Mixed finite element method for the Heat diffusion equation in a random medium

L. Paquet * and R. Korikache
University of Valenciennes and Hainaut-Cambrésis, LAMAV, EA 4015, ISTV, Le Mont Houy, 59313-Valenciennes cedex 9 (France)
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We introduce the dual mixed method for the heat evolution equation in a polygonal domain $D$ with a random diffusion coefficient $K$ and heat flux $\vec{\gamma} = K \circ \nabla u$, $\circ$ denoting the Wick product. We prove a-priori error estimates for the semi-discrete solution $(\vec{\gamma}_h, u_h)$ of lowest order of the dual mixed method having $K$-dimensional polynomial chaos expansion of degree $N$. Due to the reentrant corner of the polygonal domain $D$, appropriate refinement rules must be imposed on the family of triangulations in order to recapture convergence of order one in space.

Keywords: Finite element method; Stochastic partial differential equations; Wick product; White noise analysis;

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1. Introduction

In this paper, we investigate the dual mixed method for the stochastic heat diffusion equation:

$$\begin{align*}
& u_t - \text{div} (K \circ \nabla u) = f \quad \text{in} \quad Q := ]0,T[ \times D \\
& u = 0 \quad \text{on} \quad ]0,T[ \times \partial D \\
& u|_{t=0} = g \quad \text{on} \quad D.
\end{align*}$$

(1)

Here $D$ denotes a bounded polygonal domain in $\mathbb{R}^2$, $f$ the random heat source, $u$ the random temperature, $g$ its random initial value and $K$ the random diffusion coefficient. $g$ and $K$ belong to some stochastic vector distributions spaces $[17]$; in particular it is assumed that the stochastic diffusion coefficient $K$ does not depend on the time variable $t$. Both $f$ and $u$ are functions of the time with values in stochastic vector distributions spaces $[17]$. The question of a Wick product, here between the random heat diffusion coefficient $K$ and $\nabla u$, the gradient of the temperature, is addressed in the papers of T. G. Theting

* Corresponding author. Email: luc.paquet@univ-valenciennes.fr
and al. [24] and [5]; see also the book of Holden and al. [17]. The classical variational formulation of the stochastic heat diffusion equation (1) and its numerical discretization have been studied in [33]. A stochastic version of the dual mixed formulation for the corresponding stationary problem to (1) has been studied in [24] and a priori error estimates have been derived but for “regular solutions in the space variable” only (i.e. belonging to the stochastic Sobolev space $S^{-1,k,H^m(D)}$ (see (4) for its definition)).

Our contribution here consists in introducing a stochastic version of the dual mixed formulation (see [11] for the nonstochastic case) for the stochastic heat diffusion equation (1) in a polygonal domain with a reentrant corner and proving a-priori optimal error estimates for the semi-discretized problem. Thus additionally to the unknown random temperature $u$, in the mixed formulation, the random heat flux $\overline{\nabla}u$ is considered as an additional unknown. Denoting by $t \mapsto (\overline{\nabla}h(t), u_h(t))$ the solution of the semi-discretized problem, we establish rates of convergence for $u_h(.)$ and $\overline{\nabla}h(.)$ in terms of the mesh width $h$ of the triangulation, the dimension $K$ of the homogeneous polynomial chaoses and their maximum order $N \ (|13|, pp. 52-55). Using a regularity result on the solution $u$ of (1) expressed by the fact that $u$ belongs to some spatially weighted Sobolev space taking into account the singularities induced by the reentrant corner of the polygonal domain $D$, and imposing appropriate refinement rules on our regular family of triangulations $(T_h)_{h>0}$ of the polygonal domain $D$ linked to that regularity of the solution $u$ (on the spur of (15, section 8.4)), we derive $O(h)$ error estimates in the spatial directions.

We also discuss algorithmic aspects of this numerical method. In particular we show how the chaoses coefficients of each component of the semi-discretized solution $(u_h(.), \overline{\nabla}h(.))$ can be computed successively by solving a sequence of deterministic discrete evolution mixed problems.

2. Preliminaries on white noise analysis and stochastic Sobolev spaces

Let us recall some notations from [33], [24] and [17]. $\mathcal{I}$ denotes the set of all sequences $\alpha = (\alpha_j)_{j \geq 1} \in (\mathbb{N}_0)^N$ with compact support (for the discrete topology on $\mathbb{N}_0$) i.e. such that $\exists j_0 \in \mathbb{N}_0: \alpha_j = 0, \forall j \geq j_0$ (! we use the notations of the “Norway-School” [17]: in particular $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). For $\alpha \in \mathcal{I}$:

$$(2N)_{\alpha} := \prod_{j=1}^{+\infty} (2j)^{\alpha_j};$$

let us observe that this is in fact a finite product as $\alpha$ has compact support. For $\omega \in \mathcal{S}'(\mathbb{R}^2)$ i.e. a tempered distribution on $\mathbb{R}^2$ ($\mathcal{S}(\mathbb{R}^2)$ denotes the Fréchet space of rapidly decreasing functions on $\mathbb{R}^2$ and $\mathcal{S}'(\mathbb{R}^2)$ its dual [27] p.133), and $\alpha \in \mathcal{I}$, we set:

$$H_{\alpha}(\omega) = \prod_{i=1}^{+\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle),$$

where $h_{\alpha_i}(\cdot)$ denotes the $\alpha_i$-th order Hermite polynomial on $\mathbb{R}$

$$h_{\alpha_i}: \mathbb{R} \to \mathbb{R} : x \mapsto (-1)^{\alpha_i}e^{\frac{1}{2} x^2} \frac{d^{\alpha_i}}{dx^{\alpha_i}}(e^{-\frac{1}{2} x^2})$$

([17], p.18, 207, 208) monic and orthogonal with respect to the normalized Gauss measure

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

and where $(\eta_i)_{i \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^2)$ denotes the orthonormal basis in $L^2(\mathbb{R}^2; dx_1 \otimes dx_2)$ constructed by taking tensor products of the $1-D$ Hermite functions $\xi_n(\cdot): \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-1/2}e^{-\frac{1}{2} x^2}h_{n-1}(\sqrt{2}x)$$
\[ \forall x \in \mathbb{R}, \forall n = 1, 2, 3, \ldots \quad (\text{[17] p.19, 208}) \text{ (it is well known that the } 1 - D \text{ Hermite functions } (\xi_n (\cdot))_{n \geq 1} \text{ which are the eigenfunctions of the "Harmonic Oscillator" since} \]

\[ \frac{d^2 \xi_n}{dx^2} + x^2 \frac{d\xi_n}{dx} = (2n - 1) \xi_n, \quad \forall n \geq 1, \]

belong to \( S (\mathbb{R}) \) the space of rapidly decreasing functions on \( \mathbb{R} \), and form an orthonormal basis of \( L^2 (\mathbb{R}; dx) \) (\text{[27] p.142}) \( (\text{[17] pp.207-208}) \)). Moreover if \( \phi \in S (\mathbb{R}^2) \), the expansion \( \sum_{j=1}^{+\infty} \langle \phi | \eta_j \rangle_{L^2 (\mathbb{R}^2;\mu)} \eta_j \) converges in \( S (\mathbb{R}^2) \) (\text{[27] p.143}), (this is also true for \( S' (\mathbb{R}^2) \) [27] p.143). By theorem 2.2.3 p.21 in \[ \text{[17]}, \quad (H_\alpha)_{\alpha \in \mathcal{I}} \text{ is an orthogonal basis of our basic probability space: } " \text{the } 1 \text{-dimensional (2-parameter) Gaussian white noise probability space }" \]

\[ L^2 (S' (\mathbb{R}^2), B_{S' (\mathbb{R}^2)}, \mu) \] \( (\text{[17] p.21}) \). \( B_{S' (\mathbb{R}^2)} \) denotes the Borel \( \sigma \)-algebra of \( S' (\mathbb{R}^2) \) i.e. the \( \sigma \)-algebra of \( S' (\mathbb{R}^2) \) generated by all subsets of \( S' (\mathbb{R}^2) \) of the form \( \{ \omega \in S' (\mathbb{R}^2); \langle \omega, \phi_1 \rangle \in B_1, \ldots, \langle \omega, \phi_n \rangle \in B_n \} \) for arbitrary numbers of functions \( \phi_1, \ldots, \phi_n \in S (\mathbb{R}) \) and arbitrary Borel sets \( B_1, \ldots, B_n \) of \( \mathbb{R} \). \( \mu \) is the normalized Gaussian measure on \( S' (\mathbb{R}^2) \) also often called the 1-dimensional (2-parameter) Gaussian white noise measure and may be defined by the property that for an arbitrary orthonormal set \( \{ \phi_1, \ldots, \phi_n \} \subset S (\mathbb{R}^2) \) orthonormal with respect to the \( L^2 (\mathbb{R}^2) \) scalar product, that its image by the mapping

\[ S' (\mathbb{R}^2) \rightarrow \mathbb{R}^n: \omega \mapsto (\langle \omega, \phi_1 \rangle, \ldots, \langle \omega, \phi_n \rangle) \]

is the normalized Gauss measure on \( \mathbb{R}^n: (\text{[17] p.12}) \)

\[ (2\pi)^{-\frac{n}{2}} \exp^{-\frac{1}{2}(x_1^2 + \cdots + x_n^2)} \mu \] \( \otimes \ldots \otimes \mu \).

Thus every \( f \in L^2 (\mu) \) possesses a unique expansion: (\text{[17] p.23})

\[ f = \sum_{\alpha \in \mathcal{I}} \frac{(f|H_\alpha)_{L^2(\mu)}}{\alpha!} H_\alpha \quad \text{and} \quad \|f\|_{L^2(\mu)}^2 = \sum_{\alpha \in \mathcal{I}} \frac{|(f|\alpha)_{L^2(\nu)}|^2}{\alpha!}, \]

as \( \|H_\alpha\|_{L^2(\mu)}^2 = \alpha ! := \alpha_1! \alpha_2! \alpha_3! \ldots \). This expansion is called the Wiener-Itô chaos expansion of \( f \) (\text{[17] p.23}, \text{[13] p.42}, \text{[25] p.6}, \text{[36]}). The vector subspace spanned by the \( \{H_\alpha, |\alpha| = p\} \) is called the polynomial chaos of order \( p \) and its closure in \( L^2 (\mu) \) the Wiener homogeneous chaos of order \( p \) (\text{[13] p.44}, \text{[21] p.4}, \text{[25] p.6}, \text{see also [36]}).

**Remark 2.1:** (i) The Wiener-Itô chaos expansion theorem remains valid for rather general \( L^2 \)-spaces. Let us consider a general \( (\Omega, \mathcal{A}, \nu) \) probability space and let \( (\eta_i)_{i \geq 1} \) denote an i.i.d. (independent, identically distributed) sequence of normalized Gaussian random variables. Similarly, as previously, we define the polynomial chaos

\[ H_\alpha (\omega) = \prod_{i=1}^{\infty} h_{\alpha_i} (\eta_i (\omega)), \quad \forall \omega \in \Omega, \forall \alpha \in \mathcal{I}. \]

Let us denote by \( \mathcal{G} \subset \mathcal{A} \), the \( \sigma \)-algebra generated by the family of standard Gaussian random variables \( (\eta_i)_{i \geq 1} \). Then for every \( f \in L^2 (\Omega, \mathcal{G}, \nu) \), we have also \( [36] \) (\text{[25], p. 6}):

\[ f = \sum_{\alpha \in \mathcal{I}} \frac{(f|H_\alpha)_{L^2(\nu)}}{\alpha!} H_\alpha \quad \text{and} \quad \|f\|_{L^2(\nu)}^2 = \sum_{\alpha \in \mathcal{I}} \frac{|(f|\alpha)_{L^2(\nu)}|^2}{\alpha!}. \quad (3) \]

Consequently, all the results (except specific examples using explicitly the probability space \( (S' (\mathbb{R}^2), B_{S' (\mathbb{R}^2)}, \mu) \) which follow in this paper remain valid when replacing \( L^2 \left( S' (\mathbb{R}^2), B_{S' (\mathbb{R}^2)}, \mu \right) \) by
$L^2(\Omega, \mathcal{G}, \nu)$. This flexibility is useful in theory; for example the i.i.d. sequence of standard Gaussian random variables $(\eta_i)_{i \geq 1}$ may appear from the Karhunen-Loève expansion \cite{13}, \cite[pp. 37-43]{16}, theorem 5 p. 251 of the random diffusion coefficient $\mathcal{K}$ assumed to be a Gaussian random field on a bounded open set of $\mathbb{R}^2$ and in this case, may not be imposed a priori. However, from the numerical point of view, this is of no importance: the only important fact is that $(\eta_i)_{i \geq 1}$ is a i.i.d. (independent identically distributed) sequence of standard Gaussian random variables. When simulating, we generate a sequence of numbers that behaves as if each number where independently selected at random with the normal distribution $N(0, 1)$ \cite[pp. 117, ..]{18} to obtain a realization of the sequence $(\eta_i)_{i \geq 1}$. This last numerical procedure does not take into account the peculiarity of the i.i.d. sequence of standard Gaussian random variables $(\eta_i)_{i \geq 1}$.

Let us now recall the definition of the stochastic Sobolev spaces introduced by Y. Kondratiev \cite{19}, \cite{17} that we will need to explain the classical variational formulation and the mixed variational formulation for the Cauchy problem (initial boundary value problem with homogeneous Dirichlet boundary condition) of the stochastic heat diffusion equation \eqref{eq:1}. Let $(V, (\ldots)_V)$ denotes any real separable Hilbert space and $k \in \mathbb{R}$, $\rho \in [-1, 1]$ be given parameters. We define the space

\begin{equation}
\mathcal{S}^{\rho,k,V} := \left\{ f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha; \ f_\alpha \in V \text{ for } \alpha \in \mathcal{I} \text{ and } \|f\|_{\rho,k,V} < +\infty \right\}
\end{equation}

where

\begin{equation}
\|f\|_{\rho,k,V}^2 := \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_V^2 (2N)^{k\alpha} (\alpha!)^{1+\rho}.
\end{equation}

Clearly the norm defined by equality \eqref{eq:5} is induced by the scalar product:

\begin{equation}
(f \mid g)_{\rho,k,V} := \sum_{\alpha \in \mathcal{I}} (f_\alpha \mid g_\alpha)_V (2N)^{k\alpha} (\alpha!)^{1+\rho}.
\end{equation}

If $k \geq 0$ and $\rho \geq 0$, then it follows immediately from $\|f\|_{\rho,k,V} < +\infty$, that $\sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_V^2 \alpha! < +\infty$. Thus in this case, the series $\sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha$ is a Cauchy sequence and thus convergent in $L^2(\mu, V)$. But in other situations for the parameters $k$ and $\rho$, the series $\sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha$ does not converge in $L^2(\mu; V)$ and consequently must be considered as a “formal series” satisfying the summability condition \eqref{eq:5}. In other words, we could define $\mathcal{S}^{\rho,k,V}$ in the following manner:

\begin{equation}
\mathcal{S}^{\rho,k,V} := \left\{ f = (f_\alpha)_{\alpha \in \mathcal{I}}; \ f_\alpha \in V, \forall \alpha \in \mathcal{I} \text{ and } \|f\|_{\rho,k,V} < +\infty \right\}
\end{equation}

and thus $\mathcal{S}^{\rho,k,V}$ may be seen as an orthogonal countable direct sum of Hilbert spaces (copies of $V$) with the positive weights $(2N)^{k\alpha} (\alpha!)^{1+\rho}$, $\alpha \in \mathcal{I}$ (wich is countable) \cite[volume 4, p.114]{12}, \cite[pp.40]{27}, \cite[p.114, last § of the introduction]{35}.

For $k \geq 0$ and $\rho \geq 0 : \mathcal{S}^{\rho,k,V} \subset L^2(\mu; V)$. On the other hand $f \in L^2(\mu; V) \implies \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_V^2 \alpha! < +\infty$, and if $\rho \leq 0$ and $k \leq 0$, then a fortiori:

\begin{equation}
\sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_V^2 (2N)^{k\alpha} (\alpha!)^{1+\rho} < +\infty.
\end{equation}

Thus for $\rho \leq 0$ and $k \leq 0$, we have the inclusion in the reverse order: $L^2(\mu; V) \subset \mathcal{S}^{\rho,k,V}$. Let us also
observe that, for $k \in \mathbb{R}$, $\rho \in [-1,1]$, that

$$S^{\rho,k,V} = S^{\rho,k,R} \otimes V$$

(9)

where $\otimes$ denotes the algebraic tensor product completed for the projective norm ([30], p.93-94).

Let $D \subset \mathbb{R}^2$ be an open bounded set in $\mathbb{R}^2$. If $V = L^2(D)$, then we will note more shortly the Hilbert space $S^{\rho,k,L^2(D)}$ by $S^{\rho,k,0}(D)$ or $S^{\rho,k,0}$. If $V = H^{1}(D)$ (resp. $H^{1}(D)$), then we will note more shortly the Hilbert space $S^{\rho,k,H^1(D)}$ (resp. $S^{\rho,k,H^1(D)}$) by $S^{\rho,k,1}(D)$ or $S^{\rho,k,1}$ (resp. by $S^{\rho,k,1}(D)$ or $S^{\rho,k,1}$).

$S^{\rho,0,0}(D) \equiv L^2(\mu; L^2(D))$ is not closed under the “Wick multiplication” $\Diamond$ defined by

$$\Diamond : (f,g) \mapsto f \Diamond g := \sum_{\gamma \in \mathcal{I}} \left( \sum_{\alpha, \beta \in \mathcal{I}} f_{\alpha} g_{\beta} \right) H_{\gamma}$$

[35]. To provide conditions on $f$ such that $g \mapsto f \Diamond g$ is a continuous linear operator in $S^{-1,k,0}(D)$, we introduce the Banach space $F_l(D)$ [35]. Given $l \in \mathbb{R}$, we define the Banach space [35], [32]

$$F_l(D) = \left\{ f = \sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha} ; f_{\alpha} : D \to \mathbb{R} \text{ measurable } \forall \alpha \in \mathcal{I} \right\}$$

and

$$\| f \|_{l,\alpha} := \text{ess sup} \left( \sum_{x \in D} |f_{\alpha}(x)| (2\eta)^{\alpha} \right) < +\infty$$

(10)

$F_l(D)$ is a commutative Banach algebra for the Wick product ([35], prop.6, p.123) with 1 as unity. Moreover, if $f \in F_l(D)$ and $g \in S^{-1,k,0}(D)$ with $k \leq 2l$, then the Wick product is a well defined element of $S^{-1,k,0}(D)$ and $\| f \Diamond g \|_{-1,k,0} \leq \| f \|_{l,\alpha} \| g \|_{-1,k,0}$ ([35], prop.4, p.120) [32].

For the mixed formulation, we will also need the space $S^{\rho,0,V}$ with $V = H(\text{div}; D)$; more shortly we will denote it as in [24] p. 609, $\mathcal{H}(\text{div}; D)$.

Finally if $f = \sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$ is in $L^2(\mu; V) \implies \sum_{\alpha \in \mathcal{I}} \| f_{\alpha} \|^2 \alpha! < +\infty$, then by using lemma 2.1.2 p.12 of [17], it follows that $E(f) = f(0,0,...)$ where $E(f)$ denotes the mathematical expectation of $f$ with respect to the white noise Gaussian measure $\mu$.

For that reason if $f = \sum_{\alpha \in \mathcal{I}} f_{\alpha} H_{\alpha}$ belongs to $S^{\rho,k,V}$, we will call generalized expectation of $f$, the coefficient $f(0,0,...)$ and we will denote it $E[f]$ ([17] p.64).

Note also that $E[f \Diamond g] = E[f] E[g]$ ([17] p.64 and p.30).

**Remark 2.2:** The vector space $E$ generated by the stochastic monomials of order $p$

$$\{ H(\alpha_1,...,\alpha_M,0,0,...) ; (\alpha_1,...,\alpha_M) \in \mathbb{N}_0^M, \alpha_1 + \cdots + \alpha_M = p \}$$

coincides with the vector space $F$ generated by the

$$\left\{ h_p \left( \sum_{i=1}^{M} t_i \langle \cdot , \eta_i \rangle \right) ; (t_1,\ldots,t_M) \in \mathbb{R}^M, t_1^2 + \cdots + t_M^2 = 1 \right\},$$

where $h_p(\cdot)$ denotes the $1 - D$ Hermite polynomial of order $p$ (it is in this manner that the polynomial chaos of order $p$ in the random variables $\langle \cdot , \eta_1 \rangle, \ldots, \langle \cdot , \eta_M \rangle$ is described in ([25] p. 6)).

That $F$ is contained in $E$ follows from proposition D.2 p. 210 of [17] which tells us that

$$h_p \left( \sum_{i=1}^{M} t_i \langle \cdot , \eta_i \rangle \right) = \sum_{\alpha=(\alpha_1,\alpha_2,...,\alpha_M), \alpha_1 + \cdots + \alpha_M = p} \frac{p!}{\alpha!} t^{\alpha} H_{\alpha}(\cdot)$$
for every \( t = (t_1, \ldots, t_M) \in \mathbb{R}^M \) such that \( t_1^2 + \cdots + t_M^2 = 1 \). To show that in fact \( F = E \), in view of

that proposition it suffices to show that if \( y \in \mathbb{R}^d \) (\( d = \frac{p(p+1)}{2} \) being the number of all multi-indices \( (\alpha_1, \ldots, \alpha_M) \in \mathbb{N}_0^M \) such that \( \alpha_1 + \cdots + \alpha_M = p \)) is orthogonal to all the vectors of \( \mathbb{R}^d \) of the form

\[
\left( \frac{t_1^\alpha}{\alpha!} \right)_{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M), \quad \alpha_1 + \cdots + \alpha_M = p}
\]

formed by the coefficients appearing in the right-hand side of the preceding equation, that \( y = 0 \). By homogeneity \( y \) is orthogonal to every vector \( \left( \frac{t_1^\alpha}{\alpha!} \right)_{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M), \quad \alpha_1 + \cdots + \alpha_M = p} \) with \( t_1^2 + \cdots + t_M^2 = 1 \).

This implies that

\[
D_t^\beta \left( \sum_{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M), \quad \alpha_1 + \cdots + \alpha_M = p} y_\alpha \frac{t_1^\alpha}{\alpha!} \right) \equiv y_\beta = 0,
\]

for every \( \beta = (\beta_1, \ldots, \beta_M) \in \mathbb{N}_0^M \) such that \( \beta_1 + \cdots + \beta_M = p \). Thus \( y = 0 \). What was to be proved.

**Remark 2.3:** If \( \varphi \in S(\mathbb{R}^2) \), then we can view it as the element \( \langle ., \varphi \rangle \in L^2(\mu) \) defined by:

\[
\langle ., \varphi \rangle : S'(\mathbb{R}^2) \to \mathbb{R} : \omega \mapsto \langle \omega, \varphi \rangle.
\]

It follows from lemma 2.1.2 of \([17]\) p. 12) that \( \| \varphi \|_{L^2(\mathbb{R}^2)} = \| \langle ., \varphi \rangle \|_{L^2(\mu)} \). Thus if \( (\varphi_n)_{n \geq 1} \) is a sequence in \( S(\mathbb{R}^2) \) that converges to some function \( \psi \in L^2(\mathbb{R}^2) \), the sequence \( (\langle ., \varphi_n \rangle)_{n \geq 1} \) is also a Cauchy sequence in the Hilbert space \( L^2(\mu) \) and thus converges to some element in \( L^2(\mu) \) that we still note \( \langle ., \psi \rangle \) \((17,\) p.13). The converse is also true. If \( (\varphi_n)_{n \geq 1} \) is a sequence in \( S(\mathbb{R}^2) \) such that the sequence \( (\langle ., \varphi_n \rangle)_{n \geq 1} \) converges to some element \( \psi \in L^2(\mu) \), the sequence \( (\langle ., \varphi_m \rangle)_{n \geq 1} \) is a Cauchy sequence in \( L^2(\mu) \) and thus by the equality

\[
\| \varphi_n - \varphi_m \|_{L^2(\mathbb{R}^2)} = \| \langle ., \varphi_n - \varphi_m \rangle \|_{L^2(\mu)} = \| \langle ., \varphi_n \rangle - \langle ., \varphi_m \rangle \|_{L^2(\mu)}
\]

the sequence \( (\varphi_n)_{n \geq 1} \) is a Cauchy sequence in \( L^2(\mathbb{R}^2) \) and thus converges to some element \( \psi \in L^2(\mathbb{R}^2) \) what implies

\[
\psi = \langle ., \psi \rangle.
\]

This proves that the Gaussian Hilbert space generated by the i.i.d. sequence of standard normal variables \( (\langle ., \eta _j \rangle)_{j \geq 1} \) is the space \( L^2(\mathbb{R}^2) \) isometrically imbedded in \( L^2(\mu) \) by the mapping \( L^2(\mathbb{R}^2) \to L^2(\mu) : \psi \mapsto \langle ., \psi \rangle \). Denoting by \( H^1 \): the closed vector subspace of \( L^2(\mu) \) generated by the random variables \( \langle ., \psi \rangle \), \( \psi \) running among \( L^2(\mathbb{R}^2) \), the so called homogeneous Wiener chaos of order 1 \((21,\) p.4) \((13,\) p.44), we may write by identifying every \( \psi \in L^2(\mathbb{R}^2) \) with \( \langle ., \psi \rangle \) that \( H^1 = H := L^2(\mathbb{R}^2) \). Let us observe
that for every $\psi \in L^2(\mathbb{R}^2) \setminus \{0\}$, that the square integrable random variable defined on the probability space $(\mathcal{S}'(\mathbb{R}^2), \mathcal{B}_{\mathcal{S}'(\mathbb{R}^2)}, \mu)$

$$\langle ., \psi \rangle: \mathcal{S}'(\mathbb{R}^2) \rightarrow \mathbb{R} : \omega \mapsto \langle \omega, \psi \rangle$$

is a normal variable with mean 0 and variance $\|\psi\|_{L^2(\mathbb{R}^2)}$. It is also interesting to remark that the mapping which sends every Borel set $B$ of $\mathbb{R}^2$ of finite Borel measure onto the square integrable random variable defined on the probability space $(\mathcal{S}'(\mathbb{R}^2), \mathcal{B}_{\mathcal{S}'(\mathbb{R}^2)}, \mu)$

$$\langle ., 1_B \rangle: \mathcal{S}'(\mathbb{R}^2) \rightarrow \mathbb{R} : \omega \mapsto \langle \omega, 1_B \rangle$$

is a random orthogonal measure with the Borel measure on $\mathbb{R}^2$ as reference measure ( [16], p. 255), ( [20], p. 40).

**Remark 2.4:** It follows immediately from the above definition of the Wick product that for every $\eta_j \in \mathcal{S}(\mathbb{R}^2)$ belonging to the orthonormal basis of $L^2(\mathbb{R}^2)$ constructed by taking tensor products of 1-D Hermite functions that

$$\eta_j^{\otimes n} = \eta_j \otimes \eta_j \otimes \cdots \otimes \eta_j = h_n(\eta_j)$$

where $h_n$ denote the Hermite polynomial of order $n$ (e.g. $\eta_j^{\otimes 2} = \eta_j^2 - 1$, $\eta_j^{\otimes 3} = \eta_j^3 - 3\eta_j$, $\eta_j^{\otimes 4} = \eta_j^4 - 6\eta_j^2 + 3$, $\eta_j^{\otimes 5} = \eta_j^5 - 10\eta_j^3 + 15, \cdots$). In particular $\eta_j^{\otimes n}$ belongs to the polynomial chaos of order $n$ ( [21] p.7). The so called homogeneous Wiener chaos of order two $H^{2,2}$ ( [21], p. 4) ( [13], p. 44) is the closed vector space generated by the $h_2(\eta_j) = \eta_j^{\otimes 2}$ and the $h_1(\eta_j) h_1(\eta_j) = \eta_j \eta_j = \eta_j \otimes \eta_j$ for $i \neq j$. $\eta_j^2$ whose meaning is in fact $<., \eta_j>$ being the square of a standard normal random variable is a chi-square random variable with one degree of freedom from which it is easy to derive that the probability density function of the random variable $\eta_j^{\otimes 2} = h_2(\eta_j) = \eta_j^2 - 1$ is given by the function

$$R \rightarrow \mathbb{R}^+ : x \mapsto \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{1+x}} & \text{for } x > -1, \\ 0 & \text{for } x \leq -1. \end{cases}$$

On the other hand by using the advanced change of variables by area formula (theorem 1.12 page 5 of [23]) and the fact that that $(<., \eta_i>)_{i \in \mathbb{N}}$ is a i.i.d. sequence of standard Gaussian random variables, we find that the product random variables $\eta_i \eta_j$ (thus in fact $<., \eta_i> <., \eta_j>$) have for $i \neq j$ as probability density function: $\frac{1}{2} K_0(|.|)$, where $K_0$ denotes the modified Hankel function of order 0 ([1] pp. 374-375) ($K_0(x)$ is for $x > 0$ the solution of the modified Bessel equation of order 0

$$y'' + \frac{1}{x} y' - y = 0$$

which has a regular singular point at 0 ([4], p.71-75), such that $K_0(x) \sim -\ln(x)$ as $x \rightarrow 0^+$. Thus already the laws of probability of the random variables of the homogeneous Wiener chaos of order 2, $H^{2,2}$, do not seem any more, at first sight at least, to belong to some common stable family of probability laws like for $H^{1}$.

We close this section by the following technical lemma, that we will need in section 3:

**Lemma 2.1:** The two-dimensional “Hermite functions” $(\eta_j)_{j \geq 1}$ on $\mathbb{R}^2$ are uniformly bounded in $j$. In particular $\|\eta_j\|_{\infty, \mathbb{R}^2} \leq 1$, $\forall j \in \mathbb{N}$. A fortiori $\|\eta^\alpha\|_{\infty, \mathbb{R}^2} \leq 1$, $\forall \alpha \in \mathcal{I}$. 

Proof: Denoting temporarily by $H_n$ the “physical form” of the Hermite polynomials orthogonal with respect to the weight $\exp\left(-x^2\right)$ and with leading coefficient $2^n$, we have by inequality 22.14.17 p.787 of [1]:

$$|H_n(x)| \leq \exp\left(\frac{x^2}{2}\right) k 2^{\frac{n}{2}} \sqrt{n!}$$

(11)

with $k \approx 1.086435$.

Now, we have the following formula which links our Hermite polynomials $h_n$ monic and orthogonal with respect to the Gaussian weight $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$, the so called “probabilistic form” of the Hermite polynomials, to the $H_n$:

$$h_n(x) = 2^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right), \forall x \in \mathbb{R}, \forall n \in \mathbb{N}_0.$$  

(12)

On the other hand, for every $n \in \mathbb{N}$, the one-dimensional Hermite function $\xi_n$ on $\mathbb{R}$ is linked to $h_{n-1}$ by the formula ( [17], (2.2.2), p.18):

$$\xi_n(x) = \pi^{-\frac{1}{4}} ((n - 1)!)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x^2\right) h_{n-1}\left(\sqrt{2}x\right), \forall x \in \mathbb{R}.$$  

(13)

From formulas (11) to (13) follows that:

$$|\xi_n(x)| \leq \frac{k}{\pi^{\frac{3}{4}}} \leq 0.82, \forall x \in \mathbb{R}.$$  

Thus $\|\xi_n\|_{\infty, \mathbb{R}} \leq 1, \forall n \in \mathbb{N}$. As the Hermite functions $\eta_j, j \in \mathbb{N}$, on $\mathbb{R}^2$ are simply tensor products of the Hermite functions $\xi_n$ on $\mathbb{R}$, we have also $\|\eta_j\|_{\infty, \mathbb{R}^2} \leq 1, \forall j \in \mathbb{N}$. This implies also that $\|\eta^\alpha\|_{\infty, \mathbb{R}^2} \leq 1, \forall \alpha \in \mathcal{I}$.

3. Existence, uniqueness and time regularity of the solution of the classical variational formulation of the heat diffusion equation in a random medium

We firstly recall the (classical) variational formulation of the heat diffusion equation in a random medium and T.G Theting’s result on existence and uniqueness [33]. Then we will give a time regularity result for the solution. Let $D \subset \mathbb{R}^2$ be an open bounded set in $\mathbb{R}^2$ (we will restrict ourselves later to polygonal domains in $\mathbb{R}^2$). Let $T$ be a positive real number, fixed. As already said in the previous section, for $k \in \mathbb{R}$, $S^{-1,k,1}_0(D)$ (or $S_0^{-1,k,1}$) denotes the space $S^{-1,k,H^1(D)}$ and $S^{-1,k,0}_0(D)$ (or $S^{-1,k,0}$) the space $S^{-1,k,L^2(D)}$.

In the following, $\|\cdot\|_{1,k,1}$ (resp. $\|\cdot\|_{1,k,0}$) denotes the norm in $S^{-1,k,1}_0$ (resp. $S^{-1,k,0}$) and $(\cdot, \cdot)_{1,k,1}$ (resp. $(\cdot, \cdot)_{1,k,0}$) denotes the scalar product in $S^{-1,k,1}_0$ (resp. $S^{-1,k,0}$).

By (2.12) p.5 of [33] the dual space of $S^{-1,k,1}_0(D)$ may be identified to $S^{1,-k,H^{-1}}(D)$ (we will denote sometimes this space more simply $S^{1,-k,-1}$) under the pairing:

$$\langle (F, f) \rangle = \sum_{\alpha \in \mathcal{I}} (F_\alpha, f_\alpha) \alpha!.$$  

It is immediately seen that this series is absolutely convergent and

$$\sum_{\alpha \in \mathcal{I}} |(F_\alpha, f_\alpha)| \alpha! \leq \sum_{\alpha \in \mathcal{I}} \|F_\alpha\|_{H^{-1}(D)} \alpha! (2\mathbb{N})^{-\alpha} \|f_\alpha\|_{H^1(D)} (2\mathbb{N})^{\frac{\alpha}{2}}$$

$$\leq \left( \sum_{\alpha \in \mathcal{I}} \|F_\alpha\|_{H^{-1}(D)}^2 (\alpha!)^2 (2\mathbb{N})^{-\alpha k} \right)^{\frac{1}{2}} \left( \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|_{H^1(D)}^2 (2\mathbb{N})^{\alpha k} \right)^{\frac{1}{2}}$$
From this latter inequality follows immediately that the mapping
$S_{0}^{-1,k,1} \rightarrow \mathbb{R} : f \mapsto \langle \langle F, f \rangle \rangle$ is a continuous
linear form on $S_{0}^{-1,k,1}$. Consequently, the norm and the inner product on the dual space of $S_{0}^{-1,k,1}$ will be
denoted $\|\cdot\|_{1,-k,-1}$ and $\langle \cdot, \cdot \rangle_{1,-k,-1}$.
To introduce the classical variational formulation [33] of the stochastic heat equation, we firstly need
to recall the definition of Sobolev spaces comprising functions mapping time in Hilbert spaces ([10], p. 285, . . .) ([8], vol. 8 p. 577-579).

**Definition 3.1:** Let $X$ be a real separable Hilbert space.

By $W(0, T; X)$, we denote the space of all square-integrable function from $[0, T]$ into $X$ having a weak time
derivative square integrable from $[0, T]$ into $X'$ i.e. $W(0, T; X) = \{\psi \in L^2(0, T; X) : \psi' \in L^2(0, T; X')\}$.

If $H$ is another separable Hilbert space and if there is a continuous injection with dense image from $X$
into $H$, then ([8], vol. 8 p. 579) $W(0, T; X)$ maps continuously into $C([0, T] ; H)$, the space of continuous
functions from $[0, T]$ into $H$ endowed with the sup norm. In particular

$$W\left(0, T; S_{0}^{-1,k,1}(D)\right) \hookrightarrow C\left([0, T] ; S_{0}^{-1,k,0}(D)\right)$$

Now let us suppose that $f \in L^2\left(0, T; S_{0}^{-1,k,1}(D)\right)$ and that the initial condition $g \in S_{0}^{-1,k,0}(D)$. Let
us also suppose that the stochastic diffusion coefficient $K \in F_1(D)$ and that $k \leq 2l$. We have the following
existence and uniqueness result, which results from theorem 4.10 p. 12 of T.G. Theting’s paper [33], for the “classical”
variational formulation of the heat diffusion equation with stochastic diffusion coefficient $K$.

**Theorem 3.2:** [33] Let us assume that the stochastic diffusion coefficient $K \in F_1(D)$ for some $l \in \mathbb{R}$ and
that its generalized expectation $E[K]$ is strictly positively lower bounded i.e. that $\inf_{D} E[K] > 0$. Let us
suppose that $k \in \mathbb{R}$ is choosen sufficiently small to satisfy to the condition

$$k < 2l + \frac{2}{\ln 2} \ln \left(\frac{\inf_{D} E[K]}{\|K\|_{l,s}}\right).$$

Then $\forall f \in L^2\left(0, T; S_{0}^{-1,k,1}(D)\right)$ and $\forall g \in S_{0}^{-1,k,0}(D)$, there exists one and only one $u \in
W\left(0, T; S_{0}^{-1,k,1}(D)\right) \hookrightarrow C\left([0, T] ; S_{0}^{-1,k,0}(D)\right)$ solution of the classical variational formulation relative
to the heat equation with random diffusion coefficient $K$ (1):

$$\begin{cases}
\frac{d}{dt} (u(\cdot), v)_{-1,k,0} + \left(\mathcal{K}: \mathcal{V} u(\cdot), \mathcal{V} v\right)_{-1,k,0} = \langle f(\cdot), v \rangle_{-1,k,0}, \quad \forall v \in S_{0}^{-1,k,1}(D) \\
u(0) = g.
\end{cases}$$

(15)

Moreover we have the following energy inequality:

$$\sup_{t \in [0, T]} \|u(t)\|^2_{-1,k,0} + \int_{0}^{T} \|u(t)\|^2_{-1,k,1} dt \lesssim \left(\|g\|^2_{-1,k,0} + \int_{0}^{T} \|f(t)\|^2_{1,-k,-1} dt\right).$$

(16)

**Remark 3.1:** In (15), concerning the second term in the left-hand side, $(\cdot, \cdot)_{-1,k,0}$ denotes in fact the scalar product in $S_{0}^{-1,k,0}(D)^2$. 
Remark 3.2: Due to our hypothesis (14) which implies that \( k \leq 2l \) and lemma 4.9 p. 12 of [33], the bilinear form:

\[
S^{-1,k,1}_0 (D) \times S^{-1,k,1}_0 (D) \rightarrow \mathbb{R} : (u, v) \mapsto \left( \mathcal{K} \triangledown \nabla u; \nabla v \right)_{-1,k,0}
\]

(17)
is well defined and coercive on \( S^{-1,k,1}_0 (D) \). Theorem 3.2 is then a consequence of theorems 1 and 2 chapter XVIII, p.619 and 620 of [8]. Let us also mention that some existence and uniqueness result for the Cauchy problem (15) could also be obtained by applying Lumer-Phillips’ theorem ([26], p. 14) in the Hilbert space \( \mathcal{S}^{-1,k,0}_0 (D) \).

If we suppose the stronger condition on \( k \) that

\[
k < 2l + \frac{2}{\ln 2} \ln \left( \inf_{D} \frac{E[\mathcal{K}]}{1.5 \| \mathcal{K} \|_{L^2}} \right),
\]

(18)
it can even be shown that \( A \) generates a holomorphic semi-group ([22], theorem 1 p. 237).

Example 3.3 Let us recall firstly the definition of the singular white noise field \( W = (W(x))_{x \in D} \) (following ([16], p. 80), we prefer to say “field” instead of “process” because the parameter \( x \) runs here over \( D \) a bounded subregion of the plane): \( W(x) := \sum_{i=1}^{+\infty} \eta_i (x) H_{\varepsilon_i}, x \in D \) ([32], p. 4) ([17], p. 38) where \( \varepsilon_i \in \mathbb{I} \) denotes the multi-index whose \( i \)th component is 1 and whose other components are 0. As the two-dimensional “Hermite functions” \( (\eta_i)_{i \geq 1} \) on \( \mathbb{R}^2 \) are uniformly bounded in \( i \) by lemma 2.1, and as the series \( \sum_{i=1}^{+\infty} (2i)^l \) converges if \( l < -1 \), it results immediately from the definition of \( \mathcal{F}_l (D) \) (10) that \( W \in \mathcal{F}_l (D) \) for \( l < -1 \). But \( E[|W(x)|] = 0, \forall x \in D \). Thus if we choose for the coefficient of diffusion \( \mathcal{K} \) the white noise field, the hypotheses of T.G. Theting’s theorem 3.2 could not be verified. Let us consider rather for \( \mathcal{K} \) its Wick exponential: the so-called singular positive noise field on \( D \)

\[
\mathcal{K} = \exp \triangledown [W] := \sum_{n=0}^{+\infty} \frac{1}{n!} W^{\otimes n} \quad ([17], p. 67, p. 65, p. 166).
\]

(19)

It follows easily by using the basic algebraic properties of the Wick product ([17], lemma 2.4.5 p. 42) that

\[
\mathcal{K} = \sum_{\alpha \in \mathbb{I} \backslash \{0\}} \frac{\eta^\alpha}{\alpha!} H_{\alpha} \quad ([32], p. 17),
\]

where \( \eta^\alpha \) means \( \prod_{i=1}^{+\infty} \eta_i^{\alpha_i} \) and \( \alpha \) means \( \prod_{i=1}^{+\infty} \alpha_i \). Knowing by lemma 2.1 that \( \| \eta^\alpha \|_{\infty} \leq 1, \forall \alpha \in \mathbb{I} \), and using proposition 2.3.3 p. 31 of [17] (or [37]) which tells us that the series \( \sum_{\alpha \in \mathbb{I}} (2n)^{-\alpha q} \) converges if \( q > 1 \), it follows easily that the positive singular noise exp\( \triangledown W(\cdot) \in \mathcal{F}_l (D) \) for \( l < -1 \). Alternatively, this follows immediately from the fact that \( W \in \mathcal{F}_l (D) \) for \( l < -1 \) and proposition 6, (ii) p. 123 of [35] (an elementary operational calculus result).

Also as \( E[\mathcal{K}] = 1 \), the hypothesis \( \inf_{D} E[\mathcal{K}] > 0 \) of T.G. Theting’s theorem 3.2 is trivially satisfied. Thus T.G. Theting’s theorem 3.2 applies in this case, if we take \( k \) sufficiently negative for condition (14) to be satisfied.
Now we want to give some time regularity result:

**Theorem 3.4:** Additionally to the hypotheses of theorem 3.2, we suppose that the right-hand side \( f \) and its time derivative \( \frac{df}{dt} \) belong to \( L^2 \left( 0, T; \mathcal{S}^{-1,k,0} \left( D \right) \right) \). We also suppose that the initial condition \( g \in \mathcal{S}_0^{-1,k,1} \left( D \right) \) and satisfies \( \text{div} \left( \mathcal{K} \cdot \nabla g \right) \in \mathcal{S}^{-1,k,0} \left( D \right) \), this latter condition being satisfied for example if \( \Delta g \in \mathcal{S}^{-1,k,0} \left( D \right) \) and \( \nabla \mathcal{K} \in \mathcal{F}_1( D)^2 \). Then the time derivative \( \frac{du}{dt} \) of the solution of the classical variational formulation (15) has the following regularity properties:

\[
\frac{du}{dt} \in L^2 \left( 0, T; \mathcal{S}_0^{-1,k,1} \left( D \right) \right) \cap C \left( [0, T]; \mathcal{S}^{-1,k,0} \left( D \right) \right).
\]

**Proof:** Let us consider the Cauchy problem: find \( u \) such that:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{du}{dt} (t) + \left( \mathcal{K} \cdot \nabla u \right)(t) = f(t), u(0) = g
\end{array} \right.
\end{align*}
\]

Proof:

As \( \frac{df}{dt} \in L^2 \left( 0, T; \mathcal{S}^{-1,k,0} \right) \hookrightarrow L^2 \left( 0, T; \mathcal{S}^{1,-k,-1} \right) \) and \( g(0) \in \mathcal{S}^{-1,k,0} \) (because \( f(0) \in \mathcal{S}^{-1,k,0} \) due to the hypotheses on \( f \) and \( \frac{df}{dt} \)), it follows by theorem 3.2 that \( z \in L^2 \left( 0, T; \mathcal{S}_0^{-1,k,1} \right) \cap C \left( [0, T]; \mathcal{S}^{-1,k,0} \right) \) and

\[
\|z\|_{L^2(0,T;\mathcal{S}_0^{-1,k,1})} + \|z\|_{C([0,T];\mathcal{S}^{-1,k,0})} \leq \left\| f(0) \right\|_{\mathcal{S}_0^{-1,k,0}} + \left\| \frac{df}{dt} \right\|_{L^2(0,T;\mathcal{S}^{-1,k,0})}.
\]

Let us set \( u(t) = \int_0^t z(s)ds + g, \frac{du}{dt}(t) = z(t) \) a.e. and by integrating both sides of equation (20) from 0 to \( t \), taking into account the initial condition (20)(i) and applying Green’s formula (24), (2.10) p.611, we obtain equation (15)(i). We have also \( u(0) = g \) i.e. (15)(ii). By unicity, it is thus the solution of the Cauchy problem (15).

We have \( \frac{du}{dt} = z \) from which the stated regularity on \( \frac{du}{dt} \) follows. 

To be able to establish error estimates for the semi-discrete solution of the dual mixed method relative to the heat equation in a stochastic medium (1), we will need also some spatial regularity of its solution \( u \), in weighted Sobolev spaces.

Firstly, let us recall the definition of the weighted Sobolev spaces, \( H^{2,\alpha_w}(D) \), \( 0 < \alpha_w < 1 \).

**Henceforth, we suppose that \( D \) is a plane domain, simply connected, with a polygonal boundary \( \mathbf{\Gamma} \), the union of a finite number \( N \) of linear segments \( \mathbf{\Gamma}_i \) numbered according to the positive orientation.**

**We denote by \( \omega_i \) the aperture of the angle between \( \mathbf{\Gamma}_i \) and \( \mathbf{\Gamma}_{i+1} \) for \( i = 1, \ldots, N \) (\( \mathbf{\Gamma}_{N+1} := \mathbf{\Gamma}_1 \)).** We suppose that \( D \) possesses only one reentrant corner \( \{S_N\} = \mathbf{\Gamma}_N \cap \mathbf{\Gamma}_1 \). For simplicity we assume that \( S_N \) is situated at the origin of our cartesian frame. By \( r(\cdot) \) we denote the distance function from an arbitrary point in the plane \( \mathbb{R}^2 \) to the origin and \( \omega \) denotes the aperture of our reentrant corner.

**Definition 3.5:** (15, p.388) For \( \alpha_w \in [0,1] \), we denote by \( H^{2,\alpha_w}(D) \) the space of all functions in \( H^1(D) \) such that in addition \( r^\alpha u \in L^2(D) \) for every \( \beta \in \mathbb{N}_0^2 \) such that \( |\beta| = 2 \).

**Theorem 3.6:** We suppose that the right-hand side \( f \) and its time-derivative \( \frac{df}{dt} \) belong to \( L^2 \left( 0, T; \mathcal{S}^{-1,k,0} \left( D \right) \right) \), and that the initial condition \( g \) of the Cauchy problem (15) belongs to

\[
\left\{ g \in \mathcal{S}_0^{-1,k,1}(D); \Delta g \in \mathcal{S}^{-1,k,0}(D) \right\}.
\]

On the stochastic diffusion coefficient \( \mathcal{K} \), we suppose that its generalized expectation \( E [\mathcal{K}] \) is strictly positively lower bounded i.e. that \( \inf_D E [\mathcal{K}] > 0 \) and that \( \mathcal{K}, \mathcal{K}^{-1}, \frac{\partial \mathcal{K}}{\partial x_1}, \frac{\partial \mathcal{K}}{\partial x_2} \in \mathcal{F}_1( D) \). Finally, we suppose...
that \( k \in \mathbb{R} \) satisfies inequality (14).

Then \( u \) solution of the Cauchy problem (15) satisfies:

\[
u \in L^2 \left( 0, T; S^{-1,k<H^2,α_w}(D) \right) \quad \text{for all } \alpha_w \in \left] 1 - \frac{π}{ω}, 1 \right].\]

**Proof:** From the heat diffusion equation in a stochastic medium

\[
u_t - \text{div} \left( \mathcal{K} \diamond \nabla \nu \right) = f,
\]

follows that

\[
\mathcal{K} \diamond \Delta \nu = \nu_t - f - \nabla \mathcal{K} \diamond \nabla \nu.
\]  

(22)

By theorem 3.4: \( \nu_t \in L^2 \left( 0, T; S^{-1,k,0}(D) \right) \) and by theorem 3.2: \( \nu \in L^2 \left( 0, T; S^{-1,k,0}(D) \right) \).

Moreover \( \nabla \mathcal{K} \in \mathcal{F}_l(D)^2 \) with \( l \geq \frac{k}{2} \). Thus \( \nabla \mathcal{K} \diamond \nabla \nu \in L^2 \left( 0, T; S^{-1,k,0}(D) \right) \).

Consequently the right-hand side of equation (22) belongs to \( L^2 \left( 0, T; S^{-1,k,0}(D) \right) \). Moreover, by hypothesis, \( \mathcal{K}^{-1} \) exists and also belongs to \( \mathcal{F}_l(D) \). Thus:

\[
\Delta \nu = \mathcal{K}^{-1} \diamond \left( \nu_t - f - \nabla \mathcal{K} \diamond \nabla \nu \right) \in L^2 \left( 0, T; S^{-1,k,0}(D) \right).
\]

Let us set \( h := \mathcal{K}^{-1} \diamond \left( \nu_t - f - \nabla \mathcal{K} \diamond \nabla \nu \right) \).

\[
\| h \|_{L^2(0,T;S^{-1,k,0})} \leq \left\| \frac{du}{dt} \right\|_{L^2(0,T;S^{-1,k,0})} + \| u \|_{L^2(0,T;S^{-1,k,1})} + \| f \|_{L^2(0,T;S^{-1,k,0})}.
\]  

(23)

Setting \( u(t) = \sum_{α \in \mathcal{I}} u_α(t) H_α \) and \( h(t) = \sum_{α \in \mathcal{I}} h_α(t) H_α \) be the chaos expansions of \( u(t) \) and \( h(t) \) respectively, \( \forall t \in [0,T] \), we have:

\[
\Delta u_α(t) = h_α(t), \quad \forall α \in \mathcal{I}, \forall t \in [0,T].
\]  

(24)

As \( u_α(t) \in H_0^3(D) \) and \( h_α(t) \in L^2(D), \forall t \in [0,T] \), we have by (8,4,1,7) p. 388 of Grisvard’s book [15], that \( u_α(t) \in H^{2,α_w}(D) \) and by the closed graph theorem

\[
\| u_α(t) \|_{H^{2,α_w}(D)} \lesssim \| h_α(t) \|_{L^2(D)}, \quad \forall α \in \mathcal{I}, \forall t \in [0,T],
\]

(25)

with a constant (hidden in \( \lesssim \)) independant of \( α \) and \( t \).

Taking the squares of each side of inequality (25), multiplying both sides by \( (2N)^{kα} := \prod_{j=1}^{+∞} (2j)^{kα_j} \), summing over \( α \in \mathcal{I} \), and integrating on the time variable \( t \) from 0 to \( T \), we obtain:

\[
\int_0^T \sum_{α \in \mathcal{I}} \| u_α(t) \|_{H^{2,α_w}(D)}^2 (2N)^{kα} dt \lesssim \int_0^T \sum_{α \in \mathcal{I}} \| h_α(t) \|_{L^2(D)}^2 (2N)^{kα} dt
\]

i.e.

\[
\| u \|_{L^2(0,T;S^{-1,k,H^{2,α_w}}(D))} \lesssim \| h \|_{L^2(0,T;S^{-1,k,H^{2,α_w}}(D))}.
\]  

(26)

Thus \( u \in L^2 \left( 0, T; S^{-1,k,H^{2,α_w}}(D) \right) \) for all \( α_w \in \left] 1 - \frac{π}{ω}, 1 \right[. \)
Example 3.7 We give an example of a stochastic diffusion coefficient $K$ satisfying the hypotheses of theorem 3.6. Let $\phi \in H^1(\mathbb{R}^2)$ and let us set

$$K(x) = \exp\langle W_\phi(x,.) \rangle, \forall x \in D$$

where $W_\phi(x,.) := \langle \cdot, \phi_x \rangle, \forall x \in D$ and

$$\phi_x : \mathbb{R}^2 \to \mathbb{R} : y \mapsto \phi(x-y).$$

Let us recall that $W_\phi(x,.) := \langle \cdot, \phi_x \rangle$ is the element of $L^2(S'(\mathbb{R}^2),\mathcal{B}_{S'(\mathbb{R}^2)},\mu)$ defined by continuity and density from $S(\mathbb{R}^2)$ as explained in remark 2.3 ([17], (2.1.9) p.13). $(W_\phi(x,.) )_{x \in D} := \langle \cdot, \phi_x \rangle_{x \in D}$ is called the smoothed white noise field ([17] p.13, 18, 66). $W_\phi(x,.) := \langle \cdot, \phi_x \rangle$ has the following chaos expansion

$$W_\phi(x,.) := \langle \cdot, \phi_x \rangle = \sum_{i=1}^{+\infty} \langle \phi_x | \eta_i \rangle_{L^2(\mathbb{R}^2)} H_{\xi_i}(.), \quad (27)$$

where $\xi_i$ denotes the multi-indice belonging to $\mathcal{I}$ with 1 on entry number $i$ and 0 elsewhere. It is easy to see that this series is convergent in $L^2(\mu)$ and that its sum is a normal random variable $N\left(0, \|\phi\|^2\right)$ on the probability space $L^2(S'(\mathbb{R}^2),\mathcal{B}_{S'(\mathbb{R}^2)},\mu)$. From the definition of the space $\mathcal{F}_l(D)$, the boundedness of the coefficients in the series (27) and as the series $\sum_{i=1}^{+\infty} (2l)^i$ converges if $l < -1$, it follows immediately that

$$W_\phi \in \mathcal{F}_l(D), \forall l < -1.$$

From ([17] (2.6.48) p.66, (2.7.6) p.70) and (27), it follows that

$$K := \exp\langle W_\phi \rangle := \sum_{n=0}^{+\infty} \frac{1}{n!} (W_\phi)^n$$

$$= \sum_{\alpha \in \mathcal{I}} \frac{1}{\alpha!} \langle \phi | \eta \rangle^\alpha H_\alpha$$

where $\alpha! := \prod_{i=1}^{+\infty} \alpha_i!$ and $\langle \phi | \eta \rangle^\alpha := \prod_{i=1}^{+\infty} \langle \phi | \eta \rangle_\alpha^{\xi_i}$. By formula ([17], (2.6.49) p.65), it follows that $K(x)^{\alpha - 1} = \exp\langle -\langle \cdot, \phi_x \rangle \rangle$ and thus replacing $\phi$ by $-\phi$ in formula (28), we obtain

$$K(x)^{\alpha - 1} = \sum_{\alpha \in \mathcal{I}} \frac{1}{\alpha!} (-1)^{|\alpha|} \langle \phi_x | \eta \rangle^\alpha H_\alpha$$

(29)

where $|\alpha| := \sum_{i=1}^{+\infty} \alpha_i$. We are going to show that $K$, $K^{\alpha - 1}$, $\frac{\partial K}{\partial x_i}$, $\frac{\partial K}{\partial x_j}$ all belong to $\mathcal{F}_l(D)$ for $l < -1$. That $K$ and $K^{\alpha - 1}$ belong to $\mathcal{F}_1(D)$ for $l < -1$ results from $W_\phi \in \mathcal{F}_l(D)$ for $l < -1$ and proposition 6, (ii) p.123 of [35] (an elementary operational calculus result). By the definiton of the derivatives in the spatial variables $x_1, x_2$ on elements of stochastic Sobolev spaces (definition 3.4 p.8 [32]), it follows from (28) and (27) that for $i = 1, 2$

$$\frac{\partial K}{\partial x_i}(x) = \exp\langle W_\phi(x,.) \rangle \langle W_\phi(x,.) \rangle \frac{\partial x_i}{\partial x_i}(x)$$

$$= K(x) \langle W_\phi(x,.) \rangle \frac{\partial x_i}{\partial x_i}(x).$$

(30)
We know already that $K$ belongs to $\mathcal{F}_l(D)$ for $l < -1$. By (27):

$$W^{\phi}_{x_i} (x,.) = \sum_{j=1}^{+\infty} \left( \frac{\partial \phi}{\partial x_i} (x-) \eta_j \right)_{L^2(\mathbb{R}^2)} H_{x_i} (\cdot)$$

so that by the same reasoning as above follows that $W^{\phi}_{x_i}$ also belong to $\mathcal{F}_l(D)$ for $l < -1$ ($i=1,2$). $\mathcal{F}_l(D)$ being a commutative Banach algebra for the Wick product (prop. 6 p.123 [35])

$$\frac{\partial K}{\partial x_i} = K \phi W^{\phi}_{x_i}, \ K \in \mathcal{F}_l(D) \text{ for } l < -1.$$

From (28) follows that $E[K(x)] = 1$, $\forall x \in D$, so that the hypothesis $\inf_D E[K] > 0$ of theorem 3.6 is trivially satisfied in this case. Thus in conclusion for $\phi \in H^1(\mathbb{R}^2)$, the stochastic diffusion coefficient $K$ defined by

$$K(x) = \exp (W_{\phi}(x,.)), \ \forall x \in D$$

satisfies for $l < -1$ all the hypotheses of theorem 3.6. Thus if $k \in \mathbb{R}$ is choosen sufficiently negative to satisfy condition (14) for some $l < -1$, if $f, \frac{df}{dt} \in L^2 \left(0, T; S^{-1,k,0}(D)\right)$, and if the initial condition $g \in \left\{ g \in S^{-1,k,1}(D); \Delta g \in S^{-1,k,0}(D) \right\}$, then the weak solution in the classical variational sense of

$$\begin{cases}
  u_t - \text{div} \left( \exp W_{\phi_x} \nabla u \right) = f & \text{in } Q := ]0, T[ \times D \\
  u = 0 & \text{on } ]0, T[ \times \partial D \\
  u_{|t=0} = g & \text{on } D.
\end{cases}$$

$u \in L^2 \left(0, T; S^{-1,k,\alpha}(D)\right)$ for all $\alpha \in ]1 - \frac{2}{n}, 1[$. Moreover by theorem 3.4, $u$ and $\frac{du}{dt} \in L^2 \left(0, T; S^{-1,k,1}(D)\right) \cap C \left([0, T]; S^{-1,k,0}(D)\right)$.

**Example 3.8** In the preceding example, whatever $\omega \in S'(\mathbb{R}^2)$ is, due to the formula:

$$\exp \left[ \langle \omega, \phi_x \rangle \right] = \exp \left( \langle \omega, \phi_x \rangle - \frac{1}{2} \| \phi \|_{L^2(\mathbb{R}^2)}^2 \right)$$

which is proved to be true in [17] (lemma 2.6.16 p.66) for every function $\phi \in L^2(\mathbb{R}^2)$, the stochastic diffusion coefficient

$$K = \exp (W_{\phi})$$

is always strictly positive. Thus, it seems to be worthwhile to give an example of a diffusion coefficient $K$ which can take negative values though satisfying all the the hypotheses of theorem 3.6 for $k$ sufficiently negative. Let us consider as diffusion coefficient

$$K(x) = 1 + W_{\phi}(x,.), \ \forall x \in D$$

$\phi$ being some function belonging to $H^1(\mathbb{R}^2)$. We know already from the previous example that $K, \frac{\partial K}{\partial x_i}$ ($i=1,2$) belong to $\mathcal{F}_l(D)$ for $l < -1$. Let us find a condition on $K$ who assures us that $K^{-1}$ exists and
then also conditions (31), (32) are satisfied, it follows by theorem 3.6, that if $f$, then condition (14) is satisfied for this example. Supposing that $\phi$, then the series

$$1 - W_\phi + W_\phi^{\otimes 2} - W_\phi^{\otimes 3} + \cdots$$

converges to some element of $\mathcal{F}_l (D)$. Thus if $\|W_\phi\|_{l,s} < 1$, its Wick inverse $W_\phi^{-1}$ exists and belong to $\mathcal{F}_l (D)$ ($l < -1$). But this condition is rather abstract; we would like a condition directly on $\phi$. Thus, let us estimate $\|W_\phi\|_{l,s}$:

$$W_\phi(x, \cdot) = \sum_{i=1}^{+\infty} (\phi_x \mid \eta_i)_{L^2(\mathbb{R}^2)} H_{\epsilon_i},$$

where $\epsilon_i = (0, \ldots, 1_{(i \text{ position})}, \ldots, 0, \ldots)$, and

$$H_{\epsilon_i} : S' (\mathbb{R}^2) \rightarrow \mathbb{R} : \omega \mapsto \langle \omega, \eta_i \rangle_{S'(\mathbb{R}^2), S(\mathbb{R}^2)}.$$

From the definition of the norm in $\mathcal{F}_l (D)$ follows that

$$\|W_\phi\|_{l,s} = \sup_{x \in D} \left( \sum_{i=1}^{+\infty} \| (\phi_x \mid \eta_i)_{L^2(\mathbb{R}^2)} \| (2n)^{i} \right)$$

$$\leq \|\phi\|_{L^2(\mathbb{R}^2)} \sum_{i=1}^{+\infty} (2i)^l = \|\phi\|_{L^2(\mathbb{R}^2)} 2^l \sum_{i=1}^{+\infty} i^l,$$

this late series being convergent if $l < -1$. Thus if $l$ is choosen sufficently negative so that

$$2^l \sum_{i=1}^{+\infty} i^l < \frac{1}{\|\phi\|_{L^2(\mathbb{R}^2)}},$$

the series $\sum_{n=1}^{+\infty} (-1)^n W_\phi^{\otimes n}$ will be absolutely convergent in the Banach space $\mathcal{F}_l (D)$ to $W_\phi^{-1}$ ($l < -1$). In conclusion, if $\phi \in H^1 (\mathbb{R}^2)$ and $l < -1$, then $\mathcal{K}, \frac{\partial \mathcal{K}}{\partial x_1}, \frac{\partial \mathcal{K}}{\partial x_2} \in \mathcal{F}_l (D)$ and if $l$ satisfies moreover condition (31), then also $\mathcal{K}^{-1} \in \mathcal{F}_l (D)$. Let us observe also in this example that the generalized expectation $E [\mathcal{K}] = 1$. If $k \in \mathbb{R}$ satisfies

$$k < 2l - \frac{2}{\ln 2} \ln \left( 1 + \|\phi\|_{L^2(\mathbb{R}^2)} 2^l \sum_{i=1}^{+\infty} i^l \right),$$

(32)

then condition (14) is satisfied for this example. Supposing that $\phi \in H^1 (\mathbb{R}^2)$, that $l < -1$ and that conditions (31), (32) are satisfied, it follows by theorem 3.6, that if $f, \frac{df}{dt} \in L^2 (0, T; S^{-1,k,0}(D))$ and if the initial condition $g \in \left\{ g \in S_0^{-1,k,1}(D); \Delta g \in S^{-1,k,0}(D) \right\}$, then the weak solution $u$ in the classical variational sense of

$$\begin{cases}
    u_t - \nabla \cdot \left( (1 + W_\phi) \nabla u \right) = f & \text{in } Q := ]0, T[ \times D \\
    u = 0 & \text{on } ]0, T[ \times \partial D \\
    u|_{t=0} = g & \text{on } D.
\end{cases}$$

belongs to $L^2 (0, T; S^{-1,k,\mathcal{H}^2,\alpha_0}(D))$ for all $\alpha_0 \in ]1 - \frac{n}{2}, 1[$. Moreover by theorem 3.4, $u$ and $\frac{du}{dt} \in$
$L^2 \left( 0, T; S^{-1,k,1}_0(D) \right) \cap C \left( [0, T]; S^{-1,k,0}(D) \right)$.

4. The mixed dual formulation for the heat diffusion equation in a stochastic medium

In the following, to alleviate the notations, we will denote by $\mathcal{H}(\text{div}; D)$ the space $S^{-1,k,H(\text{div};D)}$ where $H(\text{div}; D) = \left\{ \tilde{\psi} \in L^2(D); \text{div} \tilde{\psi} \in L^2(D) \right\}$ this latter space being endowed with its natural norm and $\mathcal{H}(\text{div}; D) := S^{-1,k,H(\text{div};D)}$ with the corresponding norm. Let us introduce the new variable $\tilde{p} := \mathcal{K}\nabla u$. Under the hypotheses of T.G. Theting’s theorem 3.2, we have that $u \in L^2 \left( 0, T; S^{-1,k,1}_0(D) \right), \mathcal{K} \in \mathcal{F}_1(D)$ and $k \leq 2l$; consequently $\tilde{p} := \mathcal{K}\nabla u \in L^2 \left( 0, T; \left( S^{-1,k,0}(D) \right)^2 \right)$.

Let us now assume that the stronger hypotheses of theorem 3.4 are verified. In particular $\frac{du}{dt} \in L^2 \left( 0, T; S^{-1,k,0}(D) \right)$. Applying equation (15)(i), it follows that:

$$\text{div} \tilde{p} = u_t - f \in L^2 \left( 0, T; S^{-1,k,0}(D) \right).$$

Thus $\tilde{p} \in L^2 \left( 0, T; \left( S^{-1,k,0}(D) \right)^2 \right)$ and $\text{div} \tilde{p} \in L^2 \left( 0, T; S^{-1,k,0}(D) \right)$. Equivalently

$$\tilde{p} \in L^2 \left( 0, T; \mathcal{H}(\text{div}; D) \right) \equiv L^2 \left( 0, T; S^{-1,k,H(\text{div};D)} \right).$$

Now, let us take some $\tilde{q} \in \mathcal{H}(\text{div}; D)$. We now assume in addition to the hypotheses of theorem 3.4 that $\mathcal{K}^{-1} \in \mathcal{F}_1(D)$. For $\forall t \in [0, T]$, we have the equation $\mathcal{K}^{-1} \hat{\tilde{p}}(t) - \nabla u(t) = 0$. Taking the scalar product with $\tilde{q}$ in $S^{-1,k,L^2(\Omega)^2}$ and then applying Green’s formula:

$$\left( \mathcal{K}^{-1} \hat{\tilde{p}}(t), \tilde{q} \right)_{-1,k,0} = \sum_{\alpha \in \mathcal{I}} \left( \nabla u_\alpha(t), \tilde{q}_\alpha \right)_0 (2N)^{\kappa_\alpha} = - \sum_{\alpha \in \mathcal{I}} (u_\alpha(t), \text{div} \tilde{q}_\alpha)_0 (2N)^{\kappa_\alpha} = - (u(t), \text{div} \tilde{q})_{-1,k,0},$$

we obtain the equation

$$\left( \mathcal{K}^{-1} \hat{\tilde{p}}(t), \tilde{q} \right)_{-1,k,0} + (u(t), \text{div} \tilde{q})_{-1,k,0} = 0, \quad \forall \tilde{q} \in \mathcal{H}(\text{div}; D). \quad (33)$$

Taking the scalar product in $S^{-1,k,0}(D)$ of both sides of the equation

$$\text{div} \tilde{p}(t) = -(f(t) - u_t(t))$$

with any $v \in S^{-1,k,0}(D)$, we obtain $\forall t \in [0, T]$ the equilibrium equation:

$$\left( \text{div} \tilde{p}(t), v \right)_{-1,k,0} = - (f(t) - u_t(t), v)_{-1,k,0}, \quad \forall v \in S^{-1,k,0}(D). \quad (34)$$

Equations (33) and (34) form the mixed formulation of the stochastic heat equation with random diffusion coefficient $\mathcal{K}$ (and random heat sources and initial temperature also) (1).

More precisely, the mixed formulation of the Cauchy problem in the polygonal domain $D$ with random heat source $f$ and random initial temperature $g$, is the following problem: find $\tilde{p} \in L^2 (O, T; \mathcal{H}(\text{div}; D))$, ...
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\[ u \in H^1 \left( 0, T; \mathcal{S}^{-1,k,0}(D) \right) \text{ such that } \forall t \in [0, T]: \]
\[
\begin{cases}
(K^{\diamond} - 1,0,0)(t), \hat{q})_{-1,k,0} + (u(t), \text{div} \, \hat{q})_{-1,k,0} = 0, & \forall \hat{q} \in H(\text{div}; D), \\
(\text{div} \, \hat{p}(t), v)_{-1,k,0} = - (f(t) - ut(t), v)_{-1,k,0}, & \forall v \in \mathcal{S}^{-1,k,0}(D), \\
\text{and} \\
u(0) = g.
\end{cases}
\]

(35)

We have already proved that under the hypotheses of theorem 3.4 and \( K^{\diamond} - 1 \in \mathcal{F}_1(D) \), that problem (35) possesses at least one solution. It remains to prove uniqueness:

**Lemma 4.1:** Assuming that \( K^{\diamond} - 1 \in \mathcal{F}_1(D) \) and that \( k \) verifies condition (14), the bilinear form
\[
a(\cdot, \cdot) : \left( \mathcal{S}^{-1,k,0}(D) \right)^2 \times \left( \mathcal{S}^{-1,k,0}(D) \right)^2 \to \mathbb{R} : (\hat{p}, \hat{q}) \mapsto (K^{\diamond} - 1,0,0)(\hat{p}, \hat{q})_{-1,k,0}
\]
is also coercive.

**Proof:** It suffices of course to prove that the bilinear form
\[
\mathcal{S}^{-1,k,0}(D) \times \mathcal{S}^{-1,k,0}(D) \to \mathbb{R} : (u, v) \mapsto (K^{\diamond} - 1,0,0)(u, v)_{-1,k,0}
\]
is coercive. Let us set \( w = K^{\diamond} - 1,0,0 u \in \mathcal{S}^{-1,k,0}(D) \). Then:
\[
(K^{\diamond} - 1,0,0)(u, u)_{-1,k,0} = (w, K^{\diamond} w)_{-1,k,0} = (K^{\diamond} w, w)_{-1,k,0} \\
\geq c \| w \|^2_{-1,k,0}
\]
where \( c > 0 \) is the constant of coercivity of the bilinear form
\[
\mathcal{S}^{-1,k,0}(D) \times \mathcal{S}^{-1,k,0}(D) \to \mathbb{R} : (h, d) \mapsto (K^{\diamond} h, d)_{-1,k,0}
\]
(see remark 3.2 or lemma 4.9 p. 12 of [33]).

But
\[
\| u \|_{-1,k,0} = \left\| K^{\diamond} \left( K^{\diamond} - 1,0,0 u \right) \right\|_{-1,k,0} \leq \| K \|_{L^\infty} \| u \|_{-1,k,0}.
\]
Thus
\[
\| w \|_{-1,k,0} \geq \| K \|_{L^\infty}^{-1} \| u \|_{-1,k,0},
\]
Putting together these inequalities, it follows that:
\[
(K^{\diamond} - 1,0,0)(u, u)_{-1,k,0} \geq c \| K \|_{L^\infty}^{-2} \| u \|^2_{-1,k,0}, \quad \forall u \in \mathcal{S}^{-1,k,0}(D).
\]

This proves the coercivity of the bilinear form \( a(\cdot, \cdot) \). \( \square \)

**Theorem 4.2:** Under the hypotheses of theorem 3.4 and assuming also that \( K^{\diamond} - 1 \in \mathcal{F}_1(D) \), the mixed formulation (35) possesses one and only one solution. Denoting by \( u \in W(0, T; \mathcal{S}^{-1,k,0}(D)) \) the unique
solution of the classical variational solution (15), the unique solution of the mixed formulation (35) is given by

\[(\vec{p}(t), u(t)) = \left( K^\triangledown \nabla u(t), u(t) \right), \forall t \in [0, T]. \]

**Proof:** We have already proved that if \( u \in W(0, T; S^{-1,k,1}_0(D)) \) is the unique solution of the classical variational solution (15), that

\[(\vec{p}(t), u(t)) := \left( K^\triangledown \nabla u(t), u(t) \right), \forall t \in [0, T] \]

is solution of the mixed formulation (35).

It remains to prove unicity. Thus we suppose that \( f = 0 \) in (35)(ii) and that \( g = 0 \) in (35)(iii). From (35) follows:

\[\left( K^\triangledown \nabla \vec{p}(t), \vec{p}(t) \right)_{-1,k,0} = - (u_t(t), u(t))_{-1,k,0}. \]  \(36\)

Due to hypothesis (14) on \( k \), the bilinear form in the left-hand side of (36) is coercive. This implies that

\[
\frac{d}{dt} \|u(t)\|_{-1,k,0}^2 \leq 0.
\]

As \( u \in H^1(0, T; S^{-1,k,0}_0(D)) \) by theorem 3.4, \( u \) is absolutely continuous from \([0, T]\) with values in the separable Hilbert space \( S^{-1,k,0}_0(D) \) ( [33], lemma 2.3, p.5). It follows by integration then, that

\[\|u(t)\|_{-1,k,0}^2 \leq \|u(0)\|_{-1,k,0}^2 = 0.\]

Thus \( u = 0 \). By (36) and the coercivity of the bilinear form of its left-hand side, we now obtain \( \vec{p} = 0 \). \( \square \)

5. **Semi-Discrete solution of the dual mixed formulation for the heat diffusion equation in a stochastic medium**

Let us consider a family of triangulations \((T_h)_{h>0}\) on the polygonal domain \( D \subset \mathbb{R}^2 \) (let us recall that \( D \) possesses one and only one reentrant corner at the origin of \( \mathbb{R}^2 \)). For \( K \) a triangle belonging to the triangulation \( T_h \), let us denote by \( h_K \) the diameter of \( K \) and by \( \rho_K \) the interior diameter of \( K \) i.e. the diameter of the biggest disc included in \( K \). As in theorem 8.4.1.6 p. 392 of [15], we suppose that the family of triangulations \((T_h)_{h>0}\) has the property that \( \max_{K \in T_h} \frac{h_K}{\rho_K} \) is bounded by a positive constant independent of the parameter \( h \); in that case, one says usually that the family of triangulations is regular (see for example [7] (17.1) p.131). In accordance with the tradition (see [7] remark 17.1 p 131) the parameter \( h \) has also another significance: it may denotes instead \( \max_{K \in T_h} h_K \). The true significance of \( h \) is always clear from the context.

Let us now define the semi-discretized problem. Firstly, let us define the following finite dimensional vector subspaces \( X_h \) of \( X := H(\text{div}; D) \), respectively \( M_h \) of \( M := L^2(D) \):

\[ X_h := \left\{ \vec{q}_h \in H(\text{div}; D) : \forall K \in T_h : \vec{q}_h|_K \in RT_0(K) \right\}, \]

\[ M_h := \left\{ v_h \in L^2(D) : \forall K \in T_h : v_h|_K \in P_0(K) \right\}, \]
where $RT_0 (K) := P_0 (K)^2 \oplus P_0 (K) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ denotes the 3-dimensional vectorial space on $\mathbb{R}$ of all Raviart-Thomas vectorfields of degree 0 on the triangle $K$ i.e vectorfields of the form

$$K \to \mathbb{R}^2 : x = (x_1, x_2) \mapsto (a + cx_1, b + cx_2)$$

where $a, b, c$ are arbitrary real numbers. $P_0 (K)$ denotes the 1-dimensional vectorial space on $\mathbb{R}$ of all constant functions on the triangle $K$ (note that $RT_0 (K)$ is denoted $D_1 (K)$ in [29] p. 550).

Now for $N, K \in \mathbb{N}$, we define the “cutting” $I_{N,K} \subset \mathcal{I}$ by

$$I_{N,K} = \{ (0, \ldots, 0, \ldots) \} \cup \left( \bigcup_{n=1}^{N} \bigcup_{k=1}^{K} \{ \alpha \in \mathbb{N}_0^k : |\alpha| = n \text{ and } \alpha_k \neq 0 \} \right)$$

that is to say $I_{N,K}$ is the set of all multi-indices $\alpha$ such that their index (index $\alpha := \max\{j; \alpha_j \neq 0\}$) is smaller than or equal to $K$ and their modulus ($|\alpha| := \sum_{j=1}^{+\infty} \alpha_j$) is smaller than or equal to $N$. This set can be shown to contain $\frac{(N+K)!}{N!K!}$ different multi-indices ([34] p. 9) ([13] p. 82). We are now in a position to define finite dimensional vector subspaces of $\mathcal{H} (\text{div}; D)$, respectively of $S^{-1,k,0} (D)$:

$$X_h^{(N,K)} := \left\{ \tilde{q}_h = \sum_{\alpha \in I_{N,K}} \tilde{q}_{h,\alpha} H_\alpha ; \tilde{q}_{h,\alpha} \in X_h, \forall \alpha \in I_{N,K} \right\},$$

$$M_h^{(N,K)} := \left\{ v_h = \sum_{\alpha \in I_{N,K}} v_{h,\alpha} H_\alpha ; v_{h,\alpha} \in M_h, \forall \alpha \in I_{N,K} \right\}.$$  

Note that these spaces do not depend on $k$.

We can now define the semi-discretized problem corresponding to the mixed formulation of the stochastic heat equation (35):

find $(\tilde{p}_h, u_h) \in L^2 (0, T; X_h^{(N,K)}) \times H^1 (0, T; M_h^{(N,K)})$ such that, $\forall t \in [0, T]$:

$$\begin{cases}
(K^{0-1} \diamond \tilde{p}_h (t), \tilde{q}_h)_{-1,k,0} + (u_h (t), \text{div} \tilde{q}_h)_{-1,k,0} = 0, \quad \forall \tilde{q}_h \in X_h^{(N,K)}, \\
(\text{div} \tilde{p}_h (t), v_h)_{-1,k,0} = - (f(t) - u_{h,t} (t), v_h)_{-1,k,0}, \quad \forall v_h \in M_h^{(N,K)},
\end{cases}$$

(37)

and

$$u_h (0) = g_h \in M_h^{(N,K)}.$$  

The initial condition $g_h \in M_h^{(N,K)}$ will be precised later. Let us first show that the above problem (37) possesses one and only one solution in $L^2 (0, T; X_h^{(N,K)}) \times H^1 (0, T; M_h^{(N,K)})$:

**Theorem 5.1:** Doing the hypotheses of theorem 4.2, problem (37) possesses one and only one solution:

$$(\tilde{p}_h, u_h) \in L^2 (0, T; X_h^{(N,K)}) \times H^1 (0, T; M_h^{(N,K)}).$$

Moreover $\tilde{p}_h \in L^2 (0, T; X_h^{(N,K)})$. 

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Proof: Let $\bar{q}_h^{(1)}, \ldots, \bar{q}_h^{(J)}$ be a basis of $X_h$ and $\bar{v}_h^{(1)}, \ldots, \bar{v}_h^{(L)}$ be the special basis of $M_h$ formed by the characteristic functions of every triangle $K \in T_h$. Then the $\bar{q}_h^{(j)} H_{\alpha} \in X_h^{(N,K)}$, $j = 1, \ldots, J$, $\alpha \in \mathcal{I}_{N,K}$ form a basis of $X_h^{(N,K)}$ and the $\bar{v}_h^{(k)} H_{\alpha}$, $k = 1, \ldots, L$, $\alpha \in \mathcal{I}_{N,K}$ form a basis of $M_h^{(N,K)}$. Expanding $\vec{p}_h(t)$, respectively $\vec{u}_h(t)$ in these respective basis, we obtain:

$$\vec{p}_h(t) = \sum_{j=1}^{J} \sum_{\alpha \in \mathcal{I}_{N,K}} (p_h(t))_{j,\alpha} \bar{q}_h^{(j)} H_{\alpha}$$

and

$$\vec{u}_h(t) = \sum_{k=1}^{L} \sum_{\alpha \in \mathcal{I}_{N,K}} (u_h(t))_{k,\alpha} \bar{v}_h^{(k)} H_{\alpha},$$

where $(p_h(t))_{j,\alpha}$, respectively $(u_h(t))_{k,\alpha}$ are some real coefficients. Note that $J$ is equal to the number of edges of the triangulation $T_h$ on $\bar{D}$ and that $L$ is equal to the number of triangles.

Equation (37)_i is equivalent to the set of $J \times \frac{(N+K)!}{N!K!} = J \times C_{N+K}^{K}$ equations obtained by taking for $\bar{q}_h \in X_h^{(N,K)}$ an arbitrary element $\bar{q}_h^{(j)} H_{\alpha}$ of the basis \{${q}_h^{(j)} H_{\alpha}$; $j = 1, \ldots, J$, $\alpha \in \mathcal{I}_{N,K}$\} of $X_h^{(N,K)}$:

$$\sum_{j=1}^{J} \sum_{\beta+\gamma=\alpha} \left( (K^{\odot-1})_{\gamma} \bar{q}_h^{(j)} : \bar{q}_h^{(l)} \right)_{0,D} (p_h(t))_{j,\beta} + \sum_{k=1}^{L} (v_h^{(k)} : \vec{q}_h^{(l)})_{0,D} (u_h(t))_{k,\alpha} = 0$$

$\forall l = 1, \ldots, J$, $\forall \alpha \in \mathcal{I}_{N,K}$.

Each equation in (40) is a linear homogeneous equation in the unknowns $(p_h(t))_{j,\beta}$, $j = 1, \ldots, J$, $\beta \in \mathcal{I}$ with $\beta \leq \alpha$ i.e. $\beta_j \leq \alpha_j$, $\forall j \in \mathbb{N}$. For each $\alpha$ fixed in $\mathcal{I}_{N,K}$, we have $J$ equations of the type (40). Let us rewrite these $J$ equations in a matrix form. In this respect for each $\gamma \in \mathcal{I}$, $\gamma \leq \alpha$, let us introduce the square symmetric matrix of dimension $J$

$$B_\gamma = (b_{j,l}^{\gamma})_{1 \leq j,l \leq J}$$

where

$$b_{j,l}^{\gamma} = \left( (K^{\odot-1})_{\gamma} \bar{q}_h^{(j)} : \bar{q}_h^{(l)} \right)_{0,D} = \int_D (k^{\odot-1})_{\gamma} \bar{q}_h^{(j)} \cdot \bar{q}_h^{(l)} \, dx,$$

$\forall j, l = 1, \ldots, J$, and the rectangular matrix $C$ with $J$ rows and $L$ columns:

$$C = (C_{l,k}^{(\gamma)})_{1 \leq l \leq J, 1 \leq k \leq L}$$

where

$$C_{l,k}^{(\gamma)} = (v_h^{(k)} : \vec{q}_h^{(l)})_{0,D} = \int_D v_h^{(k)} \cdot \vec{q}_h^{(l)} \, dx.$$

In a matrix form, the set of equations (40) for $j = 1, \ldots, J$ and every $\alpha$ fixed in $\mathcal{I}_{N,K}$ may be rewritten:

$$\sum_{(\gamma,\beta) \in \mathcal{I}_{N,K}^2: \gamma + \beta = \alpha} B_\gamma (p_h(t))_{j,\beta} + C (u_h(t))_{k,\alpha} = 0$$

$$\sum_{(\gamma,\beta) \in \mathcal{I}_{N,K}^2: \gamma + \beta = \alpha} B_\gamma (p_h(t))_{j,\beta} + C (u_h(t))_{k,\alpha} = 0, \quad 1 \leq j \leq J, 1 \leq k \leq L.$$
Heat diffusion equation in a random medium

Let us now examine the heat balance equation (37)\((iii)\).

Equation (37)\((iii)\) is equivalent to the set of \(L \times C_{N+K}^K\) equations obtained by taking for \(v_h \in M_{h_{(N,K)}}\) an arbitrary element \(v_h^{(k)}H_\alpha\) of the basis \(\{v_h^{(k)}H_\alpha; k = 1, \ldots, L, \alpha \in I_{N,K}\}\) of \(M_{h_{(N,K)}}\):

\[
\sum_{j=1}^J \left( \text{div} \tilde{q}_h^{(j)}, v_h^{(k)} \right)_{0,D} (p_h(t))_{j,\alpha} - \frac{d}{dt} \left( (u_h(t))_{\alpha}, v_h^{(k)} \right)_{0,D} = - \left( f_\alpha(t), v_h^{(k)} \right)_{0,D} \tag{42}
\]

\(\forall k = 1, \ldots, L, \forall \alpha \in I_{N,K}\) where \(f_\alpha(t)\) denotes the \(\alpha\text{th}\) coefficient of the expansion of \(f(t)\) in chaos polynomials.

Each equation in (42) is a linear inhomogeneous equation in the \(J + 1\) unknowns \((p_h(t))_{j,\alpha}, j = 1, \ldots, J\) and \(\frac{d}{dt} (u_h(t))_{k,\alpha}\), the right-hand side being in fact the opposite of the integral of \(f_\alpha(t)\) on the triangle of \(T_h\) whose \(v_h^{(k)}\) is the characteristic function. Denoting that triangle of \(T_h, K_k\), equation (42) can be rewritten

\[
\sum_{j=1}^J \left( \text{div} \tilde{q}_h^{(j)}, v_h^{(k)} \right)_{0,K_k} (p_h(t))_{j,\alpha} - |K_k| \frac{d}{dt} (u_h(t))_{k,\alpha} = - \int_{K_k} f_\alpha(t) \, dx_1 \otimes dx_2, \quad \forall k = 1, \ldots, L, \forall \alpha \in I_{N,K}.
\]

Introducing the diagonal matrix \(D\) of order \(L\), whose diagonal elements are \(|K_1|, \ldots, |K_L|\) and the vector \(F_\alpha(t)\) of \(\mathbb{R}^L\) whose components are \(\int_{K_1} f_\alpha(t) \, dx_1 \otimes dx_2, \ldots, \int_{K_L} f_\alpha(t) \, dx_1 \otimes dx_2\), the system of \(L\) equations (43) for an arbitrary fixed \(\alpha \in I_{N,K}\), can be rewritten:

\[
C^T \left[ (p_h(t))_{j,\alpha} \right]_{1 \leq j \leq J} - D \frac{d}{dt} \left[ (u_h(t))_{k,\alpha} \right]_{1 \leq k \leq L} = -F_\alpha(t). \tag{44}
\]

From (37)\((iii)\) we have also the set of \(J \times C_{N+K}^K\) initial conditions:

\[
(u_h(0))_{k,\alpha} = (g_h)_{k,\alpha}, \quad \forall k = 1, \ldots, J, \forall \alpha \in I_{N,K}. \tag{45}
\]

Let us first consider the case \(\alpha = 0\).

In this case the system of equations (41), (44), (45) become:

\[
\begin{cases}
B_0[p_h(t)_{j,0}]_{1 \leq j \leq J} + C \left[ (u_h(t))_{k,0} \right]_{1 \leq k \leq L} = 0, \\
\frac{d}{dt} \left[ (u_h(t))_{k,0} \right]_{1 \leq k \leq L} = D^{-1}C^T \left[ (p_h(t))_{j,0} \right]_{1 \leq j \leq J} + D^{-1}F_0(t), \\
\left[ (u_h(0))_{k,0} \right]_{1 \leq k \leq L} = \left[ (g_h)_{k,0} \right]_{1 \leq k \leq L}.
\end{cases}
\]

From the definition of the matrix \(B_0\), it follows that \(B_0\) is a square matrix of order \(J\) whose elements are:

\[
(B_0)_{j,l} := \left( \frac{1}{E|K|} \tilde{q}_h^{(j)} \cdot \tilde{q}_h^{(l)} \right)_{0,D} = \int_D \frac{1}{E|K|} \tilde{q}_h^{(j)} \cdot \tilde{q}_h^{(l)} \, dx, \quad \forall j, l = 1, \ldots, J.
\]

\[\text{as } B_\gamma = B_\gamma^T.\]

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But by the hypothesis inf_{D} E [\mathcal{K}] > 0, from which it follows that:

$$\sum_{j=1}^{J} \sum_{l=1}^{J} \left( \frac{1}{E [\mathcal{K}]} d_{j}^{(i)}, d_{l}^{(i)} \right)_{0,D} \xi_{j} \xi_{l} \geq \inf_{D} E [\mathcal{K}] \left\| \sum_{j=1}^{J} \varphi_{j}^{(i)} \xi_{j} \right\|_{0,D}^{2} > 0,$$

\(\forall \xi = (\xi_{j})_{j=1}^{J} \in \mathbb{R}^{J} \setminus \{0\}\), where

\[
\varphi_{j}^{(i)} := \frac{1}{E [\mathcal{K}]} q_{j}^{(i)}, \quad \forall j = 1, \ldots, J
\]

((\cdot, \cdot)_{0,D} (resp. \| \cdot \|_{0,D}) denotes the scalar product (resp. the norm) in \(L^{2}(D)\)). This shows that \(B_{0}\) is a symmetric positive definite matrix and thus invertible. It now follows from equations (46) that

$$\frac{d}{dt} \left[ (u_{h}(t))_{k,0} \right]_{1 \leq k \leq L} = -D^{-1} C^{T} B_{0}^{-1} C \left[ (u_{h}(t))_{k,0} \right]_{1 \leq k \leq L} + D^{-1} F_{0}(t). \quad (48)$$

It is equivalent to rewrite (48) by multiplying both sides by \(D^{\frac{1}{2}}\) to the left, in the “symmetric form”:

$$\frac{d}{dt} D^{\frac{1}{2}} \left[ (u_{h}(t))_{k,0} \right]_{1 \leq k \leq L} = -D^{-\frac{1}{2}} C^{T} B_{0}^{-1} C D^{-\frac{1}{2}} \cdot D^{\frac{1}{2}} \left[ (u_{h}(t))_{k,0} \right]_{1 \leq k \leq L} + D^{-\frac{1}{2}} F_{0}(t), \quad (49)$$

this time the linear operator \(-D^{-\frac{1}{2}} C^{T} B_{0}^{-1} C D^{-\frac{1}{2}}\) being symmetric. Using the fact that the divergence operator from \(X_{h}\) into \(M_{h}\) is in fact surjective (\([31]\) p.612), it is easy to see that the linear operator \(C^{T} B_{0}^{-1} C : \mathbb{R}^{L} \rightarrow \mathbb{R}^{L}\) is still positive definite despite the fact that \(L < J\). Thus the linear operator \(-D^{-\frac{1}{2}} C^{T} B_{0}^{-1} C D^{-\frac{1}{2}}\) in \(\mathbb{R}^{L}\) is symmetric negative definite and thus generates a contraction semigroup \((P_{t})_{t \geq 0}\) on \(\mathbb{R}^{L}\) endowed with the euclidian norm. The solution of the inhomogeneous linear system of differential equations (49) with the initial conditions (46)(iii) is given in terms of this semigroup by:

$$D^{\frac{1}{2}} \left[ (u_{h}(t))_{k,0} \right]_{1 \leq k \leq L} = P_{t} D^{\frac{1}{2}} \left[ (g_{h})_{k,0} \right]_{1 \leq k \leq L} + \int_{0}^{t} P_{t-s} D^{\frac{1}{2}} F_{0}(s) \, ds. \quad (50)$$

i.e.

$$\left[ (u_{h}(t))_{k,0} \right]_{1 \leq k \leq L} = D^{\frac{1}{2}} P_{t} D^{\frac{1}{2}} \left[ (g_{h})_{k,0} \right]_{1 \leq k \leq L} + \int_{0}^{t} D^{\frac{1}{2}} P_{t-s} D^{\frac{1}{2}} F_{0}(s) \, ds. \quad (51)$$

By \([28]\) p.256 and theorem 3.1 p.110 of \([26]\), \(\left[ (u_{h}(\cdot))_{k,0} \right]_{1 \leq k \leq L}\) is Hölder continuous of exponent \(\frac{1}{2}\) on \([0,T]\). Moreover, its derivative being given by the right-hand side of (48) is in \(L^{2}(0,T;\mathbb{R}^{L})\). Thus \(\left[ (u_{h}(\cdot))_{k,0} \right]_{1 \leq k \leq L} \in H^{1}(0,T;\mathbb{R}^{L})\).

From equation (46)(ii) and the fact that, as we have seen previously, \(B_{0}\) is symmetric positive definite and thus invertible, we have also that \(\left[ (p_{h}(\cdot))_{j,0} \right]_{1 \leq j \leq J} \in H^{1}(0,T;\mathbb{R}^{J})\).

Let us now consider the case \(\alpha \neq 0\). Reasoning by recurrence, we may suppose that we have already computed all the

\(\left[ (u_{h}(\cdot))_{k,\beta} \right]_{1 \leq k \leq L}\) and \(\left[ (p_{h}(\cdot))_{j,\beta} \right]_{1 \leq j \leq J}\) for \(\beta < \alpha\).
Equations (41), (44) and the initial conditions (45), give us the following system:

\[
\begin{cases}
B_0[p_h(t)_{j,\alpha}]_{1 \leq j \leq J} + C \begin{bmatrix} (u_h(t))_{k,\alpha} \end{bmatrix}_{1 \leq k \leq L} = -\sum_{\beta < \alpha} B_{\alpha - \beta} [p_h(t)_{j,\beta}]_{1 \leq j \leq J}, \\
\frac{d}{dt} \begin{bmatrix} (u_h(t))_{k,\alpha} \end{bmatrix}_{1 \leq k \leq L} = D^{-1}C^\top \begin{bmatrix} (p_h(t))_{j,\alpha} \end{bmatrix}_{1 \leq j \leq J} + D^{-1}F_\alpha(t), \\
\begin{bmatrix} (u_h(0))_{k,\alpha} \end{bmatrix}_{1 \leq k \leq L} = \begin{bmatrix} (g_h)_{k,\alpha} \end{bmatrix}_{1 \leq k \leq L}.
\end{cases}
\]

(52)

We can now proceed similarly as in the case \( \alpha = 0 \), using equation (52)_{(i)} to eliminate \([p_h(t)_{j,\alpha}]_{1 \leq j \leq J}\) in equation (52)_{(ii)} obtaining

\[
\frac{d}{dt} \begin{bmatrix} (u_h(t))_{k,\alpha} \end{bmatrix}_{1 \leq k \leq L} = -D^{-1}C^\top B_0^{-1}C \begin{bmatrix} (u_h(t))_{k,\alpha} \end{bmatrix}_{1 \leq k \leq L} + D^{-1}F_\alpha(t),
\]

(53)

where

\[
F_\alpha(t) = F_\alpha(t) - \sum_{\beta < \alpha} C^\top B_0^{-1}B_{\alpha - \beta} [p_h(t)_{j,\beta}]_{1 \leq j \leq J}.
\]

(54)

Note that equation (53) is completely analogous to (48) and \( F_\alpha(t) \) is completely known. Reasoning like in (48) to (51), it follows from (53) and (52)_{(iii)} that \([u_h(\cdot))_{k,\alpha}]_{1 \leq k \leq L} \in H^1(0,T;\mathbb{R}^L).\)

Using (52)_{(i)} and the invertibility of \( B_0 \), we obtain that \([p_h(\cdot))_{j,\beta}]_{1 \leq j \leq J} \in H^1(0,T;\mathbb{R}^J)\). Having determined the \( J \times C_{N+K}^K \) coefficients \( p_h(\cdot)_{j,\beta}, \ j = 1,\ldots,J, \ \beta \in \mathcal{I}_{N,K} \) and the \( L \times C_{N+K}^K \) coefficients \( u_h(\cdot))_{k,\alpha}, \ k = 1,\ldots,L, \ \alpha \in \mathcal{I}_{N,K} \), and plugging them in the formulas (38) and (39), we obtain \( \tilde{p}_h \in H^1(0,T;\mathcal{X}^{(N,K)}_h) \) and \( u_h \in H^1(0,T;\mathcal{M}^{(N,K)}_h) \) satisfying the equations (37).

\( \square \)

6. Error estimates for the dual mixed formulation in the stationary case

We will need these error estimates for the “elliptic projection” of the dual mixed formulation relative to the heat equation with a random diffusion coefficient in a polygonal domain with a reentrant corner. The dual mixed formulation for the stationary problem has been studied in [24] but a priori error estimates have been derived only for “regular solutions in the space variable” i.e. belonging at least to the stochastic Sobolev space \( S^{-1,k,H^1(D)} \) which is not the case here due to the reentrant corner of our polygonal domain \( D \).

The following hypotheses are always tacitly assumed in this section:

on the stochastic diffusion coefficient \( K \), we suppose that \( K, K^{\circ} \in \mathcal{F}_l(D) \) and that its generalized expectation \( E[K] \) is strictly positively lower bounded on \( D \) i.e. that \( \inf_D E[K] > 0 \); finally, we suppose that \( k \in \mathbb{R} \) satisfies inequality (14). These hypotheses will be strengthened when necessary.

We present in this section two methods to derive the error estimates in the stationary case as the first method has the defect to require regularity on the spatial derivatives of the right-hand side \( f \). Thus, exceptionally in this section, we consider the system of equations: find \( \tilde{p} \in \mathcal{H}(\text{div};D), \ u \in S^{-1,k,0}(D) \) such that:

\[
\begin{cases}
(K^{\circ} \mathcal{G} \tilde{p}, \tilde{q})_{-1,k,0} + (u, \text{div} \tilde{q})_{-1,k,0} = 0, \quad \forall \tilde{q} \in \mathcal{H}(\text{div};D), \\
(\text{div} \tilde{p}, v)_{-1,k,0} = -(f, v)_{-1,k,0}, \quad \forall v \in S^{-1,k,0}(D),
\end{cases}
\]

(55)
and its discretization: find \( \tilde{p}_h \in X_h^{(N,K)} \), \( u_h \in M_h^{(N,K)} \) such that:

\[
\begin{cases}
(K^{-1} \tilde{p}_h, \tilde{q}_h)_{-1,k,0} + (u_h, \text{div } \tilde{q}_h)_{-1,k,0} = 0, & \forall \tilde{q}_h \in X_h^{(N,K)}, \\
(\text{div } \tilde{p}_h, v_h)_{-1,k,0} = - (f, v_h)_{-1,k,0}, & \forall v_h \in M_h^{(N,K)}.
\end{cases}
\tag{56}
\]

Under the above hypotheses, we have seen in lemma 4.1 that the bilinear form

\[
a (\cdot, \cdot) : (S^{-1,k,0}(D))^2 \times (S^{-1,k,0}(D))^2 \to \mathbb{R} : (\tilde{p}, \tilde{q}) \mapsto (K^{-1} \tilde{p}, \tilde{q})_{-1,k,0},
\tag{57}
\]

is coercive. For the bilinear form

\[
b (\cdot, \cdot) : S^{-1,k,0}(D) \times \mathcal{H} (\text{div}; D) \to \mathbb{R} : (v, \tilde{q}) \mapsto b (v, \tilde{q}) := (v, \text{div } \tilde{q})_{-1,k,0},
\tag{58}
\]

the inf-sup inequality:

\[
\sup_{\tilde{q} \in \mathcal{H} (\text{div}; D)} \frac{b (v, \tilde{q})}{\| \tilde{q} \|_{-1,k,\text{div}}} \geq \| v \|_{-1,k,0},
\tag{59}
\]

is proved in [24] (lemma 3.7 p. 615) and in fact follows easily by applying the construction used in the “deterministic case” to prove it for each coefficient \( v_\alpha \in L^2(D) \) of the chaos expansion of \( v \). Thus by corollary 4.1 p. 61 of [14], problem (55) is well-posed (the above coercivity of \( a (\cdot, \cdot) \) on \( (S^{-1,k,0}(D))^2 \) implying of course the ellipticity in the sense of the norm of \( \mathcal{H} (\text{div}; D) = S^{-1,k,H(\text{div};D)} \) on this subspace of divergence free vectorfields).

To prove that the discrete problem (56) is well-posed, being a finite dimensional problem, it suffices to prove unicity. So let us suppose that \( f = 0 \) in (56)(ii). Taking \( \tilde{q}_h = \tilde{p}_h \) in (56)(i), using (56)(ii) with \( v_h = u_h \) and using the coercivity of \( a (\cdot, \cdot) \), we obtain \( \tilde{p}_h = 0 \). That \( u_h = 0 \) follows now from (56)(i), knowing that \( \tilde{p}_h = 0 \) and the following proposition:

**Proposition 6.1:** (uniform inf-sup inequality [24] p.620)

Let \( (\mathcal{T}_h)_{h>0} \) be a regular family of triangulations over \( D \). Then, there exists a constant \( c > 0 \) independent of \( h, N \) and \( K \) such that:

\[
\sup_{\tilde{q}_h \in X_h^{(N,K)}} \frac{b (v_h, \tilde{q}_h)}{\| \tilde{q}_h \|_{-1,k,\text{div}}} \geq c \| v_h \|_{-1,k,0}, \quad \forall v_h \in M_h^{(N,K)}.
\tag{60}
\]

**Proof:** As the domain \( D \) presents geometric singularities (\( D \) is a polygonal domain in \( \mathbb{R}^2 \) with one reentrant corner at the origin), we indicate our proof, based on our work [31], which seems to us somewhat more clear than the proof given in [24]. Let \( v_h \in M_h^{(N,K)} \) and let us consider its expansion in chaos polynomials \( v_h = \sum_{\alpha \in \mathcal{I}_{N,K}} (v_h)_\alpha H_\alpha \). By lemma 1.14 of [31], there exists for each \( \alpha \in \mathcal{I}_{N,K} \) some \( (\tilde{q}_h)_\alpha \in X_h \) such that \( \text{div } (\tilde{q}_h)_\alpha = (v_h)_\alpha \) and

\[
\| (\tilde{q}_h)_\alpha \|_{L^2(D)^2} \leq c \| (v_h)_\alpha \|_{L^2(D)}
\tag{61}
\]

with a constant \( c > 0 \) independent of \( h \).

Let us set \( \tilde{q}_h = \sum_{\alpha \in \mathcal{I}_{N,K}} (\tilde{q}_h)_\alpha H_\alpha; \tilde{q}_h \in X_h^{(N,K)} \) and:

\[
b (v_h, \tilde{q}_h) = (v_h, \text{div } \tilde{q}_h)_{-1,k,0}
\]

\[
= \left( \sum_{\alpha \in \mathcal{I}_{N,K}} (v_h)_\alpha H_\alpha, \sum_{\alpha \in \mathcal{I}_{N,K}} \text{div } (\tilde{q}_h)_\alpha H_\alpha \right)_{-1,k,0}
\]
Let us suppose that the stochastic diffusion coefficient

Theorem 6.2: 

concerned with the stationary case): 

Thus we are reduced to bound the right-hand side of the previous inequality. To do that, we need some 

By this observation and proposition 6.1 all the hypotheses of theorem II .1.1 p. 114 of [14] are verified. 

the bilinear form 

∀ 

⃗ q 

by inequality (61) and the fact that div (⃗ q) = (vh), Thus:

Thus:

By inequalities (62) and (63):


\[
\frac{b (vh, \bar{q}_h)}{\|\bar{q}_h\|_{-1,k,\text{div}}} \geq \frac{1}{c} \|vh\|_{-1,k,0}.
\]

Let us observe that if for some element \( \bar{q}_h \in X_h^{(N,K)} \), \( b (vh, \bar{q}_h) = (vh, \text{div} \bar{q}_h)_{-1,k,0} \equiv 0 \) for every \( vh \in M_h^{(N,K)} \), then as \( \text{div} \bar{q}_h \) itself belongs to \( M_h^{(N,K)} \), it follows that \( \text{div} \bar{q}_h \equiv 0 \) and thus

\[
\|\bar{q}_h\|_{-1,k,\text{div}}^2 = \|\bar{q}_h\|_{-1,k,0}^2 \lesssim a(\bar{q}_h, \bar{q}_h),
\]

\[
\forall \bar{q}_h \in \{ \bar{q}_h \in X_h^{(N,K)}; b (vh, \bar{q}_h) = 0, \forall vh \in M_h^{(N,K)} \} \text{ with a constant (hidden in } \lesssim \text{) independent of } h. \text{ Thus,}
\]

the bilinear form \( a(.,.) \) is uniformly coercive on the family of subspaces \( X_h^{(N,K)} \) of \( \mathcal{H} (\text{div}; D) \).

By this observation and proposition 6.1 all the hypotheses of theorem II.1.1 p. 114 of [14] are verified. Thus:

\[
\|\bar{p} - \bar{p}_h\|_{-1,k,\text{div}} + \|u - u_h\|_{-1,k,0} \lesssim \inf_{\bar{q}_h \in X_h^{(N,K)}} \|\bar{p} - \bar{q}_h\|_{-1,k,\text{div}} + \inf_{vh \in M_h^{(N,K)}} \|u - vh\|_{-1,k,0}.
\]

Thus we are reduced to bound the right-hand side of the previous inequality. To do that, we need some 

spatial regularity on \( \bar{p} \); we have the following result (analogous to theorem 3.6, but in this section we are 

concerned with the stationary case):

**Theorem 6.2:** Let us suppose that the stochastic diffusion coefficient \( K \) satisfies:

\[
\inf_D E[K] > 0, \quad K \frac{\partial K}{\partial x_1}, \frac{\partial K}{\partial x_2}, K^{\alpha - 1} \in \mathcal{F}_1(D), \text{ and that } k \in \mathbb{R} \text{ satisfies inequality (14). We also suppose that }
\]

\( f \in S^{-1,k,0}(D) \). Then the weak solution \( u \in S_0^{-1,k,1}(D) \) := \( S^{-1,k,H^1(D)} \) of the stationary equation with 

Dirichlet boundary condition:

\[
\begin{cases}
- \text{div} \left( K \nabla u \right) = f & \text{in } D, \\
u|_{\partial D} = 0 & \text{on } \partial D,
\end{cases}
\]

belongs to \( S^{-1,k,H^{\alpha \omega}}(D) \) for all \( \alpha \omega > 1 - \frac{\omega}{2} \) (\( \omega \) denoting the opening of the reentrant corner of the polygonal 

domain \( D \) at the origin). Consequently:

\[
\bar{p} = K \nabla u \in \left( S^{-1,k,H^{1-\alpha \omega}}(D) \right)^2.
\]
Proof: From (65) follows:

\[ K \vartriangleleft \Delta u = -f - \nabla K \vartriangleleft \nabla u. \]  

Let us set \( g = f + \nabla K \vartriangleleft \nabla u \). As by hypothesis:

\[ \frac{\partial K}{\partial x_1}, \frac{\partial K}{\partial x_2} \in F_1(D), \]

\( g \in S^{-1,k,0}(D) \) and

\[ \|g\|_{-1,k,0} \lesssim \|f\|_{-1,k,0}. \]

As by hypothesis \( K \vartriangleleft^{-1} \in F_1(D) \) by prop. 4 p. 120 of [35] (or proposition 2.4 of [24]):

\[ K \vartriangleleft^{-1} g \in S^{-1,k,0}(D) \]

and

\[ \left\| K \vartriangleleft^{-1} g \right\|_{-1,k,0} \lesssim \|g\|_{-1,k,0}. \]

Expanding \( u \) and \( g \) in chaos polynomials, we have:

\[ -\Delta u_\alpha = (K \vartriangleleft^{-1} g)_\alpha, \forall \alpha \in \mathcal{I}. \]

By (8,4,1,7) p. 388 of Grisvard’s book [15], \( u_\alpha \in H^{2,\alpha_w}(D) \) and \( \|u_\alpha\|_{H^{2,\alpha_w}(D)} \lesssim \left\| (K \vartriangleleft^{-1} g)_\alpha \right\|_{L^2(D)}, \) for every \( \alpha \in \mathcal{I} \) with a constant (hidden in \( \lesssim \)) independent of \( \alpha \). Consequently:

\[ \sum_{\alpha \in \mathcal{I}} \|u_\alpha\|_{H^{2,\alpha_w}(D)}^2 (2N)^{k_\alpha} \lesssim \sum_{\alpha \in \mathcal{I}} \left\| (K \vartriangleleft^{-1} g)_\alpha \right\|_{L^2(D)}^2 (2N)^{k_\alpha}, \]

i.e.

\[ \|u\|_{-1,k,H^{2,\alpha_w}(D)} \lesssim \left\| K \vartriangleleft^{-1} g \right\|_{-1,k,0}. \]

But, we have seen above that \( \left\| K \vartriangleleft^{-1} g \right\|_{-1,k,0} \lesssim \|g\|_{-1,k,0} \lesssim \|f\|_{-1,k,0}. \) Thus

\[ \|u\|_{-1,k,H^{2,\alpha_w}(D)} \lesssim \|f\|_{-1,k,0}. \]

and by prop. 4 p. 120 of [35] (or proposition 2.4 of [24]) applied to \( \bar{p} = K \vartriangleleft \nabla u : \)

\[ \|\bar{p}\|_{-1,k,H^{1,\alpha_w}(D)}^2 \lesssim \|f\|_{-1,k,0}. \]

□

Using (64), the preceding regularity result, and imposing appropriate refinement rules on our regular family of triangulations \( \{T_h\}_{h>0} \) linked to the regularity of the solution (6.2), we are going to derive \( O(h) \) error estimates in the spatial directions; however to be able to proceed in this way we will have to suppose that \( f \in S^{-1,k,1}(D) := S^{-1,k,H^1(D)}. \)

Theorem 6.3: Under the hypotheses of theorem 6.2, supposing that our regular family of triangulations \( \{T_h\}_{h>0} \) satisfies the following refinement rules:

(R1) \( h_K \leq \sigma h^{-\alpha_w} \) for every triangle \( K \in T_h \) which has one of its vertices at the origin;

(R1) \( h_K \leq \sigma (\inf_{x \in K} r^{\alpha_w}(x)) h \) for every triangle \( K \in T_h \) without any vertex at the origin,
the constant $\sigma > 0$ being independent of the triangle $K$ and $h$, and finally supposing that the right-hand side

$$f \in S^{-1,k,1}(D) \cap S^{-1,k+r,0}(D),$$

for some $k < 0$ and $r > 1$ such that $k + r \in \mathbb{R}$ satisfies inequality (14), we have the following a priori error estimate (with a constant hidden in $\lesssim$ independent of $h$, $N$, the chaos dimension $K$ and $r$):

$$\| \bar{p} - \bar{p}_h \|_{-1,k,\text{div}} + \| u - u_h \|_{-1,k,0} \lesssim B_{N,K} \left( \| u \|_{-1,k+r,0} + \| \bar{p} \|_{-1,k+r,\text{div}} \right) + h \left( \| u \|_{-1,k,1} + \| \bar{p} \|_{-1,k, H^{1,\omega}(D)}^2 + |f|_{-1,k,1} \right), \quad (67)$$

where [6]

$$B_{N,K} = \sqrt{A(r) \frac{1}{K^{r-1}}} + B(r) \frac{1}{2^r N},$$

$$A(r) = e^{r\frac{r-1}{r}} \frac{r}{r-1}, \quad B(r) = e^{2^{r-1}(r-1)} \frac{1}{2^r (r-1)},$$

$K$ denoting the dimension of the polynomial chaos and $N$ its degree.

**Proof:** We have to bound the right-hand side of (64). Firstly:

$$\inf_{\bar{q}_h \in X_h^{(N,K)}} \| \bar{p} - \bar{q}_h \|_{-1,k,\text{div}}$$

$$\leq \left\| \bar{p} - \sum_{\alpha \in I_{N,K}} \bar{p}_\alpha H_\alpha \right\|_{-1,k,\text{div}} + \inf_{\bar{q}_h \in X_h^{(N,K)}} \left\| \sum_{\alpha \in I_{N,K}} \bar{p}_\alpha H_\alpha - \bar{q}_h \right\|_{-1,k,\text{div}}$$

$$\leq B_{N,K} \| \bar{p} \|_{-1,k+r,\text{div}} + \left\| \sum_{\alpha \in I_{N,K}} (\bar{p}_\alpha - \Pi_h \bar{p}_\alpha) H_\alpha \right\|_{-1,k,\text{div}},$$

by [6] (a substantial improvement of [3]) and where $\Pi_h$ denotes the Raviart-Thomas interpolation operator of degree 0 [31]). Thus using our hypothesis that $f \in S^{-1,k,1}(D)$:

$$\inf_{\bar{q}_h \in X_h^{(N,K)}} \| \bar{p} - \bar{q}_h \|_{-1,k,\text{div}}$$

$$\leq B_{N,K} \| \bar{p} \|_{-1,k+r,\text{div}} + \left[ \sum_{\alpha \in I} \| \bar{p}_\alpha - \Pi_h \bar{p}_\alpha \|^2_{H(\text{div};D)} \right]^{\frac{1}{2}} (2N)^{k\alpha}$$

$$\leq B_{N,K} \| \bar{p} \|_{-1,k+r,\text{div}} + ch \left[ \sum_{\alpha \in I} (2N)^{k\alpha} \left( \| \bar{p}_\alpha \|^2_{H^{1,\omega}(D)} + |f_\alpha|^2_{H^{1}(D)} \right) \right]^{\frac{1}{2}}$$

$$\leq B_{N,K} \| \bar{p} \|_{-1,k+r,\text{div}} + ch(\| \bar{p} \|_{-1,k, H^{1,\omega}(D)}^2 + |f|_{-1,k, H^1(D)}^2)$$

as by ((31) p.620 of [31])

$$\| \bar{p}_\alpha - \Pi_h \bar{p}_\alpha \|_{0,D} \leq ch |\bar{p}_\alpha|_{H^{1,\omega}(D)}^2$$

and $\text{div}(\bar{p}_\alpha - \Pi_h \bar{p}_\alpha) = -(f_\alpha - P_h f_\alpha)$, (where $P_h$ denotes the orthogonal projection in $L^2(D)$ on $M_h$) which implies by inequality (45) of [31] that also: $\| \text{div}(\bar{p}_\alpha - \Pi_h \bar{p}_\alpha) \|_{0,D} \leq ch |f_\alpha|_{H^{1}(D)}$. 


Secondly:

\[
\inf_{v_h \in M_h^{(N, K)}} \| u - v_h \|_{-1,k,0} \\
\leq \left\| u - \sum_{\alpha \in I_h} u_\alpha H_\alpha \right\|_{-1,k,0} + \inf_{v_h \in M_h^{(N, K)}} \left\| \sum_{\alpha \in I_h} u_\alpha H_\alpha - v_h \right\|_{-1,k,0} \\
\leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + \left\| \sum_{\alpha \in I_h} u_\alpha H_\alpha - \sum_{\alpha \in I_h} P_h u_\alpha H_\alpha \right\|_{-1,k,0} \\
\leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + \left\| \sum_{\alpha \in I_h} (u_\alpha - P_h u_\alpha) H_\alpha \right\|_{-1,k,0} \\
\leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + \left[ \sum_{\alpha \in I_h} \left\| u_\alpha - P_h u_\alpha \right\|_0^2 \left( 2^{N(2k^2)} \right) \right]^{1/2} \\
\leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + ch \left[ \sum_{\alpha \in I_h} \left| u_\alpha \right|^2 \left( 2^{N(2k^2)} \right) \right]^{1/2} \\
\leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + ch \left| u \right|_{-1,k,1} \\
\leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + ch \left| u \right|_{-1,k,1}.
\]

by (45) of [31]. Thus:

\[
\inf_{v_h \in M_h^{(N, K)}} \| u - v_h \|_{-1,k,0} \leq B_{N,K} \left\| u \right\|_{-1,k+r,0} + ch \left| u \right|_{-1,k,1}.
\]

(67) follows from inequalities (68), (69) and (64).

\[\square\]

**Remark 6.1:** Let us observe that

\[
B_{N,K} = \sqrt{A(r) \frac{1}{K^{r-1}}} + B(r) \frac{1}{2^{rN}}
\]

(70)
tends to 0 exponentially with \( N \): the order of the chaos and only polynomially with \( K \): the dimension of the polynomial chaos.

Now, we present another method to derive error estimates, which does not require \( f \) to belong to \( S^{-1,k,1}(D) \).

**Proposition 6.4:**

\[
\left\| \tilde{p}_h - \sum_{\alpha \in I_h} \Pi_h \tilde{p}_\alpha H_\alpha \right\|_{-1,k,0} \lesssim \left\| \tilde{p} - \sum_{\alpha \in I_h} \Pi_h \tilde{p}_\alpha H_\alpha \right\|_{-1,k,0}.
\]

**Proof:** Let us set \( \tilde{q}_h = \sum_{\alpha \in I_h} \Pi_h \tilde{p}_\alpha H_\alpha \). By the coercivity of the bilinear form

\[
a(\cdot, \cdot) : \left( S^{-1,k,0}(D) \right)^2 \times \left( S^{-1,k,0}(D) \right)^2 \to \mathbb{R} : (\tilde{p}, \tilde{q}) \mapsto \left( K^{r-1} \hat{\langle \tilde{p}, \tilde{q} \rangle} \right)_{-1,k,0},
\]

(71)
on the Hilbert space \( \left( S^{-1,k,0}(D) \right)^2 \):

\[
\left\| \tilde{p}_h - \tilde{q}_h \right\|_{-1,k,0}^2 \lesssim a(\tilde{p}_h - \tilde{q}_h, \tilde{p}_h - \tilde{q}_h).
\]

(72)
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On the other hand from equations (55)\(_{(i)}\) and (56)\(_{(i)}\) follows by subtraction:
\[
a (\bar{p} - \bar{p}_h, \bar{p}_h - \bar{q}_h) + (u - u_h, \text{div} (\bar{p}_h - \bar{q}_h))_{-1,k,0} = 0. \tag{73}
\]

By equation (56)\(_{(ii)}\):
\[
\text{div} \bar{p}_h = - \sum_{\alpha \in I_{N,K}} P_h f_\alpha H_\alpha.
\]

By equation (55)\(_{(ii)}\):
\[
\text{div} \bar{q}_h = \sum_{\alpha \in I_{N,K}} P_h \text{div} \bar{p}_\alpha H_\alpha = - \sum_{\alpha \in I_{N,K}} P_h f_\alpha H_\alpha,
\]

Thus \(\text{div} (\bar{p}_h - \bar{q}_h) = 0\). Thus it follows from equation (73) that:
\[
a (\bar{p} - \bar{p}_h, \bar{p}_h - \bar{q}_h) = 0. \tag{74}
\]

Adding (74) to the right-hand side of (72), we obtain:
\[
\|\bar{p}_h - \bar{q}_h\|_{-1,k,0}^2 \lesssim a (\bar{p} - \bar{q}_h, \bar{p}_h - \bar{q}_h).
\]

Using the continuity of the bilinear form \(a (\cdot, \cdot)\), it now follows that:
\[
\|\bar{p}_h - \bar{q}_h\|_{-1,k,0} \lesssim \|\bar{p} - \bar{q}_h\|_{-1,k,0}.
\]

\[\square\]

**Corollary 6.5:**

\[
\|\bar{p}_h - \bar{q}_h\|_{-1,k,0} \leq \left\| \bar{p} - \sum_{\alpha \in I_{N,K}} \Pi_h \bar{p}_\alpha H_\alpha \right\|_{-1,k,0}.
\]

**Proof:** By the triangular inequality:

\[
\|\bar{p}_h - \bar{q}_h\|_{-1,k,0} \leq \left\| \bar{p} - \sum_{\alpha \in I_{N,K}} \Pi_h \bar{p}_\alpha H_\alpha \right\|_{-1,k,0} + \left\| \bar{p}_h - \sum_{\alpha \in I_{N,K}} \Pi_h \bar{p}_\alpha H_\alpha \right\|_{-1,k,0}
\]
\[
\lesssim \left\| \bar{p} - \sum_{\alpha \in I_{N,K}} \Pi_h \bar{p}_\alpha H_\alpha \right\|_{-1,k,0},
\]

by proposition 6.4. \[\square\]

**Theorem 6.6:** We assume that \(f \in S^{-1,k+r,0}(D)\) for some \(k < 0\) and \(r > 1\) such that \(k + r \in \mathbb{R}\) satisfies inequality (14), and that the stochastic diffusion coefficient \(\mathcal{K}\) satisfies the same hypotheses as in theorem 6.2. We suppose that our regular family of triangulations \((\mathcal{T}_h)_{h>0}\) satisfies the refinement rules (R1) and (R2) of theorem 6.3.

Then
\[
\|\bar{p}_h - \bar{q}_h\|_{-1,k,0} \lesssim B_{N,K} \|\bar{p}\|_{-1,k+1,r,0} + h \|\bar{p}\|_{-1,k,H^1,\text{div}(D)}^2
\]
\[
\lesssim B_{N,K} \|u\|_{-1,k+1,r,1} + h \|u\|_{-1,k,H^2,\text{div}(D)},
\]
where the constants hidden in the symbol $\lesssim$ are independent of $h, N, K$ and $r$, and where $B_{N,K}$ has been defined in theorem 6.3.

**Proof:** By corollary 6.5, we are reduced to bound $\left\| \vec{p} - \sum_{\alpha \in I_{N,K}} \Pi_h \vec{p}_\alpha H_\alpha \right\|_{-1,k,0}$.

By the triangular inequality:

$$
\left\| \vec{p} - \sum_{\alpha \in I_{N,K}} \Pi_h \vec{p}_\alpha H_\alpha \right\|_{-1,k,0} \leq \left\| \vec{p} - \sum_{\alpha \in I_{N,K}} \vec{p}_\alpha H_\alpha \right\|_{-1,k,0} + \left\| \sum_{\alpha \in I_{N,K}} (\vec{p}_\alpha - \Pi_h \vec{p}_\alpha) H_\alpha \right\|_{-1,k,0}
$$

$$
\leq B_{N,K} \left\| \vec{p} \right\|_{-1,k+1,r,0} + \left[ \sum_{\alpha \in I} c^2 h^2 \left\| \vec{p}_\alpha \right\|^2_{H^{1,\omega}(D)} (2h)^{k}\right]^{\frac{1}{2}}
$$

by [6] (a substantial improvement of [3]) and by (31) p. 620 of [31], $c$ denoting a strictly positive constant. Thus:

$$
\left\| \vec{p} - \sum_{\alpha \in I_{N,K}} \Pi_h \vec{p}_\alpha H_\alpha \right\|_{-1,k,0} \leq B_{N,K} \left\| \vec{p} \right\|_{-1,k+1,r,0} + ch \left\| \vec{p} \right\|_{-1,k,H^{1,\omega}(D)^2} \lesssim B_{N,K} \left\| \vec{u} \right\|_{-1,k+1,r,1} + h \left\| \vec{u} \right\|_{-1,k,H^{2,\omega}(D)}
$$

(77)

as $\vec{p} = K \hat{\nabla} \vec{v}$ and $K \in \mathcal{F}_l(D)$ with $l \geq 2(k + r)$ because by hypothesis $k + r$ satisfies inequality (14). $\Box$

To obtain an error estimate on $\vec{u}_h$, we need the uniform inf-sup inequality:

**Proposition 6.7:** $(T_h)_{h>0}$ being supposed to be a regular family of triangulations, one has:

$$
\sup_{\vec{q}_h \in X_h^{(N,K)}} \frac{(v_h, \text{div}(\vec{q}_h))_{-1,k,0}}{\left\| \vec{q}_h \right\|_{-1,k,0}} \gtrsim \left\| v_h \right\|_{-1,k,0}, \quad \forall v_h \in M_h^{(N,K)},
$$

(78)

with a constant (hidden in $\gtrsim$) independent of $h, N$ and $K$.

**Proof:** In proposition 6.1, we have proved that

$$
\sup_{\vec{q}_h \in X_h^{(N,K)}} \frac{b(v_h, \vec{q}_h)}{\left\| \vec{q}_h \right\|_{-1,k,\text{div}}} \geq c \left\| v_h \right\|_{-1,k,0}, \quad \forall v_h \in M_h^{(N,K)}.
$$

As $\left\| \vec{q}_h \right\|_{-1,k,0} \leq \left\| \vec{q}_h \right\|_{-1,k,\text{div}}$, this late inequality implies a fortiori inequality (78). $\Box$

**Corollary 6.8:** Let us denote by $P_h^{(N,K)}$ the orthogonal projection in the space $S^{-1,k,0}(D)$ onto the subspace $M_h^{(N,K)}$. $(T_h)_{h>0}$ being supposed to be a regular family of triangulations on $D$, we have:

$$
\left\| P_h^{(N,K)} \vec{u} - \vec{u}_h \right\|_{-1,k,0} \lesssim \left\| \vec{p} - \vec{p}_h \right\|_{-1,k,0}.
$$

(79)

**Proof:** From equation (55) follows a fortiori for every $\vec{q}_h \in X_h^{(N,K)}$:

$$
(K^{\hat{\nabla}^{-1}} \hat{\nabla} \vec{p}, \vec{q}_h)_{-1,k,0} + (u, \text{div} \vec{q}_h)_{-1,k,0} = 0.
$$

As $\text{div} \vec{q}_h \in M_h^{(N,K)}$: $(u - P_h^{(N,K)} \vec{u}, \text{div} \vec{q}_h)_{-1,k,0} = 0$, and thus we can replace $(u, \text{div} \vec{q}_h)_{-1,k,0}$ in the
preceeding equation by \((P_{h}^{(N,K)} u, \text{div} \, \tilde{q}_{h})_{-1,k,0}\) getting in this way:
\[
(K_{-1}^{\text{op}} \rho, \tilde{q}_{h})_{-1,k,0} + (P_{h}^{(N,K)} u, \text{div} \, \tilde{q}_{h})_{-1,k,0} = 0.
\]
Subtracting equation (56)\((i)\) from the preceding equation we obtain:
\[
(K_{-1}^{\text{op}} (\bar{p} - \bar{p}_{h}), \tilde{q}_{h})_{-1,k,0} + (P_{h}^{(N,K)} u - u_{h}, \text{div} \, \tilde{q}_{h})_{-1,k,0} = 0,
\]
for every \(\tilde{q}_{h} \in X_{h}^{(N,K)}\). By the uniform inf-sup inequality: proposition 6.7, we now obtain:
\[
\|P_{h}^{(N,K)} u - u_{h}\|_{-1,k,0} \leq \|K_{-1}^{\text{op}} (\bar{p} - \bar{p}_{h})\|_{-1,k,0}
\]
\[
\lesssim \|\bar{p} - \bar{p}_{h}\|_{-1,k,0}
\]
by the hypothesis stated at the beginning of this section on \(K_{-1}^{\text{op}}\).

**Theorem 6.9:** We assume that \(f \in S^{-1,k+r,0}(D)\) for some \(k < 0\) and \(r > 1\) such that \(k + r < \mathbb{R}\) satisfies inequality (14) and that the stochastic diffusion coefficient \(\mathcal{K}\) satisfies the same hypotheses as in theorem 6.2. We suppose that our regular family of triangulations \((T_{h})_{h>0}\) satisfies the refinement rules (R1) and (R2) of theorem 6.3.

Then, the following error estimates hold on \(u_{h}\):
\[
\|P_{h}^{(N,K)} u - u_{h}\|_{-1,k,0} \lesssim B_{N,K} \|u\|_{-1,k+1} + h \|u\|_{-1,k,H^{2,\alpha_w}(D)}, \tag{80}
\]
\[
\|u - u_{h}\|_{-1,k,0} \lesssim B_{N,K} \|u\|_{-1,k+1} + h \|u\|_{-1,k,H^{2,\alpha_w}(D)}, \tag{81}
\]
where \(B_{N,K}\) has been defined in theorem 6.3 and \(P_{h}^{(N,K)}\) denotes the orthogonal projection in the space \(S^{-1,k,0}(D)\) onto the subspace \(M_{h}^{(N,K)}\).

**Proof:** (80) follows from (79) and (76). On the other hand:
\[
\|u - P_{h}^{(N,K)} u\|_{-1,k,0} = \left\| \sum_{\alpha \in I} (u_{\alpha} - P_{h} u_{\alpha}) H_{\alpha} \right\|_{-1,k,0}
\]
\[
= \left[ \sum_{\alpha \in I} \|u_{\alpha} - P_{h} u_{\alpha}\|_{0,D}^{2} (2N)^{k\alpha} \right]^{\frac{1}{2}}
\]
\[
\lesssim \left[ \sum_{\alpha \in I} h^{2} |u_{\alpha}|_{1,D}^{2} (2N)^{k\alpha} \right]^{\frac{1}{2}} \lesssim h |u|_{-1,k,1} \tag{82}
\]
by (45) of [31].

From (80) and (82), we obtain (81).

**7. The elliptic projection in the context of the dual mixed formulation**

We will always assume in the following of this section, at least that the coefficient of diffusion \(\mathcal{K}(\cdot) \in \mathcal{F}(D)\) satisfies \(\text{inf}_{D} E[\mathcal{K}] > 0\), that \(f \in L^{2}(0,T;S^{-1,k+r,0}(D))\) for some \(r > 1\) and \(k < 0\) such
that \( k + r \in \mathbb{R} \) satisfies inequality (14), and that our regular family of triangulations \( (\mathcal{T}_h)_{h>0} \) satisfies the refinement rules (R1) and (R2) of theorem 6.3. To get regularity on the time derivative of the solution \( \frac{du}{dt} \), we also assume more regularity on the data \( f \) and \( g \); we assume also, that \( \frac{df}{dt} \in L^2(0,T;\mathcal{S}^{-1,k+r,0}(D)) \) and that the initial condition

\[
g \in \left\{ g \in \mathcal{S}^{-1,k+r,1}(D); \text{ div } \left( \mathcal{K} \otimes \nabla g \right) \in \mathcal{S}^{-1,k+r,0}(D) \right\}.
\]

Under these hypotheses, we know by theorem 3.4 that:

\[
\frac{du}{dt} \in L^2 \left( 0, T; \mathcal{S}^{-1,k+r,1}_{0}(D) \right) \cap C \left( [0,T]; \mathcal{S}^{-1,k+r,0}(D) \right).
\]

We can now introduce the concept of elliptic projection in the setting of the dual mixed method:

**Definition 7.1:** We call elliptic projection at the fixed time \( t \) of the exact solution \( (\overline{p}(\cdot),u(\cdot)) \) of the mixed formulation of the evolution problem (35), the solution denoted \( (\overline{p}_h(t),\overline{u}_h(t)) \in X^{(N,K)}_h \times M^{(N,K)}_h \) of the discretized mixed formulation of the stationary problem (56) with right-hand side \( f(t) - \frac{du}{dt}(t) \), i.e.

\[
\begin{cases}
\left( \mathcal{K}^{\omega-1} \overline{p}_h(t), \overline{q}_h \right)_{-1,k,0} + (\overline{u}_h(t), \text{ div } \overline{q}_h)_{-1,k,0} = 0, & \forall \overline{q}_h \in X^{(N,K)}_h, \\
(\text{ div } \overline{p}_h(t), v_h)_{-1,k,0} = -(f(t) - \frac{du}{dt}(t), v_h)_{-1,k,0}, & \forall v_h \in M^{(N,K)}_h.
\end{cases}
\]

(83)

Note that due to our hypotheses, \( \forall t \in [0,T] : f(t) - \frac{du}{dt}(t) \in \mathcal{S}^{-1,k+r,0}(D) \).

Comparing \( (\overline{p}_h(t),\overline{u}_h(t)) \) with \( (\overline{p}(t),u(t)) \), we have the following error estimate (to give a self-contained statement, we recall all the hypotheses done at the beginning of this section):

**Theorem 7.2:** We suppose that the generalized expectation \( E[\mathcal{K}] \) of the stochastic diffusion coefficient \( \mathcal{K} \), is strictly positively lower bounded i.e. that \( \inf_D E[\mathcal{K}] > 0 \) and that \( \mathcal{K} \in F_1(D) \). We suppose that our regular family of triangulations \( (\mathcal{T}_h)_{h>0} \) satisfies the refinement rules (R1) and (R2) of theorem 6.3. We assume that \( f, \frac{df}{dt} \in L^2(0,T;\mathcal{S}^{-1,k+r,0}(D)) \) and that the initial condition

\[
g \in \left\{ g \in \mathcal{S}^{-1,k+r,1}_{0}(D); \text{ div } \left( \mathcal{K} \otimes \nabla g \right) \in \mathcal{S}^{-1,k+r,0}(D) \right\}
\]

for some \( r > 1 \) and \( k < 0 \) such that \( k + r \in \mathbb{R} \) satisfies inequality (14).

Then \( \forall t \in [0,T] : \)

\[
\left\| \overline{p}_h(t) - \overline{p}(t) \right\|_{-1,k,0} \lesssim B_{N,K} \left\| \overline{p}(t) \right\|_{-1,k+r,0} + h \left\| \overline{p}(t) \right\|_{-1,k,H^1} + \left\| u(t) \right\|_{-1,k,H^2}.
\]

(84)

\[
\left\| P_h^{(N,K)} u(t) - \overline{u}_h(t) \right\|_{-1,k,0} \lesssim B_{N,K} \left\| u(t) \right\|_{-1,k+r,1} + h \left\| u(t) \right\|_{-1,k,H^2}.
\]

(85)

\[
\left\| u(t) - \overline{u}_h(t) \right\|_{-1,k,0} \lesssim B_{N,K} \left\| u(t) \right\|_{-1,k+r,1} + h \left\| u(t) \right\|_{-1,k,H^2}.
\]

(86)

where \( P_h^{(N,K)} \) denotes the orthogonal projection in the space \( \mathcal{S}^{-1,k,0}(D) \) onto the subspace \( M^{(N,K)}_h \).
**Theorem 7.3:** Let us be given some \( r > 1 \) and \( k < 0 \) such that \( k + r \in \mathbb{R} \) satisfies inequality \((14)\). Let us assume that \( f, \frac{df}{dt}, \frac{d^2 f}{dt^2} \in L^2 \left(0, T; S^{-1,k+r,0}(D)\right) \) and for the initial condition \( g \) that

\[
\begin{align*}
g &\in S^{-1,k,r,1}(D), \quad \text{div} \left(K \nabla \phi \right) \in S^{-1,k+r,0}(D), \\
f(0) + \text{div} \left(K \nabla \phi \right) &\in S^{-1,k+r,1}(D), \quad \text{and} \quad \text{div} \left(K \nabla \phi \left(f(0) + \text{div} \left(K \nabla \phi \right)\right)\right) \in S^{-1,k+r,0}(D).
\end{align*}
\]

Then for \( m = 0, 1, 2 \):

\[
\frac{d^m u}{dt^m} \in L^2 \left(0, T; S^{-1,k,r,1}(D)\right) \cap C \left([0, T]; S^{-1,k+r,0}(D)\right).
\]

**Proof:** We know already by theorem 3.4 that the thesis is true for \( m = 0, 1 \). Let us consider the Cauchy problem: find \( \zeta \in W \left(0, T; S^{-1,k,r,1}(D)\right) \) such that:

\[
\begin{cases}
\frac{d}{dt} (\zeta(s),v)_{-1,k+r,0} + \left(K \nabla \phi \zeta(s), \nabla v\right)_{-1,k+r,0} = \left(\frac{d^2 f}{dt^2} (s),v\right)_{-1,k+r,0}, & \forall v \in S^{-1,k+r,1}(D), \\
\zeta(0) = \frac{df}{dt}(0) + \text{div} \left(K \nabla \phi \left(f(0) + \text{div} \left(K \nabla \phi \right)\right)\right) & \in S^{-1,k+r,0}(D).
\end{cases}
\]

As by hypothesis \( \frac{df}{dt} \) and \( \frac{d^2 f}{dt^2} \) belong to \( L^2 \left(0, T; S^{-1,k+r,0}(D)\right) \), \( \frac{df}{dt} \in C \left([0, T]; S^{-1,k+r,0}(D)\right) \) and \( \frac{d^2 f}{dt^2} \in S^{-1,k+r,0}(D) \). Thus \( \zeta(0) \in S^{-1,k+r,0}(D) \).

By theorem 3.2, \( \zeta(\cdot) \in L^2 \left(0, T; S^{-1,k,r,1}(D)\right) \cap C \left([0, T]; S^{-1,k+r,0}(D)\right) \) and:

\[
\|\zeta\|_{C([0,T];S^{-1,k+r,0}(D))} + \|\zeta\|_{L^2(0,T;S^{-1,k,r,1}(D))} \sim \left\|\frac{d^2 f}{dt^2}\right\|_{L^2(0,T;S^{-1,k+r,0}(D))} + \|\zeta(0)\|_{S^{-1,k+r,0}(D)}.
\]

Let us set

\[
z(t) = \int_0^t \zeta(s) ds + f(0) + \text{div} \left(K \nabla \phi \right).
\]

Due to our hypothesis that \( f(0) + \text{div} \left(K \nabla \phi \right) \in S^{-1,k+r,1}(D) \),

\[
z \in C \left([0, T]; S^{-1,k,r,1}(D)\right), \quad z(0) = f(0) + \text{div} \left(K \nabla \phi \right), \quad \text{(88)}
\]

The purpose of the next result is to prove under some assumptions, some regularity on \( \frac{d^2 u}{dt^2} \), which will be needed to bound the norm in \( S^{-1,k,0}(D) \) of \( \frac{d^2 u}{dt^2} \) in proposition 7.5.
Integrating both sides of equation (87)(i) from 0 to t, we obtain:

\[
(\zeta (t), v)_{1,k+r,0} - (\zeta (0), v)_{1,k+r,0} + \left( K \nabla \zeta (t), \nabla v \right)_{1,k+r,0} = \left( \frac{df}{dt}(t), v \right)_{1,k+r,0},
\]

for all \( v \in S_0^{-1,k+r,1}(D) \) and \( t \in [0,T] \).

By Green's formula in the stochastic spaces \( S^{-1,k+r,H(\text{div})}, S^{-1,k+r,H^1(D)} \) (24), (2.10) p. 611 and (87)(ii), the above equation simplifies to:

\[
\left( \frac{dz}{dt}(t), v \right)_{1,k+r,0} + \left( K \nabla \zeta (t), \nabla v \right)_{1,k+r,0} = \left( \frac{df}{dt}(t), v \right)_{1,k+r,0},
\]

for all \( v \in S_0^{-1,k+r,1}(D) \) and \( t \in [0,T] \).

Comparing (89) and (88) with the Cauchy problem stated in the proof of theorem 3.4 shows us that \( z = \frac{du}{dt} \).

Thus

\[
\frac{d^2 u}{dt^2} = \zeta \in L^2 \left( [0,T] ; S_0^{-1,k+r,1}(D) \right) \cap C \left( [0,T] ; S_0^{-1,k+r,0}(D) \right).
\]

\[\Box\]

**Corollary 7.4:** Under the hypotheses of theorem 7.3, and supposing also that \( \frac{\partial K}{\partial x_i}, \frac{\partial K}{\partial x_j}, K^0-1 \in \mathcal{F}_1(D) \), then:

\[
\frac{du}{dt} \in C \left( [0,T] ; S_0^{-1,k+r,H^2(u)}(D) \right)
\]

(this is already known to be true for \( u(.) \) by theorem 3.6).

**Proof:** By the proof of theorem 7.3, \( z = \frac{da}{dt} \in C \left( [0,T] ; S_0^{-1,k+r,1}(D) \right) \) and satisfies:

\[
\left( K \nabla \zeta (t), \nabla v \right)_{1,k+r,0} = \left( \frac{df}{dt}(t) - \frac{dz}{dt}(t), v \right)_{1,k+r,0}, \quad \forall v \in S_0^{-1,k+r,1}(D),
\]

for all \( t \in [0,T] \).

By theorem 7.3, \( \frac{dz}{dt} = \frac{d^2 u}{dt^2} \in C \left( [0,T] ; S_0^{-1,k+r,0}(D) \right) \) and as by hypothesis: \( \frac{df}{dt}, \frac{d^2 f}{dt^2} \in L^2 \left( [0,T] ; S_0^{-1,k+r,0}(D) \right) \), we have also that \( \frac{du}{dt} \in C \left( [0,T] ; S_0^{-1,k+r,0}(D) \right) \). Thus the right-hand side in equation (90) belongs to \( S_0^{-1,k+r,0}(D) \), \( \forall t \in [0,T] \).

From equation (90) follows that in the weak sense

\[
- \text{div} \left( K \nabla \Delta z (t) \right) = \frac{df}{dt}(t) - \frac{dz}{dt}(t) \in S_0^{-1,k+r,0}(D), \quad \forall t \in [0,T].
\]

From the above considerations follows that \( \frac{df}{dt} - \frac{dz}{dt} \in C \left( [0,T] ; S_0^{-1,k+r,0}(D) \right) \). This implies that the mapping \( [0,T] \to S_0^{-1,k+r,0}(D) : t \mapsto - \text{div} \left( K \nabla \Delta z (t) \right) \) is a continuous mapping. By theorem 6.2, and
the “closed graph theorem” follows that the mapping \( t \mapsto z(t) = \frac{df}{dt}(t) \) is continuous from \([0,T]\) into \( S^{-1,k+r,H^{2,\alpha_w}(D)} \), for all \( \alpha_w > 1 - \frac{n}{2} \).

**Proposition 7.5.** Under the hypotheses of corollary 7.4 and supposing that our regular family of triangulations \((\mathcal{T}_h)_{h>0}\) satisfies the refinement rules (R1) and (R2) of theorem 6.3, we have:

\[
\left\| \frac{du}{dt}(t) - \frac{d\tilde{u}_h}{dt}(t) \right\|_{-1,k,0} \lesssim B_{N,K} \left\| \frac{du}{dt}(t) \right\|_{-1,k+1} + h \left\| \frac{du}{dt}(t) \right\|_{-1,k,H^{2,\alpha_w}(D)},
\]

\( \forall t \in [0,T] \), where the constant hidden in \( \lesssim \) is independent of \( h, N, K, t \).

**Proof:** As a consequence of our hypotheses on \( f \), \( \frac{df}{dt}, \frac{d^2f}{dt^2} \), it follows that \( f \in C^1([0,T];S^{-1,k+r,0}(D)) \). By theorem 7.3, \( \frac{du}{dt} \in C^1([0,T];S^{-1,k+r,0}(D)) \).

If we consider the finite dimensional stationary problem: given a linear form \( F_h \) on \( M_h^{(N,K)} \), find \( \overline{p}_h \in X_h^{(N,K)}, u_h \in M_h^{(N,K)} \) such that:

\[
\begin{align*}
\left\{ \begin{array}{l}
(K^{-1} \circ \tilde{p}_h, \tilde{q}_h)_{-1,k,0} + (u_h, \text{div} \tilde{q}_h)_{-1,k,0} = 0, \forall \tilde{q}_h \in X_h^{(N,K)}, \\
(\text{div} \tilde{p}_h, v_h)_{-1,k,0} = -F_h(v_h), \forall v_h \in M_h^{(N,K)}.
\end{array} \right.
\end{align*}
\]

(92)

(it is clear from the proof of theorem 5.1, that this problem does not depend on the particular value of \( k \in \mathbb{R} \)), and introduce the linear operator

\[
A_h : \left( M_h^{(N,K)} \right)' \to X_h^{(N,K)} \times M_h^{(N,K)} : F_h \mapsto (\overline{p}_h, u_h)
\]

solving the preceding problem ( \( A_h \) being a linear operator between finite dimensional spaces is automatically also continuous), we see that \( \forall t \in [0,T] \):

\[
\overline{p}_h(t), \tilde{u}_h(t) = A_h \circ P_h^{(N,K)} \left( f(t) - \frac{du}{dt}(t) \right).
\]

Consequently,

\[
\overline{p}_h(\cdot), \tilde{u}_h(\cdot) \in C^1([0,T];X_h^{(N,K)} \times M_h^{(N,K)}),
\]

and \( \forall t \in [0,T] \):

\[
\left( \frac{d\overline{p}_h}{dt}(t), \frac{d\tilde{u}_h}{dt}(t) \right) = A_h \circ P_h^{(N,K)} \left( \frac{df}{dt}(t) - \frac{d^2u}{dt^2}(t) \right).
\]

By theorem 7.3:

\[
\frac{df}{dt}(\cdot) - \frac{d^2u}{dt^2}(\cdot) \in C \left( [0,T];S^{-1,k+r,0}(D) \right)
\]

and thus a fortiori:

\[
\frac{df}{dt}(t) - \frac{d^2u}{dt^2}(t) \in S^{-1,k+r,0}(D),
\]
\( \forall t \in [0,T] \).

Thus we are allowed to apply theorem 6.9, which gives us:

\[ \left\| \frac{du}{dt}(t) - \frac{d\bar{u}_h}{dt}(t) \right\|_{-1,k,0} \lesssim B_{N,K} \left\| \frac{du}{dt}(t) \right\|_{-1,k+r,1} + h \left\| \frac{du}{dt}(t) \right\|_{-1,k,H^{2+\omega}(D)}, \]

as \( \frac{du}{dt}(t) \) is the solution of the exact stationary problem at the fixed time \( t \) corresponding to (92) with datum

\[ F(v) = \left( \frac{df}{dt}(t) - \frac{d^2u}{dt^2}(t), v \right)_{-1,k,0}, \forall v \in S^{-1,k,0}(D). \]

(as can be seen by a similar reasoning for the exact problem as we have done for the approximate problem).

\( \Box \)

8. A priori error estimates for the semi-discrete solution

In view to compare the solution at time \( t \) of the dual mixed semi-discretized problem with the solution of the elliptic projection at time \( t \), let us introduce the following quantities:

\[ \xi_h(t) := \bar{p}_h(t) - \bar{p}_h(t) \text{ and } \theta_h(t) := u_h(t) - \bar{u}_h(t). \]

Subtracting equation (83)(i) from equation (37)(i) and equation (83)(ii) from equation (37)(ii), we obtain the following system in the quantities \( \xi_h(t) \) and \( \theta_h(t) \):

\[
\begin{align*}
&\left( \mathcal{K}^{N-1} \xi_h(t), q_h \right)_{-1,k,0} + (\theta_h(t), \text{div } q_h)_{-1,k,0} = 0, \quad \forall q_h \in X^{(N,K)}_h, \\
&(\text{div } \xi_h(t), v_h)_{-1,k,0} + \left( \frac{d(u - u_h)}{dt}(t), v_h \right)_{-1,k,0} = 0, \quad \forall v_h \in M^{(N,K)}_h.
\end{align*}
\]

(93)

Moreover, as we choose \( u_h(0) = \tilde{u}_h(0) \) as initial condition for the semi-discretized problem, we have:

\[ \theta_h(0) = 0. \]

(94)

Choosing \( q_h = \xi_h(t) \) in (93)(i) and \( v_h = \theta_h(t) \) in (93)(ii), we obtain:

\[
\left( \mathcal{K}^{N-1} \xi_h(t), \xi_h(t) \right)_{-1,k,0} + (\theta_h(t), \text{div } \xi_h(t))_{-1,k,0} = 0
\]

(95)

\[
(\text{div } \xi_h(t), \theta_h(t))_{-1,k,0} + \left( \frac{d(u - u_h)}{dt}(t), \theta_h(t) \right)_{-1,k,0} = 0.
\]

(96)

From equation (96) and (95), we obtain:

\[
\left( \mathcal{K}^{N-1} \xi_h(t), \xi_h(t) \right)_{-1,k,0} + \frac{1}{2} \frac{d}{dt} \left\| \theta_h(t) \right\|^2_{-1,k,0} = \left( \frac{d}{dt} (u - \tilde{u}_h)(t), \theta_h(t) \right)_{-1,k,0}.
\]

(97)
Integrating both sides of this equation from 0 to $t$, taking into account (94), we obtain:

$$
\int_0^t \left( K^{\diamond-1} \odot \varepsilon_h (s), \varepsilon_h (s) \right)_{-1,k,0}^2 ds + \frac{1}{2} \| \theta_h (t) \|_{-1,k,0}^2 = \int_0^t \left( \frac{d}{ds} (u - \tilde{u}_h) (s), \theta_h (s) \right)_{-1,k,0}^2 ds.
$$

(98)

By Cauchy-Schwarz and Young inequalities, we obtain for $\epsilon > 0$:

$$
\int_0^t \left( K^{\diamond-1} \odot \varepsilon_h (s), \varepsilon_h (s) \right)_{-1,k,0}^2 ds + \frac{1}{2} \| \theta_h (t) \|_{-1,k,0}^2 \leq \epsilon^2 \int_0^t \| \theta_h (s) \|_{-1,k,0}^2 ds + \frac{1}{\epsilon^2} \int_0^t \left\| \frac{d}{ds} (u - \tilde{u}_h) (s) \right\|_{-1,k,0}^2 ds.
$$

(99)

Due to hypothesis (14) and lemma 4.1, $\exists C_\alpha > 0$ such that:

$$
\left( K^{\diamond-1} \odot \varepsilon_h (s), \varepsilon_h (s) \right)_{-1,k,0}^2 \geq C_\alpha \| \varepsilon_h (s) \|_{-1,k,0}^2.
$$

(100)

To be able to absorb the term $\epsilon^2 \int_0^t \| \theta_h (s) \|_{-1,k,0}^2 ds$ in the right-hand side of inequality (99) by $C_\alpha \int_0^t \| \varepsilon_h (s) \|_{-1,k,0}^2 ds$, term implicitly contained in the left-hand side of inequality (99) due to (100), let us firstly prove that

$$
\| \theta_h (s) \|_{-1,k,0} \lesssim \| \varepsilon_h (s) \|_{-1,k,0}.
$$

(101)

By (63), there exists $\tilde{\theta}_h (s) \in X_h^{(N,K)}$ such that $\text{div} \tilde{\theta}_h (s) = \theta_h (s)$ and

$$
\| \tilde{\theta}_h (s) \|_{-1,k,0} \lesssim \| \theta_h (s) \|_{-1,k,0}.
$$

(102)

Equation (93) \((t)\) (with $t$ replaced by $s$) is valid for any $\tilde{\theta}_h \in X_h^{(N,K)}$. Thus we may choose $\tilde{\theta}_h = \tilde{\theta}_h (s)$, which gives us:

$$
\left( K^{\diamond-1} \odot \varepsilon_h (s), \tilde{\theta}_h (s) \right)_{-1,k,0} + \| \theta_h (s) \|_{-1,k,0}^2 = 0.
$$

This last equation implies that:

$$
\| \theta_h (s) \|_{-1,k,0} \leq \left\| K^{\diamond-1} \odot \varepsilon_h (s) \right\|_{-1,k,0} \| \tilde{\theta}_h (s) \|_{-1,k,0}
\lesssim \left\| K^{\diamond-1} \odot \varepsilon_h (s) \right\|_{-1,k,0} \| \theta_h (s) \|_{-1,k,0} \text{ by (102)}.
$$

Thus

$$
\| \theta_h (s) \|_{-1,k,0} \lesssim \left\| K^{\diamond-1} \odot \varepsilon_h (s) \right\|_{-1,k,0}
\lesssim \| \varepsilon_h (s) \|_{-1,k,0}.
$$

This proves (101). From (99), (100) and (101) follows the following result:

**Proposition 8.1:** Supposing that $K$, $K^{\diamond-1} \in F_l (D)$ and that $k \in \mathbb{R}$ satisfies to hypothesis (14), the following inequality holds:

$$
\int_0^t \| \varepsilon_h (s) \|_{-1,k,0}^2 ds + \| \theta_h (t) \|_{-1,k,0}^2 \lesssim \int_0^t \left\| \frac{d}{ds} (u - \tilde{u}_h) (s) \right\|_{-1,k,0}^2 ds.
$$
where $\tilde{\mathbf{e}}_h (s) = \tilde{\mathbf{p}}_h (s) - \tilde{\mathbf{p}}_h (s)$ and $\theta_h (s) = u (s) - \tilde{u}_h (s)$.

**Corollary 8.2:** Under the hypotheses of proposition 7.5

$$
\left\| \tilde{\mathbf{p}}_h (\cdot) - \tilde{\mathbf{p}}_h (\cdot) \right\|_{L^2 (0, T; (\mathbb{S}^{-1, k, 0})^2)} + \sup_{0 \leq t \leq T} \| u_h (\cdot) - \tilde{u}_h (\cdot) \|_{-1, k, 0} \\
\lesssim B_{N, K} \left\| \frac{du}{dt} (t) \right\|_{L^2 (0, T; \mathbb{S}^{-1, k, r, 1} (D))} + h \left\| \frac{du}{dt} (t) \right\|_{L^2 (0, T; \mathbb{S}^{-1, k, r, 1} (D))}.
$$

**Proof:** This follows immediately from proposition 8.1 and proposition 7.5.

Applying corollary 8.2 in conjunction with theorem 7.2, we obtain the following a priori error estimates on $\tilde{\mathbf{p}}_h (\cdot)$ and $u_h (\cdot)$ (we recall all the hypotheses):

**Theorem 8.3:** We suppose:

(i) that the stochastic diffusion coefficient $\mathcal{K}(\cdot)$, its Wick inverse $\mathcal{K}^{-1}$, and its partial derivatives $\frac{\partial \mathcal{K}}{\partial x_1}, \frac{\partial \mathcal{K}}{\partial x_2}$ all belong to $\mathcal{F}_1 (D)$ and that its generalized mean $\mathbb{E} [\mathcal{K}]$ is lower bounded by a strictly positive constant on $D$;

(ii) that $k < 0, r > 1$, and that

$$
k + r < 2l + \frac{2}{\ln 2} \ln \left( \frac{\inf_D \mathbb{E} [\mathcal{K}]}{\| \mathcal{K} \|_{l, 4}} \right);
$$

(iii) that $f, \frac{df}{dt}, \frac{df}{dx} \in L^2 (0, T; \mathbb{S}^{-1, k, r, 0} (D))$ and that the initial condition $g$ satisfies

$$
g \in \mathbb{S}_0^{-1, k, r, 1} (D), \quad \text{div} \left( \mathcal{K} \Box \nabla g \right) \in \mathbb{S}_0^{-1, k, r, 0} (D),
$$

$$
f (0) + \text{div} \left( \mathcal{K} \Box \nabla g \right) \in \mathbb{S}_0^{-1, k, r, 1} (D),
$$

$$
\text{div} \left( \mathcal{K} \Box \nabla (f (0) + \text{div} \left( \mathcal{K} \Box \nabla g \right)) \right) \in \mathbb{S}_0^{-1, k, r, 0} (D);
$$

(iv) that our regular family of triangulation $(T_h)_{h > 0}$ satisfies the refinement rules (R1) and (R2) stated in theorem 6.3 for some $\alpha_w \in (1 - \frac{r}{2}, 1]$.

Then:

$$
\| \tilde{\mathbf{p}}_h - \tilde{\mathbf{p}} \|_{L^2 (0, T; (\mathbb{S}^{-1, k, 0} (D))^2)} + \| u_h - u \|_{L^2 (0, T; \mathbb{S}^{-1, k, 0} (D))} \\
\lesssim B_{N, K} \left\| \frac{du}{dt} \right\|_{L^2 (0, T; \mathbb{S}^{-1, k, r, 1} (D))} + h \left\| \frac{du}{dt} \right\|_{L^2 (0, T; \mathbb{S}^{-1, k, r, 1} (D))} + \| u \|_{C (0, T; \mathbb{S}^{-1, k, r, 1} (D))};
$$

$$
\| u_h - u \|_{C (0, T; \mathbb{S}^{-1, k, 0} (D))} \\
\lesssim B_{N, K} \left\| \frac{du}{dt} \right\|_{L^2 (0, T; \mathbb{S}^{-1, k, r, 1} (D))} + h \left\| \frac{du}{dt} \right\|_{L^2 (0, T; \mathbb{S}^{-1, k, r, 1} (D))} + \| u \|_{C (0, T; \mathbb{S}^{-1, k, r, 1} (D))}.
$$