a maximal operator in \( \mathcal{H} \), i.e. for all \((f, g, h)^T \in \mathcal{H}\), there exists \((u, v, z)^T \in \mathcal{D}(\tilde{A}_-(t))\) such that

\[
(I + \tilde{A}_-(t))(u, v, z)^T = (f, g, h)^T.
\]

**Proof.** Given \((f, g, h)^T \in \mathcal{H}\), we seek \(U = (u, v, z)^T \in \mathcal{D}(\tilde{A}_-(t))\) solution of

\[
\begin{cases}
(1 + \kappa(t))u - v = f \\
(1 + \kappa(t))v - \Delta u = g \\
(1 + \kappa(t))z + \frac{1}{\tau'(t)}z_\rho = h.
\end{cases}
\]

(2.66)

In the beginning of this proof we follow the proof of Theorem 2.3. Suppose that we have found \(u\) with the appropriated regularity. Then \(v\) is given by

\[
v := (1 + \kappa(t))u - f \in V,
\]

(2.67)

and \(z\) by

\[
z(x, \rho) = (1 + \kappa(t))u(x)e^{-(1+\kappa(t))\tau(t)} - f(x)e^{-(1+\kappa(t))\tau(t)}
\]

\[
+ \tau(t)e^{-(1+\kappa(t))\tau(t)} \int_0^\rho h(x, \sigma)e^{(1+\kappa(t))\tau(t)}d\sigma \quad \text{on} \quad \Gamma_N \times (0, 1),
\]

(2.68)

if \(\tau'(t) = 0\), and

\[
z(x, \rho) = (1 + \kappa(t))u(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}}\ln(1-\tau'(t)\rho) - f(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}}\ln(1-\tau'(t)\rho)
\]

\[
+ e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)}}\ln(1-\tau'(t)\rho) \int_0^\rho h(x, \sigma)\tau(t)e^{-(1+\kappa(t))\tau(t)}d\sigma \quad \text{on} \quad \Gamma_N \times (0, 1)
\]

(2.69)

otherwise.

In particular, if \(\tau'(t) = 0\)

\[
z(x, 1) = (1 + \kappa(t))u(x)e^{-(1+\kappa(t))\tau(t)} + z_{0}(x), \quad x \in \Gamma_N,
\]

(2.70)
with $z_0 \in L^2(\Gamma_N)$ defined by

$$z_0(x) = -f(x)e^{-(1+\kappa(t))\tau(t)} + \tau(t)e^{-(1+\kappa(t))\tau(t)} \int_0^1 h(x, \sigma)e^{(1+\kappa(t))\sigma\tau(t)}d\sigma, \quad x \in \Gamma_N,$$

and, if $\tau'(t) \neq 0$

$$z(x, 1) = (1 + \kappa(t))u(x)e^{(1+\kappa(t))\frac{z_0(x)}{\tau(t)}} \tau(t) + z_0(x), \quad x \in \Gamma_N,$$

with $z_0 \in L^2(\Gamma_N)$ defined by

$$z_0(x) = -f(x)e^{(1+\kappa(t))\frac{z_0(x)}{\tau(t)}} \ln(1-\tau'(t))
+ e^{(1+\kappa(t))\frac{z_0(x)}{\tau(t)}} \ln(1-\tau'(t)) \int_0^1 h(x, \tau(t)e^{-(1+\kappa(t))\frac{z_0(x)}{\tau(t)}}d\sigma, \quad x \in \Gamma_N. \tag{2.73}$$

By (2.66), as in the proof of Theorem 2.3, the function $u$ satisfies

$$(1 + \kappa(t))^2u - \Delta u = g + (1 + \kappa(t))f, \tag{2.74}$$

which can be reformulated as

$$\int_\Omega ((1 + \kappa(t))^2u - \Delta u)wdx = \int_\Omega (g + (1 + \kappa(t))f)wdx, \quad \forall w \in H^1_{T,D}(\Omega). \tag{2.75}$$

Integrating by parts and since $(u, v, z)^T \in \mathcal{D}(A(t))$, we obtain

$$\int_\Omega ((1 + \kappa(t))^2uw + \nabla u\nabla w)dx + \int_{\Gamma_N} (\beta_1(v) + \beta_2(z(x, 1)))wdx = \int_\Omega (g + (1 + \kappa(t))f)wdx,$$

for all $w \in H^1_{T,D}(\Omega).

Assume that $\tau'(t) = 0$. Using (2.67) and (2.70), we get

$$\gamma(u, w) = F(w), \quad \forall w \in H^1_{T,D}(\Omega), \tag{2.76}$$

where the form $\gamma$ (linear on $w$ but not on $u$) is defined by

$$\gamma(u, w) = \int_\Omega ((1 + \kappa(t))^2uw + \nabla u\nabla w)dx
+ \int_{\Gamma_N} \left( \beta_1((1 + \kappa(t))u - f) + \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) \right)wdx,$$

and the linear form $F$ is defined by

$$F(w) = \int_\Omega (g + (1 + \kappa(t))f)wdx.$$

Introducing the (nonlinear) mapping

$$B : V \rightarrow V' : u \rightarrow Bu,$$

where $Bu(w) = \gamma(u, w)$, we see that (2.76) is equivalent to

$$Bu = F,$$

since $F$ clearly belongs to $V'$. This means that the solvability of (2.76) is equivalent to the surjectivity of $B$. This surjectivity is obtained using Corollary II.2.2 of [29], which states that $B$ is surjective if $B$ is monotone, hemicontinuous, bounded and coercive. Let us then check these properties.
We first prove that $B$ is monotone, i.e.

$$[Bu - Bv](u - v) \geq 0, \quad \forall u, v \in V. \tag{2.77}$$

In view of the definition of $B$,

$$[Bu - Bv](u - v) = \int_{\Omega} ((1 + \kappa(t))^2 (u - v)^2 + |\nabla (u - v)|^2) \, dx$$

$$+ \int_{\Gamma_N} (\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)) \, (u - v) d\Gamma$$

$$+ \int_{\Gamma_N} \left( \beta_2((1 + \kappa(t))ue^{-(1 + \kappa(t))\tau(t)} + z_0) - \beta_2((1 + \kappa(t))ve^{-(1 + \kappa(t))\tau(t)} + z_0) \right) \, (u - v) d\Gamma.$$

By (1.22),

$$\int_{\Gamma_N} \left( \beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f) \right) \, (u - v) d\Gamma$$

$$= \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \left( \beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f) \right) \, (((1 + \kappa(t))u - f) - ((1 + \kappa(t))v - f)) d\Gamma \geq 0,$$

and as $\beta_2$ is nondecreasing,

$$\int_{\Gamma_N} \left( \beta_2((1 + \kappa(t))ue^{-(1 + \kappa(t))\tau(t)} + z_0) - \beta_2((1 + \kappa(t))ve^{-(1 + \kappa(t))\tau(t)} + z_0) \right) \, (u - v) d\Gamma$$

$$= \frac{e^{(1 + \kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \left( \beta_2((1 + \kappa(t))ue^{-(1 + \kappa(t))\tau(t)} + z_0) - \beta_2((1 + \kappa(t))ve^{-(1 + \kappa(t))\tau(t)} + z_0) \right) \, (((1 + \kappa(t))u - f) - ((1 + \kappa(t))v - f)) d\Gamma \geq 0.$$

This two estimates clearly imply (2.77).

The boundedness of $B$ follows from the properties (1.20) satisfied by $\beta_1$ and $\beta_2$, the fact that $1 + \kappa(t)$ is bounded by (2.9) and (1.8), Cauchy-Schwarz’s inequality and a trace theorem (reminding that $f$ and $z_0$ are fixed).

The hemicontinuity of $B$ means that the function $s \to B(u + sw)(w)$ is continuous for each $u, w \in V$. As

$$B(u + sw)(w) = \int_{\Omega} ((1 + \kappa(t))^2 (u + sw)w + \nabla (u + sw)\nabla w) \, dx$$

$$+ \int_{\Gamma_N} (\beta_1((1 + \kappa(t))(u + sw) - f) + \beta_2((1 + \kappa(t))(u + sw)e^{-(1 + \kappa(t))\tau(t)} + z_0)) \, (u - v) d\Gamma,$$

this follows from the continuity of $\beta_1$ and $\beta_2$.

It remains to check the coerciveness of $B$, i.e.

$$\frac{Bu(u)}{\|u\|_V} \to \infty \quad \text{if} \quad \|u\|_V \to +\infty. \tag{2.78}$$

From the definition of $B$, we have, since $1 + \kappa(t) > 1$,

$$Bu(u) \geq \|u\|^2 + \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f) \, ud\Gamma + \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1 + \kappa(t))\tau(t)} + z_0) \, ud\Gamma.$$

We deduce, by (1.24) and (1.21)

$$Bu(u) \geq \|u\|^2 + \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f) \, f d\Gamma$$

$$- \frac{e^{(1 + \kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1 + \kappa(t))\tau(t)} + z_0) \, z_0 d\Gamma.$$
By Cauchy-Schwarz’s inequality, (2.9), (1.20), a trace theorem and the fact that $1 + \kappa(t)$ is bounded, we obtain that there exists $C > 0$ such that

$$\left| \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f) f d\Gamma \right| \leq C \left( \| u \|_{H^1(\Omega)} + \| f \|_{H^1(\Omega)} \right) \| f \|_{H^1(\Omega)}$$

and

$$\left| \frac{e^{(1 + \kappa(t))t} t}{1 + \kappa(t)} \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1 + \kappa(t))t} + z_0) z_0 d\Gamma \right| \leq C \left( \| u \|_{H^1(\Omega)} + \| z_0 \|_{H^1(\Omega)} \right) \| z_0 \|_{H^1(\Omega)}.$$

Then

$$Bu(u) \geq \| u \|^2_V - C \left( \| u \|_{H^1(\Omega)} + \| f \|_{H^1(\Omega)} \right) \| f \|_{H^1(\Omega)} + \left( \| u \|_{H^1(\Omega)} + \| z_0 \|_{H^1(\Omega)} \right) \| z_0 \|_{H^1(\Omega)},$$

which implies (2.78).

Therefore, by Corollary II.2.2 of [29], there exists $u \in V$ solution of (2.76). If $\tau'(t) \neq 0$, we obtain the same result by similar arguments. This function $u$ satisfies (2.74) by choosing test function in $D(\Omega)$ and then satisfies

$$\frac{\partial u}{\partial \nu} = -\beta_1(v) - \beta_2(z(.,1)) \text{ on } \Gamma_N$$

by Green’s formula.

So we have found $(u, v, z)^T \in D(\bar{A}_-(-t))$ which verifies (2.65). \qed

It remains to show (iii) of Theorem 2.5 to finish the proof of Theorem 2.6. This is the aim of the following lemma.

**Lemma 2.9** Assume that (1.8), (1.13) and (2.9) hold. Then (2.58) holds.

**Proof.** Let $\phi = (u, v, z)^T \in D(\bar{A}_-(-t))$. By definition (2.63) of $\bar{A}_-(-t)$, we have

$$\left\| \bar{A}_-(-t)\phi - \bar{A}_-(s)\phi \right\|_0 = \|(\kappa(t) - \kappa(s))\phi - (A(t)\phi - A(s))\phi\|_0 \leq |\kappa(t) - \kappa(s)| \| \phi \|_0 + \| A(t)\phi - A(s)\phi \|_0. \tag{2.79}$$

As

$$\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2 + 1)^{\frac{3}{2}}} - \frac{\tau'(t)(\tau'(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)^2},$$

is bounded on $[0, T]$ for all $T > 0$ (by (2.9) and (1.8)), by the mean value theorem there exists $K > 0$ such that

$$|\kappa(t) - \kappa(s)| \| \phi \|_0 \leq K |t - s| \| \phi \|_0. \tag{2.80}$$

Moreover

$$\| A(t)\phi - A(s)\phi \|_0^2 = \xi \tau(0) \int_{\Gamma_N} \int_{0}^{1} \left( \frac{\tau'(t)\rho - 1}{\tau(t)} - \frac{\tau'(s)\rho - 1}{\tau(s)} \right)^2 \rho^2 d\rho d\Gamma.$$

In addition

$$\left( \frac{\tau'(t)\rho - 1}{\tau(t)} \right)' = \frac{\tau''(t)\tau(t) - \tau'(t)^2 \rho + \tau'(t)}{\tau(t)^2}$$

is bounded on $[0, T]$ for all $T > 0$ by (2.9) and (1.8). By the mean value theorem, we then obtain that there exists $K > 0$ such that

$$\| A(t)\phi - A(s)\phi \|_0^2 \leq K^2 |t - s|^2 \xi \tau(0) \int_{\Gamma_N} \int_{0}^{1} \rho^2(x, \rho) d\rho d\Gamma.$$
Moreover by (2.9) and (1.13)
\[
\left(\frac{\tau(t)}{\tau'(t)\rho - 1}\right)^2 \leq \frac{\tau^2}{(1 - d)^2}.
\]
Therefore
\[
\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0^2 \leq \left(\frac{K\tau}{1 - d}\right)^2 |t - s|^2 \xi\tau(0) \int_{\Gamma_N} \int_0^1 \left(\frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho(x, \rho)\right)^2 d\rho d\Gamma.
\]
This leads to
\[
\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0^2 \leq \left(\frac{K\tau}{1 - d}\right)^2 |t - s|^2 \|\mathcal{A}(t)\phi\|_0^2,
\]
and thus to
\[
\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0 \leq \left(\frac{K\tau}{1 - d}\right) |t - s| \|\mathcal{A}(t)\phi\|_0.
\]
By definition (2.63) of \(\tilde{\mathcal{A}}(t)\), we have
\[
\|\mathcal{A}(t)\phi\|_0 \leq \|\tilde{\mathcal{A}}(t)\phi\|_0 + \kappa(t) \|\phi\|_0,
\]
with \(\kappa(t)\) bounded on \([0, T]\). Consequently there exists \(C > 0\) such that
\[
\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0 \leq C |t - s| \left(\|\tilde{\mathcal{A}}(t)\phi\|_0 + \|\phi\|_0\right). \tag{2.81}
\]
Therefore (2.79), (2.80) and (2.81) imply (2.58). \(\blacksquare\)

**Proof of Theorem 2.6.** The assumptions of Theorem 2.5 were verified for \(\tilde{\mathcal{A}}(t)\) and the inner product \(\langle \cdot, \cdot \rangle_t\) defined by (2.62). Consequently the evolution equation

\[
\begin{cases}
\dot{U}'' + \tilde{\mathcal{A}}(t)\dot{U} = 0 \\
\dot{U}(0) = U_0 \in D(\mathcal{A}(t)),
\end{cases} \tag{2.82}
\]

has a unique solution \(\dot{U} \in C([0, T]; \mathcal{H})\) such that \(\dot{U}(t)\) belongs to \(D(\tilde{\mathcal{A}}(t))\) for all \(t \in [0, T]\), its strong derivative \(\ddot{U}(t) = -\tilde{\mathcal{A}}(t)\dot{U}(t)\) exists and is continuous except at a countable numbers of values \(t\).

The requested solution of (2.55) is then given by
\[
U(t) = e^{\beta(t)}\dot{U}(t)
\]
with \(\beta(t) = \int_0^t \kappa(s) ds\). \(\blacksquare\)

## 3 Stability result for the linear problem

In this section, we will give an exponential stability result for problem (1.1) - (1.5) under the assumption (1.12). We define the energy of system (1.1) - (1.5) as
\[
E(t) := \frac{1}{2} \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx + \frac{\xi}{2} \tau(t) \int_{\Gamma_N} \int_0^1 u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma, \tag{3.1}
\]
where \(\xi\) is a positive constant such that
\[
2\mu_1 - \frac{\mu_2}{\sqrt{1 - d}} - \xi > 0, \quad \text{and} \quad \xi - \frac{\mu_2}{\sqrt{1 - d}} > 0. \tag{3.2}
\]
Note that from (1.12) such a constant \(\xi\) exists. We have the following identity.
**Proposition 3.1** For any regular solution of problem (1.1) – (1.5) we have

\[ E'(t) = -\mu_1 \int_{\Gamma_N} u_t^2(x,t) d\Gamma - \int_{\Gamma_N} \mu_2 u_t(x,t) u_t(x,t - \tau(t)) d\Gamma \]

\[ -\frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t - \tau(t))(1 - \tau'(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t) d\Gamma. \]  

(3.3)

**Proof.** Differentiating (3.1) we obtain

\[ E'(t) = \int_{\Omega} \{ u_t u_{tt} + \nabla u \nabla u_t \} dx + \frac{\xi}{2} \tau'(t) \int_0^1 \int_{\Gamma_N} u_t^2(x,t - \tau(t)\rho) d\rho d\Gamma \]

\[ + \xi \tau(t) \int_{\Gamma_N} \int_0^1 u_t(x,t - \tau(t)\rho) u_{tt}(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho d\Gamma, \]

and then, applying Green’s formula,

\[ E'(t) = \int_{\Gamma_N} u_t \frac{\partial u}{\partial \nu} d\Gamma + \frac{\xi}{2} \tau'(t) \int_0^1 \int_{\Gamma_N} u_t^2(x,t - \tau(t)\rho) d\rho d\Gamma \]

\[ + \xi \tau(t) \int_{\Gamma_N} \int_0^1 u_t(x,t - \tau(t)\rho) u_{tt}(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho d\Gamma. \]  

(3.4)

Now, observe that, if \( \tau(t) \neq 0 \),

\[ u_t(x,t - \tau(t)\rho) = -\tau^{-1}(t) u_{\rho\rho}(x,t - \tau(t)\rho), \]

and

\[ u_{tt}(x,t - \tau(t)\rho) = \tau^{-2}(t) u_{\rho\rho}(x,t - \tau(t)\rho). \]

Therefore,

\[ \int_0^1 u_t(x,t - \tau(t)\rho) u_{tt}(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \]

\[ = -\tau^{-3}(t) \int_0^1 u_{\rho \rho}(x,t - \tau(t)\rho) u_{\rho}(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \]

\[ = -\tau^{-3}(t) [ u_{\rho \rho}^2(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) ]_0^1 \]

\[ + \tau^{-3}(t) \int_0^1 u_{\rho}(x,t - \tau(t)\rho) u_{\rho}(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \]

\[ - \tau'(t) \tau^{-3}(t) \int_0^1 u_{\rho}(x,t - \tau(t)\rho) u_{\rho}(x,t - \tau(t)\rho) d\rho. \]

(3.5)

Then, from (3.5),

\[ \int_0^1 u_t(x,t - \tau(t)\rho) u_{tt}(x,t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \]

\[ = -\frac{1}{2} \tau'(t) \tau^{-3}(t) \int_0^1 u_{\rho}^2(t - \tau(t)\rho) d\rho \]

\[ - \frac{\tau^{-1}(t)}{2} u_{t}^2(x,t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_{t}^2(x,t) \]

\[ = -\frac{1}{2} \tau'(t) \tau^{-1}(t) \int_0^1 u_{t}^2(x,t - \tau(t)\rho) d\rho \]

\[ - \frac{\tau^{-1}(t)}{2} u_{t}^2(x,t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_{t}^2(x,t). \]  

(3.6)
Using (3.4), (3.6) and the boundary condition (1.3) on $\Gamma_N$, we have

$$E'(t) = -\mu_1 \int_{\Gamma_N} u_t^2(x,t) d\Gamma - \mu_2 \int_{\Gamma_N} u_t(x,t) u_t(x,t - \tau(t)) d\Gamma$$
$$\quad + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t) d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t - \tau(t))(1 - \tau'(t)) d\Gamma.$$  \hspace{1cm} (3.7)

So, for any time $t$ such that $\tau(t) \neq 0$, the identity (3.3) is proved.

Now, let $t$ be such that $\tau(t) = 0$. Then, from (3.4) we have

$$E'(t) = -(\mu_1 + \mu_2) \int_{\Gamma_N} u_t^2(x,t) d\Gamma + \frac{\xi}{2} \tau'(t) \int_{\Gamma_N} u_t^2(x,t) d\Gamma.$$ \hspace{1cm} (3.8)

Therefore, identity (3.3) is proved for all times $t > 0$.  \hfill \Box

**Proposition 3.2** For any regular solution of problem (1.1) – (1.5) the energy decays and there exists a positive constant $C$ such that

$$E'(t) \leq -C \int_{\Gamma_N} \{u_t^2(x,t) + u_t^2(x,t - \tau(t))\} d\Gamma.$$ \hspace{1cm} (3.9)

**Proof.** In the case of $\tau(t) \neq 0$, from (3.7), applying Cauchy-Schwarz’s inequality, we obtain

$$E'(t) \leq -\mu_1 \int_{\Gamma_N} u_t^2(x,t) d\Gamma + \frac{1}{\sqrt{1 - d}} \frac{\mu_2}{2} \int_{\Gamma_N} u_t^2(x,t) d\Gamma + \sqrt{1 - d} \frac{\mu_2}{2} \int_{\Gamma_N} u_t^2(x,t - \tau(t)) d\Gamma$$
$$\quad - \frac{\xi}{2} (1 - \tau'(t)) \int_{\Gamma_N} u_t^2(x,t - \tau(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t) d\Gamma,$$

from which easily follows (3.9) recalling (3.2). In the case of $\tau(t) = 0$, from (3.8) easily follows (3.9) observing that by (3.2)

$$\xi < 2\mu_1 < \frac{2(\mu_1 + \mu_2)}{d}. \quad \Box$$

**Remark 3.3** The choice to apply Cauchy-Schwarz’s inequality with a factor $\sqrt{1 - d}$ in the proof of the above proposition is made in order to give the stability result under the best assumption between $\mu_1$ and $\mu_2$.

Now, let us introduce the Lyapunov functional

$$\dot{E}(t) = E(t) + \gamma \left\{ \int_{\Omega} [2m \cdot \nabla u + (n - 1)u] u_t dx + E(t) \right\},$$ \hspace{1cm} (3.10)

where $\gamma$ is a positive small constant that we will choose later on and $E(t)$ is defined by

$$E(t) := \xi \tau(t) \int_{\Gamma_N} e^{-2\tau(t)\rho} u_t^2(x,t - \tau(t)\rho) d\rho d\Gamma.$$ \hspace{1cm} (3.11)

Note that, from Poincaré’s Theorem, the functional $\dot{E}$ is equivalent to the energy $E$, that is there exist two positive constant $d_1, d_2$ such that

$$d_1 \dot{E}(t) \leq E(t) \leq d_2 \dot{E}(t), \quad \forall \ t \geq 0.$$ \hspace{1cm} (3.12)

Moreover, we denote by $E_S(\cdot)$ the standard energy for wave equation without delay, that is

$$E_S(t) := \frac{1}{2} \int_{\Omega} (u_t^2(x,t) + |\nabla u(x,t)|^2) dx.$$ \hspace{1cm} (3.13)

The following estimate holds true.
Lemma 3.4 For any regular solution of problem (1.1) – (1.5),

\[
\frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u_t]u_t \, dx \right\} \leq -C_0 E_S(t) + C \left\{ \int_{\Gamma_N} [u_t^2(x,t) + u_t^2(x,t-\tau(t))] \, d\Gamma \right\},
\]

(3.14)

for suitable positive constants \(C_0, C\).

Proof. The standard multiplier identity gives

\[
\frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u_t]u_t \, dx \right\} = -\int_{\Omega} \left\{ u_t^2 + |\nabla u|^2 \right\} \, dx + \int_{\Gamma_N} (m \cdot \nu)(u_t^2 - |\nabla u|^2) \, d\Gamma + \int_{\Gamma_N} [2m \cdot \nabla u + (n-1)u] \frac{\partial u}{\partial \nu} \, d\Gamma.
\]

(3.15)

From (3.15) and Young’s inequality, recalling that by (1.10) \(m \cdot \nu \geq \delta\) on \(\Gamma_N\), we have

\[
\frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u_t]u_t \, dx \right\} \leq -\int_{\Omega} \left\{ u_t^2 + |\nabla u|^2 \right\} \, dx + \int_{\Gamma_N} (m \cdot \nu)u_t^2 \, d\Gamma - \delta \int_{\Gamma_N} |\nabla u|^2 \, d\Gamma + \frac{\epsilon}{\xi} \int_{\Gamma_N} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\Gamma
\]

\[
+ \epsilon \int_{\Gamma_N} \left( |\nabla u|^2 + u^2 \right) \, d\Gamma,
\]

(3.16)

for some positive constants \(\epsilon, c\). From (3.16), using the trace’s inequality and Poincaré’s Theorem, for \(\epsilon\) small enough we deduce

\[
\frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u_t]u_t \, dx \right\} \leq -C_0 E_S(t)
\]

\[
+ C \int_{\Gamma_N} u_t^2 \, d\Gamma + C \int_{\Gamma_N} \left( \frac{\partial u}{\partial \nu} \right)^2 \, d\Gamma,
\]

(3.17)

for suitable positive constants \(C_0, C\). Therefore, using the boundary condition (1.3) and Cauchy-Schwarz’s inequality in (3.17), we obtain (3.14) \(\blacksquare\).

We can also estimate the component \(E(\cdot)\) in the Lyapunov functional (3.10).

Lemma 3.5 For any regular solution of problem (1.1) – (1.5),

\[
\frac{d}{dt} E(t) \leq -2E(t) + \xi \int_{\Gamma_N} u_t^2 \, d\Gamma.
\]

(3.18)

Proof. Differentiating (3.11) we have

\[
\frac{d}{dt} E(t) = \xi \tau'(t) \int_{\Gamma_N} \int_{0}^{1} e^{-2\tau(t)\rho} u_t^2(x,t-\tau(t)\rho) \, d\rho \, d\Gamma
\]

\[
-2\xi \tau'(t) \int_{\Gamma_N} \int_{0}^{1} e^{-2\tau(t)\rho} u_t^2(x,t-\tau(t)\rho) \, d\rho \, d\Gamma
\]

\[
+ 2\xi \tau(t) \int_{\Gamma_N} \int_{0}^{1} e^{-2\tau(t)\rho} u_t(x,t-\tau(t)\rho) u_t(x,t-\tau(t)\rho)(1 - \tau'(t)\rho) \, d\rho \, d\Gamma.
\]

(3.19)
Now, let us suppose \( \tau(t) \neq 0 \) and integrate by parts the last term in (3.19). We obtain

\[
\int_0^1 e^{-2\tau(t)\rho} u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \\
= -\tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} \rho u_t(x, t - \tau(t)\rho) u_{\rho\rho}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \\
= \tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} u_{\rho}(x, t - \tau(t)\rho) u_{\rho\rho}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \\
- \tau'(t) \tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho) d\rho \\
- 2\tau^{-2}(t) \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \\
- \tau^{-3}(t) \left[ e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) \right]_0^1
\]

and then

\[
\int_0^1 e^{-2\tau(t)\rho} u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \\
= -\frac{1}{2} \tau'(t) \tau^{-1}(t) \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho) d\rho \\
- \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho \\
- \frac{\tau^{-1}(t)}{2} e^{-2\tau(t)} u_{\rho}^2(x, t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_{\rho}^2(x, t).
\]

Now, substituting identity (3.21) in (3.19), we obtain

\[
\frac{d}{dt} \mathcal{E}(t) = \xi \tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
- 2\xi \tau'(t) \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} \rho u_{\rho}^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
- \xi \tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
- 2\xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho)(1 - \tau'(t)\rho) d\rho d\Gamma \\
- \xi e^{-2\tau(t)} \int_{\Gamma_N} u_{\rho}^2(x, t - \tau(t))(1 - \tau'(t)) d\Gamma + \xi \int_{\Gamma_N} u_{\rho}^2(x, t) d\Gamma,
\]

and so

\[
\frac{d}{dt} \mathcal{E}(t) = -2\xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_{\rho}^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
- \xi e^{-2\tau(t)} \int_{\Gamma_N} u_{\rho}^2(x, t - \tau(t))(1 - \tau'(t)) d\Gamma + \xi \int_{\Gamma_N} u_{\rho}^2(x, t) d\Gamma
\]

from which immediately follows estimate (3.18) for \( t \) such that \( \tau(t) \neq 0 \). In the case of \( \tau(t) = 0 \), note
that from (3.19) we have

\[
\frac{d}{dt} \hat{E}(t) = \xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_\tau^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
\leq \xi d \int_{\Gamma_N} \int_0^1 u_\tau^2(x, t) d\rho d\Gamma = \xi d \int_{\Gamma_N} u_i^2(x, t) d\Gamma \\
= \xi d \int_{\Gamma_N} u_i^2(x, t) d\Gamma - 2\hat{E}(t).
\]  

(3.24)

Then, even in this case, we obtain (3.18). □

Now, we can deduce the exponential stability estimate for problem (1.1) – (1.5).

**Theorem 3.6** Assume (1.12). There exist positive constants \(C_1, C_2\) such that for any solution of problem (1.1) – (1.5),

\[
E(t) \leq C_1 E(0) e^{-C_2 t}, \quad \forall t \geq 0.
\]  

(3.25)

**Proof.** From Proposition 3.2, Lemma 3.4 and Lemma 3.5, we have

\[
\frac{d}{dt} \hat{E}(t) \leq -C \left\{ \int_{\Gamma_N} [u_\tau^2(x, t) + u_\tau^2(x, t - \tau(t))] d\Gamma \right\} \\
+ \gamma \left( -C_0 E_S(t) + \tilde{C} \int_{\Gamma_N} [u_\tau^2(x, t) + u_\tau^2(x, t - \tau(t))] d\Gamma - 2\hat{E}(t) \right).
\]  

(3.26)

Then, for \(\gamma\) sufficiently small, we can estimate

\[
\frac{d}{dt} \hat{E}(t) \leq -\gamma C_0 E_S(t) - 2\gamma \hat{E}(t).
\]  

(3.27)

Now, observe that by assumption (1.6) on \(\tau(t)\), we can deduce

\[
E(t) \geq \xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_i^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
\geq \frac{c_\xi \tau(t)}{2} \int_{\Gamma_N} \int_0^1 u_i^2(x, t - \tau(t)\rho) d\rho d\Gamma,
\]  

(3.28)

for some positive constant \(c\).

Therefore, from (3.27) and (3.28),

\[
\frac{d}{dt} \hat{E}(t) \leq -\gamma C_0 E_S(t) - 2\gamma \hat{E}(t) \leq -cE(t) \leq -C \hat{E}(t),
\]  

(3.29)

for suitable positive constants \(c, C\), where we used also the first inequality in (3.12). This clearly implies

\[
\hat{E}(t) \leq e^{-Ct} \hat{E}(0),
\]

and so, using (3.12),

\[
E(t) \leq C_1 e^{-C_2 t} E(0),
\]

for suitable constants \(C_1, C_2 > 0\). □
4 Nonlinear stability result

In this section we consider the problem (1.15) – (1.19) with $\beta_1, \beta_2$ satisfying (1.20), (1.24). Moreover we assume

$$\gamma_1 > \frac{c_2}{\sqrt{1-d}},$$

(4.1)

where $\gamma_1, c_2$ are the constants in (1.20) and (1.24) (which comes from (1.20) and (1.22)) and $d$ is as in (1.13).

We define the energy associated to the problem as in (3.1) with the constant $\xi$ such that

$$2\gamma_1 - \frac{c_2}{\sqrt{1-d}} - \xi > 0 \quad \text{and} \quad \xi - \frac{c_2}{\sqrt{1-d}} > 0.$$  

(4.2)

Note that, from assumption (4.1), such a constant $\xi$ exists.

Notice that (2.60) implies (4.1). Moreover the existence of $\xi$ verifying (2.61) guarantees that $\xi$ verifies (4.2), since $c_2 \leq \gamma_2$.

The following identity holds true.

**Proposition 4.1** For any regular solution of problem (1.15) – (1.18) we have

$$E'(t) = -\int_{\Gamma_N} u_t(x,t)\beta_1(u_t(x,t))d\Gamma - \int_{\Gamma_N} u_t(x,t)\beta_2(u_t(x,t - \tau(t)))d\Gamma$$

$$-\frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t - \tau(t))(1 - \tau'(t))d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t)d\Gamma.$$  

(4.3)

**Proof.** The proof is analogous to the one of Proposition 3.1, so we omit the details. $\blacksquare$

**Proposition 4.2** For any regular solution of problem (1.15) – (1.18) the energy decays and there exists a positive constant $C$ such that

$$E'(t) \leq -C \int_{\Gamma_N} \{u_t^2(x,t) + u_t^2(x,t - \tau(t))\}d\Gamma.$$  

(4.4)

**Proof.** In the case of $\tau(t) \neq 0$, from (4.3), we obtain, by (1.20) and (1.24),

$$E'(t) \leq -\gamma_1 \int_{\Gamma_N} u_t^2(x,t)d\Gamma + \int_{\Gamma_N} c_2|u_t(x,t)||u_t(x,t - \tau(t))|d\Gamma$$

$$-\frac{\xi}{2} (1 - \tau'(t)) \int_{\Gamma_N} u_t^2(x,t - \tau(t))d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t)d\Gamma.$$  

Then, applying Cauchy-Schwarz’s inequality, we have

$$E'(t) \leq -\gamma_1 \int_{\Gamma_N} u_t^2(x,t)d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x,t)d\Gamma$$

$$-\frac{\xi}{2} (1 - \tau'(t)) \int_{\Gamma_N} u_t^2(x,t - \tau(t))d\Gamma + \frac{c_2}{2\sqrt{1-d}} \int_{\Gamma_N} u_t^2(x,t)d\Gamma$$

$$+\sqrt{1-d} \frac{c_2}{2} \int_{\Gamma_N} u_t^2(x,t - \tau(t))d\Gamma.$$  

(4.5)

From (4.5) estimate (4.4) easily follows recalling that $\xi$ satisfies (4.2).
If \( t \) is such that \( \tau(t) = 0 \), then from (4.3) we deduce

\[
E'(t) = -\int_{\Gamma_N} u_t(t) \beta_1(u_t(t)) d\Gamma - \int_{\Gamma_N} u_t(t) \beta_2(u_t(t)) d\Gamma + \frac{\xi}{2} \tau'(t) \int_{\Gamma_N} u_t^2(t) d\Gamma.
\]

Then, from (1.24) and (1.21)

\[
E'(t) \leq -\left( \gamma_1 - \frac{\xi}{2} d \right) \int_{\Gamma_N} u_t^2(t) d\Gamma,
\]

and this clearly gives (4.4) observing that by (4.2)

\[
\gamma_1 > \frac{\xi}{2} + \frac{c_2}{2\sqrt{1-d}} > \frac{\xi}{2} > \frac{\xi}{2} d.
\]

Now, let \( \hat{E} \) be the Lyapounov functional introduced in (3.10) with a small enough positive constant \( \gamma \) and let \( \mathcal{E} \) be defined as in (3.11).

Even in this case lemma 3.4 holds true. Indeed inequality (3.17) is obtained without using the boundary condition (1.3) on \( \Gamma_N \). From (3.17) we easily deduce estimate (3.14) for suitable positive constants \( C_0, C \), using the boundary condition (1.17) and the assumptions (1.20) on the functions \( \beta_1, \beta_2 \).

We can estimate also the component \( \mathcal{E}(\cdot) \) in the Lyapounov functional (3.10) as in the previous case, and so analogously to the linear case, we can deduce an exponential stability estimate for problem (1.15) – (1.19).

**Theorem 4.3** Assume (4.1). There exist positive constants \( D_1, D_2 \) such that for any solution of problem (1.15) – (1.19),

\[
E(t) \leq D_1 E(0) e^{-D_2 t}, \quad \forall t \geq 0.
\]  

**References**


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