Stabilization of the Schrödinger equation with a delay term in boundary feedback or internal feedback

Serge Nicaise * and Salah-eddine Rebiai †

October 27, 2009

Abstract

In this paper, we investigate the effect of time delays in boundary or internal feedback stabilization of the multidimensional Schrödinger equation. In both cases, under suitable assumptions, we establish sufficient conditions on the delay term that guarantee the exponential stability of the solution. These results are obtained by using suitable energy functionals and some observability estimates.

Key words. Schrödinger equation, time delays, feedback stabilization.

1 Introduction

It is well known that certain infinite dimensional damped second order systems become unstable when arbitrary small time delays occur in the damping (see e.g. [4]). This lack of stability robustness was first shown to hold for the one-dimensional wave equation (see [3]). Later further examples illustrating this phenomenon were given in [2]: the two-dimensional wave equation with damping introduced through Neumann-type boundary conditions on one edge of a square boundary and the Euler-Bernoulli beam equation in one dimension with damping introduced through a specific set of boundary conditions on the right end point.

More recently, Xu et al [17] established sufficient conditions that guarantee the stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [11] extended this result to the multidimensional wave equation with a delay term in the boundary or internal feedbacks; they further underline some instability phenomena. Similar
results were obtained by Nicaise and Valein [12] for a class of second order evolution equations in one-dimensional networks with delay in unbounded feedbacks.

Motivated by the papers [17, 11, 12], we analyze in this paper the effect of time delays in internal feedback or boundary feedback stabilization of the Schrödinger equation in general domains of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a boundary $\Gamma$ of class $C^2$. Let $(\Gamma_0, \Gamma_1)$ be a partition of $\Gamma$ i.e. $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. In addition to these standard hypotheses, we assume the following.

(A) There exists a real-valued vector field $h \in (C^2(\Omega))^n$ such that

1. $h$ is coercive in $\Omega$, that is there exists $\alpha > 0$ such that the Jacobian matrix $J$ of $h$ satisfies

$\Re(J(x)\xi \cdot \xi) \geq \alpha |\xi|^2, \forall x \in \Omega, \xi \in \mathbb{C}^n$

2. $h(x) \cdot \nu(x) \leq 0$ for all $x \in \Gamma_0$.

where $\nu(x)$ is the unit normal to $\Gamma$ at $x \in \Gamma$ pointing towards the exterior of $\Omega$ and $\Re z$ means the real part of the complex number $z$.

Remark 1 A particular example of a vector field $h$ satisfying Assumption A is the radial vector field $h(x) = x - x_0$ for some $x_0 \in \mathbb{R}^n$. Another example is given by $h(x) = \nabla d(x)$ where $d$ is a real strictly convex function in $\Omega$. For further examples see [15] and the references therein.

In this paper, we are interested in the asymptotic behaviour of the solution of the initial boundary value problem

$$
\begin{align*}
&y_t(x, t) - i\Delta y(x, t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\
&y(x, 0) = y_0(x) \quad \text{in} \quad \Omega, \\
&y(x, t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \\
&\frac{\partial y}{\partial \nu}(x, t) = i\mu_1 y(x, t) + i\mu_2 y(x, t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \\
&y(x, t - \tau) = f_0(x, t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, \tau),
\end{align*}
$$

where $\frac{\partial y}{\partial \nu}$ is the normal derivative, $\tau$ is the time delay, $\mu_1$ and $\mu_2$ are positive real numbers.

In the absence of delay, that is $\mu_2 = 0$, Lasiecka et al [6] have shown that the solution of (1) decays exponentially to zero in the energy space $L^2(\Omega)$. If $\mu_2 > 0$, according to the results from [4, 3, 2, 17, 11, 12], we may expect to encounter either instability results or stability results according to the value of $\mu_2$ with respect to $\mu_1$.

Hence the main purpose of this work is to provide sufficient conditions on the coefficients $\mu_1$ and $\mu_2$ that guarantee that the system (1) remains exponentially stable. Indeed, we show as in [11, 12] that the exponential stability is preserved if

$$
(2) \quad \mu_1 > \mu_2.
$$
This is done, as in [11], by introducing the energy functional

\[ E(t) = \frac{1}{2} \int_\Omega |y(x,t)|^2 \, dx + \frac{\xi}{2} \int_{\Gamma_1} \int_0^1 |y(x,t - \tau \rho)|^2 \, d\rho \, d\sigma(x), \]  

where

\[ \tau \mu_2 < \xi < \tau (2\mu_1 - \mu_2), \]

and using an energy estimate at the \( L^2(\Omega) \) level for a fully Schrödinger equation with gradient and potential terms stated in Theorem 2.6.1 of [5] and established in Section 10 of [6]. This result can be summarized as follows: Assume that the hypothesis (A) holds and let \( y \) be a smooth solution of the partial differential equation in (1) satisfying

\[ y(x,t) = 0 \text{ on } \Gamma_0 \times (0,T) \]

Then there exists a constant \( c > 0 \) depending on \( T \) such that

\[
\int_\Omega |y(x,0)|^2 \, dx \leq c \left\{ \|y\|_{L^2(0,T;L^2(\Gamma_1))}^2 + \int_0^T \int_{\Gamma_1} \left| \frac{\partial y(x,t)}{\partial \nu} \right| |y(x,t)| \, d\sigma(x) dt + \left\| \frac{\partial y}{\partial \nu} \right\|_{H^{-1}(\Omega)}^2 + \|y\|_{H^{-1}(\Omega)}^2 \right\}
\]  

In (5), \( H^{-1}(\Omega) \) is the dual space of the space

\[ H^1((0,T) \times \Gamma_1) = H^{1/2}(0,T; L^2(\Gamma_1)) \cap L^2(0,T; H^1(\Gamma_1)) \]

with respect to the pivot space \( L^2((0,T) \times \Gamma_1) \).

On the contrary if \( \mu_2 \geq \mu_1 \), we show that some instability results may appear, namely we show that there exists a sequence of delays for which the system (1) is not asymptotically stable. To be more precise, our results concerning the system (1) are as follows.

**Theorem 2** Assume that there exists a vector field \( h \) satisfying (A), that \( \mu_1 > \mu_2 \) (see (2)) and that the energy \( E \) of the system (1) is given by (3) with \( \tau \mu_2 < \xi < \tau (2\mu_1 - \mu_2) \). Then there exist constants \( M_b \geq 1 \) and \( \delta_0 > 0 \) such that

\[ E(t) \leq M_b e^{-\delta_0 t} E(0). \]

**Theorem 3** If \( \mu_1 \leq \mu_2 \) (i.e. (2) is not satisfied), then there exists a sequence of delays for which the problem (1) is not asymptotically stable.
Remark 4 Theorem 2 remains true if the Laplacian is replaced by a second order elliptic differential operator with space variable coefficients. To this end, one invokes the Riemannian geometric approach of [16] and Remark 2.6.2. of [5].

In this paper, we also investigate the stability of the Schrödinger equation with a distributed delay term. More precisely, we consider the system described by

\[
\begin{align*}
    &y_t(x,t) = i\Delta y(x,t) - a(x)\{\mu_1 y(x,t) + \mu_2(x,t) y(x,t - \tau)\} \quad \text{in } \Omega \times (0, +\infty), \\
    &y(x,0) = y_0(x) \quad \text{in } \Omega, \\
    &y(x,t) = 0 \quad \text{on } \Gamma \times (0, +\infty), \\
    &y(x,t - \tau) = g_0(x, t - \tau) \quad \text{in } \Omega \times (0, \tau).
\end{align*}
\]

In (6) \(a(.)\) is an \(L^\infty(\Omega)\)--function which satisfies

\[
a(x) \geq 0 \ a.e. \ in \ \Omega \ \text{and} \ a(x) > a_0 > 0 \ a.e. \ in \ \omega,
\]

where \(\omega \subset \Omega\) is an open neighborhood of \(\Gamma_0\).

In [10], Machtyngier and Zuazua have shown in the case \(\mu_2 = 0\) that the \(L^2(\Omega)\)--energy of the solution of (6) decays exponentially to zero. Their proof relies on an observability inequality established previously by the first author in [9]. We use this inequality together with (2) to establish the exponential decay of the energy of the solution of the system (6) defined by

\[
F(t) = \frac{1}{2} \int_\Omega |y(x,t)|^2 \, dx + \frac{\xi}{2} \int_\Omega a(x) \int_0^1 |y(x,t - \tau \rho)|^2 \, d\rho \, dx.
\]

As before if \(\mu_2 \geq \mu_1\), we construct an explicit sequence of delays that destabilize the system.

The main results concerning the problem (6) can be summarized as follows.

**Theorem 5** Assume that there exists a vector field \(h\) satisfying (A), that \(\mu_1 > \mu_2\) (see (2)) and that the energy \(F\) of the system (6) is given by (8) with \(\tau \mu_2 < \xi < \tau (2\mu_1 - \mu_2)\). Then there exist constants \(M_d \geq 1\) and \(\delta_d > 0\) such that

\[
F(t) \leq M_d e^{-\delta_d t} F(0).
\]

**Theorem 6** If \(\mu_1 \leq \mu_2\) (i.e. (2) is not satisfied), then there exists a sequence of delays for which the problem (6) is not asymptotically stable.

The paper is organized as follows. Theorem 2 and Theorem 3 are proved in Section 2 whereas Section 3 contains the proof of Theorem 5 and Theorem 6. Both sections start with the study of the well-posedness of the system under consideration.

4
2 Stability of the Schrödinger equation with a delay term in the boundary feedback

2.1 Well-posedness of the system (1)

In order to be able to manage the boundary condition with the delay term and inspired from [17, 11] we introduce the auxiliary variable \( z(x, \rho, t) = y(x, t - \tau \rho) \). With this new unknown, problem (1) is equivalent to

\[
\begin{align*}
\begin{cases}
    y_t(x, t) - i\Delta y(x, t) = 0 & \text{in } \Omega \times (0, +\infty), \\
    z_t(x, \rho, t) + \frac{1}{\tau} z_{\rho}(x, \rho, t) = 0 & \text{on } \Gamma_1 \times (0, 1) \times (0, +\infty), \\
    y(x, 0) = y_0(x) & \text{in } \Omega, \\
    z(x, \rho, 0) = f_0(x, -\rho \tau) & \text{on } \Gamma_1 \times (0, 1), \\
    y(x, t) = 0 & \text{on } \Gamma_0 \times (0, +\infty), \\
    \frac{\partial y}{\partial \nu}(x, t) = i\mu_1 y(x, t) + i\mu_2 z(x, 1, t) & \text{on } \Gamma_1 \times (0, +\infty), \\
    z(x, 0, t) = y(x, t) & \text{on } \Gamma_1 \times (0, +\infty).
\end{cases}
\end{align*}
\]

Let us define on the Hilbert space

\[ H = L^2(\Omega) \times L^2(\Gamma_1 \times (0, 1)), \]

the inner product

\[
\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \Re \int_{\Omega} y_1(x) y_2(x) \, dx + \xi \Re \int_{\Gamma_1} \int_0^1 z_1(x, \rho) z_2(x, \rho) d\rho \, d\sigma(x)
\]

Define further

\[ H_{1,0}^1(\Omega) = \{ v \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \} \]

Setting \( Y(t) = \begin{pmatrix} y(\cdot, t) \\ z(\cdot, \cdot, t) \end{pmatrix} \), (from now on the notation \( y(\cdot, t) \) (resp. \( z(\cdot, \cdot, t) \)) means the function that maps \( x \) to \( y(x, t) \) (resp. the function that maps \( (x, \rho) \) to \( z(x, \rho, t) \))) we may rewrite problem (9) as follows

\[
\begin{align*}
\begin{cases}
    \frac{d}{dt} Y(t) = AY(t), \\
    Y(0) = (y_0, f_0^\tau),
\end{cases}
\end{align*}
\]

where \( f_0^\tau \) means the function that maps \( (x, \rho) \) to \( f_0(x, -\rho \tau) \) and the operator \( A \) is defined by

\[
A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} i\Delta & 0 \\ 0 & -\tau^{-1} \frac{\partial}{\partial \rho} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},
\]

5
with domain $D(A)$ defined by
\[
D(A) = \{ (y, z) \in (H^{3/2}(\Omega) \cap H^1_0(\Omega)) \times L^2(\Gamma_1; H^1(0, 1)) : \]
\[
\Delta y \in L^2(\Omega), \quad \frac{\partial y}{\partial \nu} = i \mu_1 y + i \mu_2 z(\cdot, 1) \text{ on } \Gamma_1, \quad y = z(\cdot, 0) \text{ on } \Gamma_1 \}.
\]

**Theorem 7** For any initial data $Y_0 \in H$, there exists a unique (weak) solution $Y \in C([0, +\infty); H)$ of (9). If in addition we assume that $Y_0 \in D(A)$, then the solution $Y \in C(0, +\infty; D(A)) \cap C^1(0, +\infty; H)$ and is called a strong solution.

**Proof.** The wellposedness of the problem (9) or its abstract version (10) follows via Lumer-Phillips Theorem (see for instance Theorem I.4.3 of [13]).

Let $Y = \begin{pmatrix} y \\ z \end{pmatrix} \in D(A)$. Then
\[
\Re \langle AY, Y \rangle = \Re \left( \begin{pmatrix} i \Delta y \\ -\tau^{-1} z_\rho \end{pmatrix} ; \begin{pmatrix} y \\ z \end{pmatrix} \right) = \Re \int_\Omega i \Delta y(x) \overline{y(x)} \, dx - \xi \tau^{-1} \Re \int_{\Gamma_1} \int_0^1 z_\rho(x, \rho) \overline{z(x, \rho)} \, d\rho \, d\sigma(x).
\]

From Green’s second theorem, we have
\[
\Re \langle AY, Y \rangle = \Re \int_{\Gamma_1} i \frac{\partial y}{\partial \nu}(x) \overline{y(x)} \, d\sigma(x) - \xi \tau^{-1} \Re \int_{\Gamma_1} \int_0^1 z_\rho(x, \rho) \overline{z(x, \rho)} \, d\rho \, d\sigma(x).
\]

Integrating by parts in $\rho$, we obtain
\[
\int_{\Gamma_1} \int_0^1 z_\rho(x, \rho) \overline{z(x, \rho)} \, d\rho \, d\sigma(x) = - \int_{\Gamma_1} \int_0^1 z(x, \rho) \overline{z_\rho(x, \rho)} \, d\rho \, d\sigma(x) + \int_{\Gamma_1} \left( |z(x, 1)|^2 - |z(x, 0)|^2 \right) \, d\sigma(x),
\]

or equivalently
\[
2\Re \int_{\Gamma_1} \int_0^1 z_\rho(x, \rho) \overline{z(x, \rho)} \, d\rho \, d\sigma(x) = \int_{\Gamma_1} \left( |z(x, 1)|^2 - |z(x, 0)|^2 \right) \, d\sigma(x).
\]

Therefore
\[
(12) \quad \Re \langle AY, Y \rangle = \Re \int_{\Gamma_1} i \frac{\partial y}{\partial \nu}(x) \overline{y(x)} \, d\sigma(x) - \frac{\xi \tau^{-1}}{2} \int_{\Gamma_1} \left( |z(x, 1)|^2 - |z(x, 0)|^2 \right) \, d\sigma(x).
\]
Insertion of the boundary conditions in (9) into (12) yields
\[ \Re \langle AY, Y \rangle = -\mu_1 \int_{\Gamma_1} |y(x)|^2 \, d\sigma(x) - \mu_2 \Re \int_{\Gamma_1} z(x, 1) y(x) \, dx - \frac{\xi \tau^{-1}}{2} \int_{\Gamma_1} \left( |z(x, 1)|^2 - |z(x, 0)|^2 \right) \, d\sigma(x), \]
from which follows, using the Cauchy-Schwarz inequality
\[ \Re \langle AY, Y \rangle \leq -\left( \mu_1 - \frac{\mu_2}{2} - \frac{\xi \tau^{-1}}{2} \right) \int_{\Gamma_1} |y(x)|^2 \, d\sigma(x) - \left( \frac{\xi \tau^{-1}}{2} - \frac{\mu_2}{2} \right) \int_{\Gamma_1} |z(x, 1)|^2 \, d\sigma(x). \]
From (4), we conclude that \[ \Re \langle AY, Y \rangle \leq 0. \]
Thus \( A \) is dissipative.

Now, we show that for a fixed \( \lambda > 0 \) and \((g, h) \in H\), there exists \( Y = (y, z) \in D(A) \) such that
\[ (\lambda I - A) \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}, \]
or equivalently
\begin{align*}
(13) & \quad \lambda y - i \Delta y = g, \\
(14) & \quad \lambda z + \tau^{-1} z_{\rho} = h.
\end{align*}
Suppose that we have found \( y \) with the appropriate regularity, then we can determine \( z \). Indeed, from (14) and the last line of (9) we have
\[ z_{\rho}(x, \rho) = -\lambda \tau z(x, \rho) + \tau h(x, \rho), \quad x \in \Gamma_1, \rho \in (0, 1), \]
\[ z(x, 0) = y(x), \quad x \in \Gamma_1. \]
The unique solution of the above initial value problem is given by
\[ z(x, \rho) = y(x)e^{-\lambda \tau} + \tau e^{-\lambda \tau \rho} \int_{0}^{\rho} h(x, \sigma)e^{\lambda \tau \sigma} \, d\sigma(x), \quad x \in \Gamma_1, \rho \in (0, 1), \]
and in particular
\[ z(x, 1) = y(x)e^{-\lambda \tau} + \tau e^{-\lambda \tau} \int_{0}^{1} h(x, \sigma)e^{\lambda \tau \sigma} \, d\sigma(x), \quad x \in \Gamma_1. \]
Problem (13) can be reformulated as follows
\[ \int_{\Omega} (\lambda y - i \Delta y) \overline{w} \, dx = \int_{\Omega} g \overline{w} \, dx, \quad \forall w \in L^2(\Omega). \]
Integrating by parts, we get

\[
\int_{\Omega} (\lambda y - i \Delta y) \overline{w} \, dx = \int_{\Omega} (\lambda y \overline{w} + i \nabla y \nabla \overline{w}) \, dx - i \int_{\Gamma_1} \frac{\partial y}{\partial \nu} \overline{w} \, d\sigma(x)
\]

\[
= \int_{\Omega} (\lambda y \overline{w} + i \nabla y \nabla \overline{w}) \, dx + \int_{\Gamma_1} (\mu_1 y \overline{w} + \mu_2 z(x, 1) \overline{w}) \, d\sigma(x) \quad \forall w \in H^1_{\Gamma_0}(\Omega).
\]

Therefore (15) can be rewritten as follows

\[
\int_{\Omega} (\lambda y \overline{w} + i \nabla y \nabla \overline{w}) \, dx + \int_{\Gamma_1} (\mu_1 y \overline{w} + \mu_2 e^{-\lambda \tau} y \overline{w}) \, d\sigma(x) = \int_{\Omega} g \overline{w} \, dx - \int_{\Gamma_1} (\tau e^{-\lambda \tau} \int_0^1 h(x, \sigma) e^{\lambda \tau \sigma} \, d\sigma(x)) \overline{w} \, d\sigma(x) \quad \forall w \in H^1_{\Gamma_0}(\Omega).
\]

Multiplying this equation by \(1 - i\), we obtain

\[
(1 - i) \int_{\Omega} (\lambda y \overline{w} + i \nabla y \nabla \overline{w}) \, dx + (1 - i) \int_{\Gamma_1} (\mu_1 + \mu_2 e^{-\lambda \tau}) y \overline{w} \, d\sigma(x) = \]

\[
(1 - i) \int_{\Omega} g \overline{w} \, dx - (1 - i) \int_{\Gamma_1} (\tau e^{-\lambda \tau} \int_0^1 h(x, \sigma) e^{\lambda \tau \sigma} \, d\sigma(x)) \overline{w} \, d\sigma(x) \quad \forall w \in H^1_{\Gamma_0}(\Omega).
\]

As the left-hand side of (16) is coercive on \(H^1_{\Gamma_0}(\Omega)\) (in the sense that if we denote this left-hand side by \(b(y, w)\), then \(\Re b(y, y) \geq \min\{1, \lambda\} \|y\|^2_{H^1(\Omega)}\) for all \(y \in H^1_{\Gamma_0}(\Omega)\)), and since the right-hand side defines a continuous linear form on \(H^1_{\Gamma_0}(\Omega)\) (since \((g, h) \in H\)) the Lax-Milgram Theorem guarantees the existence and uniqueness of a solution \(y \in H^1_{\Gamma_0}(\Omega)\) of (16).

If we consider \(w \in D(\Omega)\) in (16), then \(y\) solves in \(D'(\Omega)\)

\[
\lambda y - i \Delta y = g,
\]

and thus \(\Delta y \in L^2(\Omega)\).

Using Green’s formula in (16), we get

\[
\int_{\Gamma_1} (\mu_1 + \mu_2 e^{-\lambda \tau}) y \overline{w} \, d\sigma(x) + i \int_{\Gamma_1} \frac{\partial y}{\partial \nu} \overline{w} \, d\sigma(x) = \]

\[
\int_{\Gamma_1} (\tau e^{-\lambda \tau} \int_0^1 h(x, \eta) e^{\lambda \tau \eta} \, d\eta) \overline{w} \, d\sigma(x) \quad \forall w \in H^1_{\Gamma_0}(\Omega),
\]

from which follows

\[
i \frac{\partial y}{\partial \nu} + (\mu_1 + \mu_2 e^{-\lambda \tau}) y = \tau e^{-\lambda \tau} \int_0^1 h(x, \eta) e^{\lambda \tau \eta} \, d\eta \quad \text{on } \Gamma_1.
\]
Hence
\[ \frac{\partial y}{\partial \nu} = i(\mu_1 y + \mu_2 z(.,1)) \] on $\Gamma_1$.

As this right-hand side belongs to $L^2(\Gamma_1)$, we deduce that $\frac{\partial y}{\partial \nu} \in L^2(\Gamma_1)$ and by Theorem 2.7.4 of [8] we deduce that $y \in H^{3/2}(\Omega)$ (reminding that $\Gamma_0$ and $\Gamma_1$ are disjoint, this Theorem guarantees that if $y \in H^{1/2}_0(\Omega)$ is such that $\Delta y$ belongs to $H^{1/2}(\Omega)'$ and $\frac{\partial y}{\partial \nu} \in L^2(\Gamma_1)$, then $y \in H^{3/2}(\Omega)$).

So we have found $(y, z) \in D(A)$ which satisfies (13) and (14). By Lumer-Phillips Theorem, $A$ is the generator of a $C_0$—semigroup of contractions on $H$.

2.2 Proof of Theorem 2

Theorem 2 will be proved for smooth initial data. The general case follows by a standard density argument. We first show that the energy $E(t)$ of every solution of (1) is decreasing.

**Proposition 8** The energy corresponding to any strong solution of the problem (1) is decreasing and there exists $C > 0$ such that
\[ \frac{d}{dt} E(t) \leq -C \int_{\Gamma_1} (|y(x,t)|^2 + |y(x,t-\tau)|^2) \, d\sigma(x). \]

**Proof.** Differentiating $E(t)$ defined by (3) in time, we obtain
\[ \frac{d}{dt} E(t) = \Re \int_{\Omega} y_t \overline{y} \, dx + \xi \Re \int_{\Gamma_1} \int_0^1 y_t(x,t-\tau \rho) \overline{y}(x,t-\tau \rho) \, d\rho \, d\sigma(x) \]
\[ = \Re \int_{\Omega} (i\Delta y) \overline{y} \, dx + \xi \Re \int_{\Gamma_1} \int_0^1 y_t(x,t-\tau \rho) \overline{y}(x,t-\tau \rho) \, d\rho \, d\sigma(x). \]

Applying Green’s second Theorem and recalling the boundary conditions in (1), we obtain
\[ \frac{d}{dt} E(t) = -\mu_1 \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) - \mu_2 \Re \int_{\Gamma_1} y_t(x,t-\tau) \overline{y}(x,t) \, d\sigma(x) + \]
\[ \xi \Re \int_{\Gamma_1} \int_0^1 y_t(x,t-\tau \rho) \overline{y}(x,t-\tau \rho) \, d\rho \, d\sigma(x). \]

Now observe that
\[ y_t(x,t-\tau \rho) = -\tau^{-1} y_{\rho}(x,t-\tau \rho), \]
and
\[ \frac{d}{d\rho} |y(x,t-\tau \rho)|^2 = 2 \Re (y_{\rho}(x,t-\tau \rho) \overline{y}(x,t-\tau \rho)). \]
Insertion of (18) into (17) yields
\[
\frac{d}{dt}E(t) = -\mu_1 \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) - \mu_2 \Re \int_{\Gamma_1} y(x,t - \tau) \overline{y}(x,t) \, d\sigma(x) - \frac{\xi}{2\tau} \int_{\Gamma_1} 1 \int_0^1 \frac{d}{d\rho} |y(x,t - \tau \rho)|^2 \, d\rho \, d\sigma(x).
\]

From Cauchy-Schwarz inequality, we have
\[
\frac{d}{dt}E(t) \leq -(\mu_1 - \frac{\mu_2}{2} + \frac{\xi}{2\tau}) \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) - \left(\frac{\xi}{2\tau} - \frac{\mu_1}{2}\right) \int_{\Gamma_1} |y(x,t - \tau)|^2 \, d\sigma(x).
\]

This last inequality can be written
\[
\frac{d}{dt}E(t) \leq -C \int_{\Gamma_1} (|y(x,t)|^2 + |y(x,t - \tau)|^2) \, d\sigma(x),
\]
where
\[
C = \min\{\mu_1 - \frac{\mu_2}{2} + \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{\mu_1}{2}\},
\]
which is positive due to the assumption (4).

We now establish an observability inequality which will be used to prove the exponential decay of the energy \( E(t) \).

**Proposition 9** Let \( y \) be a strong solution of (1). Then there exists a a positive constant \( C_0 \) depending on \( T \) such that for all \( T > \tau \), the following inequality holds
\[
(19) \quad E(0) \leq C_0 \int_0^T \int_{\Gamma_1} (|y(x,t)|^2 + |y(x,t - \tau)|^2) \, d\sigma(x),
\]

**Proof.** Set
\[
E(t) = \mathcal{E}(t) + E_1(t),
\]
where
\[
\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} |y(x,t)|^2 \, dx \quad \text{and} \quad E_1(t) = \frac{\xi}{2} \int_{\Gamma_1} 1 \int_0^1 |y(x,t - \tau \rho)|^2 \, d\rho \, d\sigma(x).
\]
From Theorem 2.6.1. of [5] (see (5)), we have the following estimate

(20) \[ E(0) \leq c\{\|y\|^2_{L^2(0,T;L^2(\Gamma_1))} + \int_0^T \int_{\Gamma_1} \left| \frac{\partial y}{\partial \nu} \right| |y| \, d\sigma(x) \, dt + \left\| \frac{\partial y}{\partial \nu} \right\|^2_{H^{-1}(\Gamma_1 \times (0,T))} + \|y\|^2_{L^2(\Gamma_1 \times (0,T))}\}, \]

for \( T > 0 \) and for a suitable constant \( c \) depending on \( T \).

We now impose the boundary conditions in (9). Then (20) becomes

(21) \[ E(0) \leq c\{\int_0^T \int_{\Gamma_1} (|y(x,t)|^2 + |y(x,t - \tau)|^2) \, d\sigma(x) + \|y\|^2_{H^{-1}(\Omega \times (0,T))}\}, \]

since the \( H^{-1}_\sigma(\Gamma_1 \times (0,T)) \)-norm is dominated by the \( L^2(\Gamma_1 \times (0,T)) \)-norm.

\( E_1(t) \) can be rewritten, via a change of variable, as follows

\[ E_1(t) = \frac{\xi}{2} \int_{\Gamma_1} \int_{t-\tau}^{t} |y(x,s)|^2 \, ds \, d\sigma(x). \]

Hence

(22) \[ E_1(0) \leq c\int_{\Gamma_1} \int_{-\tau}^{0} |y(x,s)|^2 \, ds \, d\sigma(x). \]

By another change of variable in (22), we have for \( T \geq \tau \)

(23) \[ E_1(0) \leq c\int_0^T \int_{\Gamma_1} |y(x,t - \tau)|^2 \, d\sigma(x) \, dt. \]

Combining (21) and (23) we obtain for any \( T \geq \tau \)

(24) \[ E(0) \leq c\{\int_0^T \int_{\Gamma_1} (|y(x,t)|^2 + |y(x,t - \tau)|^2) \, d\sigma(x) \, dt + \|y\|^2_{H^{-1}(\Omega \times (0,T))}\}, \]

for a suitable constant \( c \) depending on \( T \).

Naturally, (24) implies a fortiori

(25) \[ E(0) \leq c\{\int_0^T \int_{\Gamma_1} (|y(x,t)|^2 + |y(x,t - \tau)|^2) \, d\sigma(x) \, dt + \|y\|^2_{L^\infty(0,T;H^{-1}(\Omega))}\}. \]

To get the requested inequality (19) from (25), we need to absorb the lower order term \( \|y\|^2_{L^\infty(0,T;H^{-1}(\Omega))} \).

To achieve this, we employ as in [11] and [15], a compactness/ uniqueness contradiction argument.
Suppose that (19) does not hold. Then there exists a sequence $y_n$ of solutions of problem (1) with $y_n(x,0) = y_{n,0}(x)$ and $y_n(x,t - \tau) = f_{n,0}(x,t - \tau)$ such that

$$(26) \quad E_n(0) > n \int_0^T \int_{\Gamma_1} (|y(x,t)|^2 + |y(x, t - \tau)|^2) d\sigma(x) dt$$

Here $E_n(0)$ is the energy corresponding to $y_n$ at time $t = 0$.

From (25), we have

$$(27) \quad E_n(0) \leq c \left\{ \int_0^T \int_{\Gamma_1} (|y_n(x,t)|^2 + |y_n(x, t - \tau)|^2) d\sigma(x) dt + \|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))} \right\}$$

(27) together with (26) yield

$$n \int_0^T \int_{\Gamma_1} (|y_n(x,t)|^2 + |y_n(x, t - \tau)|^2) d\sigma(x) dt \leq c \left\{ \int_0^T \int_{\Gamma_1} (|y_n(x,t)|^2 + |y_n(x, t - \tau)|^2) d\sigma(x) dt + \|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))} \right\}$$

that is

$$(28) \quad (n - c) \int_0^T \int_{\Gamma_1} (|y_n(x,t)|^2 + |y_n(x, t - \tau)|^2) d\sigma(x) dt \leq c \|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))}$$

Renormalizing, we obtain a sequence of solutions of problem (1) satisfying

$$(29) \quad \|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))} = 1, \forall n > c,$$

and

$$(30) \quad \int_0^T \int_{\Gamma_1} (|y_n(x,t)|^2 + |y_n(x, t - \tau)|^2) d\sigma(x) dt \leq \frac{c}{n - c}, \forall n > c.$$

From (27), (29) and (30) we deduce that the sequence $Y_{n,0} = (y_{n,0}, f_{n,0})$ is bounded in $H$. Thus there is a subsequence still denoted by $Y_{n,0}$ which converges weakly to some $Y_0 = (y_0, f_0) \in H$.

Let $\psi$ be the solution of problem (1) with such initial condition $Y_0$. We have

$$\psi \in C(0,T;L^2(\Omega))$$

from Theorem 7 and

$$\int_0^T \int_{\Gamma_1} \psi(x,t)^2 d\sigma(x) dt + \int_0^T \int_{\Gamma_1} \left| \frac{\partial \psi(x,t)}{\partial \nu} \right|^2 d\sigma(x) dt \leq Const.$$
from Proposition 8.

It then follows that

\[ y_n \rightarrow \psi \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak star} \]

\[ (y_n)_t \rightarrow \psi_t \text{ in } L^\infty(0, T; H^{-2}(\Omega)) \text{ weak star} \]

and hence

\[ \|y_n\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|(y_n)_t\|_{L^\infty(0, T; H^{-2}(\Omega))}^2 \leq \text{Const} \quad \text{for all } n \in \mathbb{N}. \tag{31} \]

Since the injection \( L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \) is compact, (31) implies (see [1] and [14]) that for \( 0 < T < +\infty \) the injection

\[ Z \hookrightarrow L^\infty(0, T, H^{-1}(\Omega)) \]

is also compact, where \( Z \) is the Banach space equipped with the norm on the left-hand side of (31), is also compact. As a consequence there is a subsequence still denoted by \( y_n \) such that

\[ y_n \rightarrow \psi \text{ in } L^\infty(0, T, H^{-1}(\Omega)) \text{ strongly}. \]

Hence by (29) we obtain

\[ \|\psi\|_{L^\infty(0, T; H^{-1}(\Omega))} = 1 \tag{32} \]

On the other hand, we have from (30)

\[ \psi(x, t) = 0 \text{ on } \Gamma_1 \times (0, T) \]

Thus \( \psi \) satisfies

\[ \begin{cases} 
\psi_t(x, t) - i\Delta \psi(x, t) = 0 & \text{in } \Omega \times (0, T), \\
\psi(x, t) = 0 & \text{on } \Gamma \times (0, T), \\
\frac{\partial \psi}{\partial n}(x, t) = 0 & \text{on } \Gamma_1 \times (0, T). 
\end{cases} \]

From Holmgren’s uniqueness Theorem (see [7], Chap. 1, Thm. 8.2), we conclude that

\[ \psi(x, t) = 0 \text{ in } \Omega \times (0, T), \]

which contradicts (32). This ends the proof of Proposition 9. \( \blacksquare \)

**Proof of Theorem 2.**

From Proposition 8, we have

\[ E(T) - E(0) \leq -C \int_0^T \int_{\Gamma_1} (|y(x, t)|^2 + |y(x, t - \tau)|^2) \, d\sigma(x). \]
The observability estimate (19) implies
\[ E(T) \leq E(0) \leq C_0 \int_0^T \int_{\Gamma_1} (|y(x, t)|^2 + |y(x, t - \tau)|^2) \, d\sigma(x) \leq C_0 C^{-1}(E(0) - E(T)). \]
Hence
\[ E(T) \leq \frac{C_0}{1 + C_0 C^{-1}} E(0). \]
Combining this estimate with the invariance by translation of the system (1), we obtain the desired conclusion.

2.3 Proof of Theorem 3

In this section we show through an example that the system (9) loses the property of exponential stability when \( \mu_2 \geq \mu_1 \).

We seek for a solution of (9) in the form
\[ y(x, t) = e^{\lambda t} \varphi(x), \]
where
\[ \lambda = -i \beta^2, \quad \beta \in \mathbb{R}. \]

Then \( \varphi \) is a solution of the eigenvalue problem
\[
\begin{align*}
-\Delta \varphi &= i\lambda \varphi \quad \text{in} \quad \Omega, \\
\varphi &= 0 \quad \text{on} \quad \Gamma_0, \\
\frac{\partial \varphi}{\partial \nu} &= i(\mu_1 + \mu_2 e^{-\lambda \tau}) \varphi \quad \text{on} \quad \Gamma_1.
\end{align*}
\] (33)

Assume that
\[ \cos(\beta^2 \tau) = -\frac{\mu_1}{\mu_2}, \]
then
\[ \mu_2 \sin(\beta^2 \tau) = \sqrt{\mu_2^2 - \mu_1^2}. \]
(35)

Inserting (34) and (35) into (33) yields
\[
\begin{align*}
-\Delta \varphi &= \beta^2 \varphi \quad \text{in} \quad \Omega, \\
\varphi &= 0 \quad \text{on} \quad \Gamma_0, \\
\frac{\partial \varphi}{\partial \nu} + \sqrt{\mu_2^2 - \mu_1^2} \varphi &= 0 \quad \text{on} \quad \Gamma_1.
\end{align*}
\]
This is a classical eigenvalue problem for the Laplacian with Dirichlet - Robin boundary conditions.
Let \( \{ \beta_n^2; n \in \mathbb{N} \} \) be the set of these eigenvalues. It is well known that \( \beta_n^2 \to +\infty \) as \( n \to +\infty \).
Taking \( 0 < \theta < 2\pi \) such that
\[
\cos \theta = -\frac{\mu_1}{\mu_2} \quad \text{and} \quad \mu_2 \sin \theta = \sqrt{\mu_2^2 - \mu_1^2},
\]
we obtain a sequence of delays
\[
\tau_{n,k} = \frac{1}{\beta_n^2} (\theta + 2k\pi), \quad n, k \in \mathbb{N},
\]
which become arbitrarily small or large for suitable choices of \( n, k \in \mathbb{N} \), and for which the problem (9) is not asymptotically stable. Indeed, the energy of the solution \( y(x, t) = e^{-i\beta_n^2 t} \varphi(x) \) is constant.

3 Stability of the Schrödinger equation with a delay term in the internal feedback

3.1 Well-posedness of the system (6)

Proceeding as in the previous section, we can see that the system (6) is equivalent to
\[
\begin{align*}
y_t(x, t) &= i\Delta y(x, t) - a(x)\{\mu_1 y(x, t) + \mu_2(x, t)z(x, 1, t)\} \quad \text{in} \quad \Omega \times (0, +\infty), \\
z_t(x, \rho, t) &= -\tau^{-1} z_\rho(x, \rho, t) \quad \text{in} \quad \Omega \times (0, 1) \times (0, +\infty), \\
y(x, 0) &= y_0(x) \quad \text{in} \quad \Omega, \\
z(x, 0) &= g_0(x, -\tau\rho) \quad \text{in} \quad \Omega \times (0, \tau), \\
y(x, t) &= 0 \quad \text{on} \quad \Gamma \times (0, +\infty), \\
z(x, 0, t) &= y(x, t) \quad \text{in} \quad \Omega \times (0, +\infty),
\end{align*}
\]
where we have set
\[
z(x, \rho, t) = y(x, t - \tau\rho), x \in \Omega, \rho \in (0, 1), t > 0.
\]
Let us introduce the operator \( A^0 \) defined by
\[
A^0 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} i\Delta y - a\mu_1 y - a\mu_2 z(\cdot, 1) \\ -\tau^{-1} z_\rho \end{pmatrix},
\]
and
\[
D(A^0) = \{(y, z) \in (H^2(\Omega) \cap H^1_\Gamma(\Omega)) \times L^2(\Omega, H^1(0, 1)); y = z(\cdot, 0) \text{ in } \Omega\}.
\]
Then we rewrite the system (37) as follows
\[
\begin{cases}
U''(t) = A^0 U(t), \\
U(0) = U_0,
\end{cases}
\]
where
\[
U(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}.
\]
Denote by \( \mathcal{H}^0 \) the Hilbert space
\[
\mathcal{H}^0 = L^2(\Omega) \times L^2(\Omega \times (0,1))
\]
equipped with the inner product
\[
\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \Re \int_{\Omega} y_1(x)\overline{y_2(x)} \, dx + \xi \Re \int_{\Omega} \int_0^1 z(x, \rho)\overline{z(x, \rho)} \, d\rho \, dx.
\]
Repeating the argument used in the proof of Theorem 7, we obtain the following well-posedness result for the problem (37).

**Theorem 10** For any \( U_0 \in \mathcal{H}^0 \), there exists a unique (weak) solution
\[
U \in C(0, +\infty; \mathcal{H}^0)
\]
of the problem (37). Moreover if \( U_0 \in D(A^0) \), then the solution \( U \) is more regular, namely
\[
U \in C(0, +\infty; D(A^0)) \cap C^1(0, +\infty; \mathcal{H}^0)
\]
and is called a strong solution.

### 3.2 Proof of Theorem 5

**Proposition 11** The energy corresponding to any strong solution of the problem (37) is decreasing and there exists a positive constant \( C \) such that
\[
\frac{d}{dt} F(t) \leq -C \int_{\Omega} a(x) \{ |y(x, t)|^2 + |y(x, t - \tau)|^2 \} \, dx.
\]

**Proof.** We differentiate \( F(t) \) in (8) and use (37) to obtain
\[
\frac{d}{dt} F(t) = \Re \int_{\Omega} (i \Delta y(x, t)) \overline{y(x, t)} \, dx \\
- \Re \int_{\Omega} a(x) \{ \mu_1 y(x, t) + \mu_2(x, t) y(x, t - \tau) \} \overline{y(x, t)} \, dx \\
+ \xi \Re \int_{\Omega} a(x) \int_0^1 y_t(x, t - \tau \rho) \overline{y(x, t - \tau \rho)} \, d\rho \, dx.
\]
Applying Green’s second Theorem and recalling the boundary condition in (37), we get
\begin{equation}
\frac{d}{dt} F(t) = -\mu_1 \Re \int_{\Omega} a(x) |y(x,t)|^2 \, dx - \mu_2 \Re \int_{\Omega} a(x)y(x,t)\overline{y(x,t)} \, dx \\
+ \xi \Re \int_{\Omega} a(x) \int_0^1 y(x,t-\tau \rho)\overline{y(x,t-\tau \rho)} \, d\rho \, dx.
\end{equation}

As in the proof of Proposition 8, we have
\begin{equation}
\Re \int_{\Omega} a(x) \int_0^1 y(x,t-\tau \rho)\overline{y(x,t-\tau \rho)} \, d\rho \, dx \\
= -\tau^{-1} \Re \int_{\Omega} a(x) \int_0^1 y(x,t-\tau \rho)\overline{y(x,t-\tau \rho)} \, d\rho \, dx \\
= -\frac{\tau^{-1}}{2} \int_{\Omega} a(x) \{ |y(x,t-\tau)|^2 - |y(x,t)|^2 \} \, dx.
\end{equation}

Insertion of (40) into (39) yields
\begin{equation}
\frac{d}{dt} F(t) = -\mu_1 \Re \int_{\Omega} a(x) |y(x,t)|^2 \, dx - \mu_2 \Re \int_{\Omega} a(x)y(x,t)\overline{y(x,t)} \, dx \\
- \frac{\xi \tau^{-1}}{2} \int_{\Omega} a(x) |y(x,t-\tau)|^2 + \frac{\xi \tau^{-1}}{2} \int_{\Omega} |y(x,t)|^2 \, dx.
\end{equation}

The desired estimate (38) follows from (41) via Cauchy-Schwarz inequality.

The key step in the proof of Theorem 5 is the following observability inequality.

**Proposition 12** Let \( y \) be a strong solution of (37). Then there exists a positive constant \( C_0 \) depending on \( T \) such that for all \( T > \tau \), the following estimate holds true
\begin{equation}
F(0) \leq C_0 \int_0^T \int_{\Omega} a(x) \{ |y(x,t)|^2 + |y(x,t-\tau)|^2 \} \, dx \, dt
\end{equation}

**Proof.** Following [10] and [11], we write the solution \( y \) of (37) as \( y = u + v \) where \( u \) solves
\begin{equation}
\begin{cases}
  u_t(x,t) = i\Delta u(x,t) & \text{in } \Omega \times (0,+\infty), \\
  u(x,0) = y_0(x) & \text{in } \Omega, \\
  u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty),
\end{cases}
\end{equation}
and \( v \) satisfies
\begin{equation}
\begin{cases}
  v_t(x,t) = i\Delta v(x,t) - a(x)\{ \mu_1 y(x,t) + \mu_2 y(x,t-\tau) \} & \text{in } \Omega \times (0,+\infty), \\
  v(x,0) = 0 & \text{in } \Omega, \\
  v(x,t) = 0 & \text{on } \Gamma \times (0,+\infty).
\end{cases}
\end{equation}
Let us denote by
\[ E_u(t) = \int_\Omega |u(x,t)|^2 \, dx \]
the energy corresponding to the solution of (43). Then, it follows from Proposition 3.1 in [10] that for all \( T > 0 \), there exists a positive constant \( c \) depending on \( T \) such that
\[ E_u(0) \leq c \int_0^T \int_\Omega |u(t,x)|^2 \, dx \, dt. \]
Using (7) we get
\[ E_u(0) \leq c \frac{\xi}{a_0} \int_0^T \int_\Omega a(x) |u(t,x)|^2 \, dx \, dt. \]
On the other hand we have for \( T > \tau \)
\[ \xi \int_\Omega a(x) \int_0^1 |y(x,-\tau \rho)|^2 \, d\rho \, dx \leq \xi \int_0^T \int_\Omega a(x) |y(x,t-\tau)|^2 \, dx \, dt. \]
Hence for \( T > \tau \)
\[ F(0) = E_u(0) + \frac{\xi}{2} \int_\Omega a(x) \int_0^1 |y(x,-\tau \rho)|^2 \, d\rho \, dx \]
\[ \leq c \int_0^T \int_\Omega a(x) \{ |u(t,x)|^2 + |y(x,t-\tau)|^2 \} \, dx \, dt \]
\[ \leq c \int_0^T \int_\Omega a(x) \{ |y(t,x)|^2 + |v(t,x)|^2 + |y(x,t-\tau)|^2 \} \, dx \, dt. \]
By classical energy estimates on Schrödinger equation we deduce that
\[ F(0) \leq C_0 \int_0^T \int_\Omega a(x) \{ |y(t,x)|^2 + |y(x,t-\tau)|^2 \} \, dx \, dt. \]

Combining the estimates (38) and (42), as in the case of a boundary feedback, we obtain the exponential stability result of Theorem 5.

### 3.3 Proof of Theorem 6

We proceed as in the case of boundary delay. We assume (34) and we look for a solution of the problem (37) in the form
\[ y(x,t) = e^{\lambda t} \phi(x) \text{ with } \lambda = -i \beta^2, \beta \in \mathbb{R}. \]
Then \( \varphi \) is a solution of the boundary value problem

\[
\begin{align*}
(-\Delta + a(x)\sqrt{\mu_2^2 - \mu_1^2})\varphi &= \beta^2 \varphi & \text{in} & \Omega \\
\varphi &= 0 & \text{on} & \Gamma
\end{align*}
\]

The operator \(-\Delta + a(x)\sqrt{\mu_2^2 - \mu_1^2}\) with Dirichlet boundary condition is positive self-adjoint in \(L^2(\Omega)\) with a compact resolvent. Let \(\{\beta_n^2; n \in \mathbb{N}\}\) be the set of its eigenvalues, then \(\beta_n^2 \to +\infty\) as \(n \to +\infty\). For \(0 < \theta < 2\pi\) given by (36), we obtain a sequence of delays

\[\tau_{n,k} = \frac{1}{\beta_n^2}(\theta + 2k\pi), \ n, k \in \mathbb{N}\]

for which the problem (37) loses its asymptotic stability.

References


