Exponential stability of second-order evolution equations with structural damping and dynamic boundary delay feedback

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We consider a stabilization problem for abstract second-order evolution equations with dynamic boundary feedback laws with a delay and distributed structural damping. We prove an exponential stability result under a suitable condition between the internal damping and the boundary laws. The proof of the main result is based on an identity with multipliers that allows to obtain a uniform decay estimate for a suitable energy functional. Some concrete examples are detailed. Some counterexamples suggest that this condition is optimal.

Keywords: second-order evolution equations; wave equation; delay feedbacks; stabilization.

1. Introduction

Delay effects arise in many applications and practical problems. On the other hand, it is well known that an arbitrarily small time delay may destabilize a system which is naturally stable (see, e.g. Datko et al., 1986; Datko, 1988, 1997; Logemann et al., 1996). Here, we consider a damped second-order evolution equation with a time delay in the boundary condition.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary $\Gamma$. We assume that $\Gamma$ is divided into two parts $\Gamma_0$ and $\Gamma_1$, i.e. $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and $\text{meas } \Gamma_0 \neq 0$. Note that the condition $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ is only made in order to simplify the presentation, hence our analysis can be performed without this assumption in a similar manner.

For $i = 0$ and 1, we consider in $\Omega$ two strongly elliptic operators $A_i$ of order $2m_i$ with a positive integer $m_0$ and $m_1 = 1$ of the form

$$A_i = \sum_{|\alpha|,|\beta| \leq m_i} (-1)^{|\beta|} D^\beta(a^{(i)}_{\alpha,\beta} D^\alpha),$$

where $a^{(i)}_{\alpha,\beta} = a^{(i)}_{\beta,\alpha}$ are in $C^\infty(\overline{\Omega})$ and such that

$$\sum_{|\alpha|=|\beta|=m_i} a^{(i)}_{\alpha,\beta}(x) \xi^\alpha \xi^\beta \geq \alpha_i |\xi|^{2m_i}, \quad \forall \xi \in \mathbb{R}^n, \ x \in \overline{\Omega}$$

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for some positive constant $a_i$. Accordingly, we can introduce the natural bilinear form

$$a_i(u, v) = \sum_{|a_i| < m_i} \int_{\Omega} a_i^{(i)} D^a u D^\beta v \, dx, \quad \forall u, v \in H^{m_i}(\Omega).$$

On $\Gamma$ we further fix a Dirichlet system $\{D_{0j}\}_{j=0}^{m_0-1}$ of order $m_0$ with the terminology of Lions & Magenes (1961) (in particular, $D_{0j}$ is an operator of order $j$) and without loss of generality, we can assume that $D_{00}$ is equal to the identity operator $I$. We also take $D_{10} = I$. According to Section 2.4 of Lions & Magenes (1961), there exists a system $\{T_{ij}\}_{j=0}^{m_i-1}$ with smooth coefficients such that the order of $T_{ij}$ is equal to $2m_i - 1 - j$ and such that the next Green formula holds

$$a_i(u, v) = \int_{\Omega} A_i uv \, dx - \sum_{j=0}^{m_i-1} \int_{\Gamma} T_{ij} u D_{ij} v \, d\Gamma, \quad \forall u \in H^2_{m_i}(\Omega), \ v \in H^{m_i}(\Omega). \quad (1.1)$$

In this domain $\Omega$, we consider the initial boundary value problem

$$u_{tt} + A_0 u + A_1 u_t = 0 \quad \text{in} \ \Omega \times (0, +\infty), \quad (1.2)$$

$$D_{0j}u = 0 \quad \text{on} \ \Gamma_0 \times (0, +\infty), \ \forall j = 0, \ldots, m_0 - 1, \quad (1.3)$$

$$T_{0j}u = 0 \quad \text{on} \ \Gamma_1 \times (0, +\infty), \ \forall j = 1, \ldots, m_0 - 1, \quad (1.4)$$

$$\mu u_{tt}(x, t) + ku_t(x, t - \tau) = T_{0j}u(x, t) + T_{1j}u_t(x, t) \quad \text{on} \ \Gamma_1 \times (0, +\infty), \quad (1.5)$$

$$u(x, 0) = u_0(x) \quad \text{and} \ u_t(x, 0) = u_1(x) \quad \text{in} \ \Omega, \quad (1.6)$$

$$u_t(x, t) = f_0(x, t) \quad \text{in} \ \Gamma_1 \times (-\tau, 0), \quad (1.7)$$

where $\tau > 0$ is the time delay, $\mu$ is a non-negative real numbers, $k$ is a real number and the initial datum $(u_0, u_1, f_0)$ belongs to a suitable space.

In the case $\mu > 0$, (1.5) is a so-called dynamic boundary condition. Dynamic boundary conditions arise in many physical applications, in particular they occur in elastic models. For instance, these conditions appear in modelling dynamic vibrations of linear viscoelastic rods and beams that have attached tip masses at their free ends. See Andrews et al. (1996), Conrad & Morgül (1998), Littman & Markus (1988) and the references therein for more details.

In (1.2), the term $A_1 u_t$ represents a viscosity term and can be interpreted as a control in feedback form. The presence of the internal damping determines the term $T_{10} u_t$ in the boundary condition (1.5). Indeed, (1.2) and the boundary conditions can be written as the following closed-loop system with interior and boundary damping:

$$u_{tt} + A_0 u = v_1 + v_2 \quad \text{in} \ V',$$

where $V$ is defined in Section 2, $v_1$ (respectively $v_2$) is the interior (respectively boundary) control input and the measured outputs are

$$y_1 = u_t \quad \text{in} \ \Omega \times (0, +\infty),$$

and

$$y_2 = u_t \quad \text{on} \ \Gamma_1 \times (0, +\infty).$$
To stabilize the system, we use the following controller and feedback laws:

$$v_1 = u_1, \quad u_1 = -A_1 y_1 \quad \text{in } \Omega \times (0, +\infty).$$  \hfill (1.8)

With this choice a formal application of Green’s formula leads to

$$T_{00} u + T_{10} u_t = v_2 \quad \text{on } T_1 \times (0, +\infty).$$

Hence, the chosen feedback law is clearly

$$v_2 = \mu u_t - ku_2, \quad u_2(\cdot, t) = -y_2(\cdot, t - \tau) \quad \text{in } T_1 \times (0, +\infty).$$  \hfill (1.9)

An alternative dynamic boundary feedback law would be (see Section II of Mörgul, 1995 or Section 5.4 of Luo et al., 1999)

$$v_2 = cz + bu_2, \quad z_t = az + du_2, \quad u_2(\cdot, t) = -y_2(\cdot, t - \tau) \quad \text{in } T_1 \times (0, +\infty),$$  \hfill (1.10)

where $a, b, c, d$ are real numbers with $a < 0, b \geq 0$ and $c \neq 0$. By eliminating $z$, this choice leads to a dynamic condition where higher-order derivatives of $u$ are involved. We refer to Corollary 1 of Mörgul (1995) or Corollary 5.39 of Luo et al. (1999) for some stability results with the boundary feedback law (1.10).

Since our law (1.9) is not considered in the literature, we now concentrate on this choice.

Note that the above system is exponentially stable in absence of time delay, i.e. if $\tau = 0$ (see, e.g. Theorem 2 of Mörgul, 1995). As said before the presence of the feedback term with a delay can generate some instabilities. Hence, the main question is to know whether the dissipative terms $A_1 u_t$ in $\Omega$ and its companion $T_{10} u_t$ on $T_1$ are sufficient to compensate this destabilizing term.

Remark that without the internal damping $A_1 u_t$ and the related boundary term $T_{10} u_t$, the above model in absence of delay, i.e. for $\tau = 0$, is exponentially stable due to the dissipative term $k u_1(t)$ in the boundary condition. However, the system is not stable in presence of time delays for every $\tau > 0$, i.e. there are arbitrarily small delays for which the system admits unstable solutions (see, e.g. Datko, 1997, Theorem 2.3 for the wave equation). Therefore, we add the internal damping $A_1 u_t$ in order to contrast the destabilizing delay effect. In other words, we are interested in giving an exponential stability result for such a problem under a suitable relation between the operator $A_1$ and the coefficient $k$.

Due to the internal damping our model is exponentially stable for $k = 0$, i.e. in absence of delay. Hence, we can expect that if $|k|$ is sufficiently small (cf. (1.11)) with respect to the other damping terms, the system remains exponentially stable for arbitrarily (small or large) delays. More precisely, we will show that, in both cases $\mu = 0$ and $\mu > 0$, under the condition

$$1 > |k| C_P,$$  \hfill (1.11)

where $C_P$ is a sort of Poincaré constant related to the operator $A_1$ and described below (see (2.4)), the energy of the solution of system (1.2)–(1.7) satisfies a uniform exponential decay estimate. This holds for every delay $\tau > 0$.

Note that the constant $C_P$ does not depend on the constant $\mu$ in the boundary condition (1.5).

We observe also that if $A_1 \to 0$, then $C_P \to +\infty$ and therefore $k$ has to tend to zero in order that (1.11) is still satisfied (cf. (4.8) for the damped wave equation).

In the 1D case, Datko et al. (1986) proved that at least for the model

$$\begin{align*}
  u_{tt}(x, t) - u_{xx}(x, t) + 2a u_t(x, t) + a^2 u(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
  u(0, t) &= 0, \quad t > 0, \\
  u_x(1, t) &= -k u_1(1, t - \tau), \quad t > 0,
\end{align*}$$

\hfill (1.12) \hfill (1.13) \hfill (1.14)
with $a, k$ positive real numbers, a standard internal damping allows to contrast the boundary delay feedback under a suitable relation between the coefficients. Indeed, through a careful spectral analysis, they have shown that, for any $a > 0$, if $k$ satisfies

$$0 < k < \frac{1 - e^{-2a}}{1 + e^{-2a}},$$

then the spectrum of the system (1.12), (1.13), (1.14) lies in $\Re \omega \leq -\beta$, where $\beta$ is a positive constant depending on the delay $\tau$ and therefore this system is exponentially stable.

Our abstract result applies to several models, in particular wave equation and Kirchoff system with Kelvin–Voight damping (see Section 4).

In a previous paper (Nicaise & Pignotti, 2006), we addressed the problem to contrast the destabilizing effect of a time delay in the boundary feedback considering the system

$$u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty),$$
$$u(x, t) = 0 \quad \text{on } \Gamma_0 \times (0, +\infty),$$
$$\frac{\partial u}{\partial \nu}(x, t) = -k_1 u_t(x, t) - k_2 u_t(x, t - \tau) \quad \text{on } \Gamma_1 \times (0, +\infty),$$

with $k_1, k_2$ positive constants and initial data in suitable spaces. We proved that under the condition

$$k_1 > k_2,$$ \hfill (1.18)

the system (1.15)–(1.17) is exponentially stable. This was obtained by introducing a suitable energy functional and by using appropriate observability inequalities. Condition (1.18) is optimal in order to have stability of the above model. Indeed, if $0 \leq k_1 \leq k_2$, there are unstable solutions, namely solutions whose energy is not decaying to zero.

Analogous problem in one dimension was proposed by Xu et al. (2006) and solved through a careful spectral analysis. Let us further mention the paper of Gugat (2010), where for the undamped 1D wave equation, it is proved that the system can be exponentially stable for certain delays $\tau > 0$ if the corresponding feedback parameter $k$ is sufficiently small.

Here, our aim is to contrast the effect of a boundary delay damping with an interior damping. The general idea is that if the dissipative law $A_1 u_t$ in (1.2) is strong enough with respect to the boundary one with a delay, then the system will be exponentially stable. On the contrary, we will show, in dimension $n = 1$, that if this condition (1.11) is not satisfied, then the system becomes unstable. We expect that similar phenomena occur in dimension $n \geq 2$, but we were not able to obtain positive results (except in tensor product situations, like a square for instance).

An opposite problem, namely the wave equation with interior delay damping and dissipative not delayed boundary condition, has been studied in Ammari et al. (2010).

The paper is organized as follows. The Section 2 deals with the well-posedness of the problem obtained by using semi-group theory. Here, we need to distinguish the case $\mu > 0$ to the case $\mu = 0$. In Section 3, we prove the exponential stability of the delayed system (1.2)–(1.7) by introducing a suitable energy functional (see (3.2)). The main result is given in Theorem 3.3. In Section 4, we illustrate some concrete examples of our abstract model: the damped wave equation and the Kirchoff model. Moreover, we illustrate some instability examples in dimension one.
2. Well-posedness of the problems

In this section, we will give well-posedness results for problem (1.2)–(1.7) using semi-group theory. We have to distinguish the two cases $\mu = 0$ and $\mu > 0$.

We assume that $a_0$ is strongly elliptic on

$$V := \{ u \in H^{m_0}(\Omega) : D_{0j}u = 0 \text{ on } \Gamma_0, \forall \ j = 0, \ldots, m_0 - 1 \},$$

(2.1)

namely, we suppose that there exists a positive constant $\beta_0$ such that

$$a_0(u, u) \geq \beta_0 \|u\|_{H^{m_0}(\Omega)}^2, \quad \forall \ u \in V.$$  

(2.2)

Similarly, we suppose that there exists a positive constant $\beta_1$ such that

$$a_1(u, u) \geq \beta_1 \|u\|_{H^1(\Omega)}^2, \quad \forall \ u \in H^1_{\Gamma_0}(\Omega),$$

(2.3)

where, as usual,

$$H^1_{\Gamma_0}(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \}.$$  

From this estimate and a trace result, we deduce that the next Poincaré estimate holds: there exists a positive constant $C_P$ such that

$$\int_{\Gamma_1} |v|^2 \ d\Gamma \leq C_P a_1(v, v), \quad \forall \ v \in H^1_{\Gamma_0}(\Omega).$$

(2.4)

In the sequel $C_P$ is defined as the smallest positive constant such that (2.4) holds.

2.1 The case $\mu = 0$

In that case system (1.2)–(1.7) enters in the abstract framework developed in Nicaise & Valein (2010). Indeed, using the notations from that paper, it suffices to take $H = L^2(\Omega), V$ given by (2.1), $A = A_0, B_1^* = \sqrt{A_1}$ that maps $H^1_{\Gamma_0}(\Omega)$ into $H$ (looking at $A_1$ as a positive self-adjoint operator from $H$ into itself) so that

$$B_1 B_1^* = A_1$$

maps $H^1_{\Gamma_0}(\Omega)$ into its dual. But since $V$ is continuously and densely embedded into $H^1_{\Gamma_0}(\Omega), B_1 B_1^* = A_1$ also maps $V$ into $V'$. For $B_2^*$, we take the trace operator up to a multiplicative factor, namely

$$B_2^* : V \rightarrow L^2(\Gamma_1) : u \rightarrow \sqrt{k} \gamma_1 u,$$

where $\gamma_1$ is the trace operator on $\Gamma_1$.

With these notations, we now show that (1.2)–(1.7) is equivalent to the system (1.2) of Nicaise & Valein (2010), namely

$$\begin{cases} 
\ddot{u}(t) + Au(t) + B_1 B_1^* \dot{u}(t) + B_2 B_2^* \dot{u}(t - \tau) = 0 & \text{in } V', t > 0, \\
\dot{u}(0) = u_0, \ \dot{u}(0) = u_1, \\
B_2^* \dot{u}(t - \tau) = f_0(t - \tau), & 0 < t < \tau.
\end{cases}$$

(2.5)
Indeed, if $u$ is solution of this last problem, then we have

$$\langle \ddot{u}(t) + Au(t) + B_1 B_1^* \dot{u}(t) + B_2 B_2^* \dot{u}(t - \tau), \varphi \rangle_{V' \rightarrow V} = 0, \quad \forall \varphi \in V. \quad (2.6)$$

Taking first test function $\varphi \in \mathcal{D}(\Omega)$, we find that (1.2) holds in the distributional sense. Coming back to (2.6) and using (1.1) (in a weak form), we find that (1.3)–(1.5) hold (with $\mu = 0$).

Now, in order to apply the existence result from Theorem 2.1 of Nicaise & Valein (2010), we need to check the assumption (2.5) from that paper that reads in our setting as follows:

$$\exists 0 < a_0 \leq 1, \quad \forall u \in V, \quad \| B_2^* u \|_{L^2(\Gamma_1)}^2 \leq a_0 \| B_1^* u \|_{L^2(\Omega)}^2. \quad (2.7)$$

From the definition of $B_1^*$ and $B_2^*$ this is equivalent to

$$\exists 0 < a_0 \leq 1, \quad \forall u \in V, \quad |k| \int_{\Gamma_1} |u|^2 \leq a_0 a_1(u, u). \quad (2.8)$$

In view of the definition (2.4), this is finally equivalent to

$$|k| C_P \leq 1. \quad (2.9)$$

Note that this condition is slightly weaker than (1.11).

At this stage, we can apply Theorem 2.1 of Nicaise & Valein (2010) and obtain the next existence result, Theorem 2.1 below. Before stating it, in order to write it in a compact form, like in Nicaise & Pignotti (2006), we introduce the auxiliary unknown

$$z(x, \rho, t) = u_t(x, t - \tau \rho), \quad x \in \Gamma_1, \quad \rho \in (0, 1), \quad t > 0. \quad (2.10)$$

Then if we denote

$$U := (u, u_t, z)^T,$$

problem (1.2)–(1.7) is equivalent to

$$\begin{cases}
U' = AU, \\
U(0) = (u_0, u_1, f_0(\cdot, - \tau))^{T},
\end{cases} \quad (2.11)$$

where the operator $A$ is defined by

$$A \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ -(A_0 u + A_1 v) \\ -\tau^{-1} z\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(A) := \left\{ (u, v, z)^T \in V \times V \times L^2(\Gamma_1, H^1(0, 1)) : \\
T_0 u = 0 \text{ on } \Gamma_1, j = 1, \ldots, m_0 - 1, \\
A_0 u + A_1 v + B_2 z(\cdot, 1) \in L^2(\Omega), \quad v = z(\cdot, 1) \text{ on } \Gamma_1 \right\}. \quad (2.12)$$

This operator is well defined on the Hilbert space

$$\mathcal{H} := V \times L^2(\Omega) \times L^2(\Gamma_1 \times (0, 1)). \quad (2.13)$$

From Theorem 2.1 of Nicaise & Valein (2010) the well-posedness result follows.
Theorem 2.1 Assume that (2.9) holds. Then for any initial datum \( U_0 \in \mathcal{H} \), there exists a unique (weak) solution \( U \in C([0, +\infty), \mathcal{H}) \) of problem (2.11). Moreover, if \( U_0 \in \mathcal{D}(A) \), then
\[
U \in C([0, +\infty), \mathcal{D}(A)) \cap C^1([0, +\infty), \mathcal{H}),
\]
called a strong solution.

2.2 The case \( \mu > 0 \)
As before we use the auxiliary unknown (2.10) hence problem (1.2)–(1.7) is equivalent to
\[
\begin{align*}
u_{t t}(x, t) + A_0 u(x, t) + A_1 u_t(x, t) & = 0 \quad \text{in } \Omega \times (0, +\infty), \\
\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) & = 0 \quad \text{in } \Gamma_1 \times (0, 1) \times (0, +\infty), \\
D_{0j} u & = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \quad \forall j = 0, \ldots, m_0 - 1, \\
T_{0j} u(x, t) & = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad \forall j = 1, \ldots, m_0 - 1, \\
u_{t t}(x, t) & = \mu^{-1}(-k z(x, 1, t) + T_{00} u(x, t) + T_{10} u_t(x, t)) \quad \text{on } \Gamma_1 \times (0, +\infty), \\
z(x, 0, t) & = u_t(x, t) \quad \text{on } \Gamma_1 \times (0, \infty), \\
u(x, 0) & = u_0(x) \quad \text{and } \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \\
z(x, \rho, 0) & = f_0(x, -\rho \tau) \quad \text{in } \Gamma_1 \times (0, 1).
\end{align*}
\]
The difference with the previous subsection is here to use
\[
U := (u, u_t, \gamma_1 u_t, z)^T,
\]
as vectorial unknown where, as above, \( \gamma_1 \) is the trace operator on \( \Gamma_1 \). Then the previous problem is formally equivalent to
\[
U' := (u_t, \nu_{t t}, \gamma_1 u_{t t}, z_t)^T
\]
\[
= (u_t, -A_0 u - A_1 u_t, \mu^{-1}(-k z, 1, \cdot) + T_{00} u + T_{10} u_t, -\tau^{-1} z_\rho)^T.
\]
Therefore, problem (2.14)–(2.21) can be rewritten as
\[
\begin{align*}
U' & = A_1 U, \\
U(0) & = (u_0, u_1, \gamma_1 u_1, f_0(\cdot, -\cdot \tau))^T,
\end{align*}
\]
where the operator \( A_1 \) is defined by
\[
A_1 \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} := \begin{pmatrix} \nu \\ -A_0 u - A_1 v \\ \mu^{-1}(-k z(\cdot, 1) + T_{00} u + T_{10} v) \\ -\tau^{-1} z_\rho \end{pmatrix},
\]
with domain
\[ D(A_1) := \{ (u, v, v_1, z)^T \in V \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0, 1)) : \]
\[ T_{0j}u = 0 \text{ on } \Gamma_0, j = 1, \ldots, m_0 - 1, \]
\[ A_0u + A_1v \in L^2(\Omega), \]
\[ T_{00}u + T_{10}v \in L^2(\Gamma_1), \quad \gamma_1 v = v_1 = z(\cdot, 0) \text{ on } \Gamma_1, \]  
(2.23)

defined as an unbounded operator in the Hilbert space \( \mathcal{H}_1 \) defined by
\[ \mathcal{H}_1 := V \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0, 1)). \]  
(2.24)

Assuming that (2.9) holds we will show that \( A_1 \) generates a \( C_0 \) semi-group on \( \mathcal{H}_1 \). From (2.9), there exists a positive real number \( \xi \) satisfying
\[ |k| \leq \frac{\xi}{\tau} \leq \frac{2}{C_p} - |k|. \]  
(2.25)

Hence, we define on the Hilbert space \( \mathcal{H}_1 \) the inner product
\[
\left\langle \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{v}_1 \\ \bar{z} \end{pmatrix} \right\rangle_{\mathcal{H}_1} := a_0(u, \bar{u}) + \int_{\Omega} v(x)\bar{v}(x)dx + \mu \int_{\Gamma_1} v_1(x)\bar{v}_1(x)d\Gamma \\
+ \xi \int_{\Gamma_1} \int_0^1 z(x, \rho)\bar{z}(x, \rho)d\rho \ d\Gamma. 
\]  
(2.26)

**Theorem 2.2** Assume that (2.9) holds. Then for any initial datum \( U_0 \in \mathcal{H}_1 \), there exists a unique (weak) solution \( U \in C([0, +\infty), \mathcal{H}_1) \) of problem (2.22). Moreover, if \( U_0 \in D(A_1) \), then
\[ U \in C([0, +\infty), D(A_1)) \cap C^1([0, +\infty), \mathcal{H}_1), \]
called a strong solution.

**Proof.** Take \( U = (u, v, v_1, z)^T \in D(A_1) \). Then
\[
\langle A_1 U, U \rangle_{\mathcal{H}_1} = a_0(v, u) - \int_{\Omega} v(x)(A_0u(x) + A_1v(x))dx \\
+ \int_{\Gamma_1} (T_{00}u + T_{10}v - kz(x, 1))v(x)d\Gamma - \xi \tau^{-1} \int_{\Gamma_1} \int_0^1 z(x, \rho)z(x, \rho)d\rho \ d\Gamma. 
\]

So, by Lemmas 2.3 and 2.4 below (generalization of Green’s formula (1.1)) and using the definition of \( D(A_1) \), we obtain
\[
\langle A_1 U, U \rangle_{\mathcal{H}_1} = -a_1(v, v) - k \int_{\Gamma_1} z(x, 1)v(x)d\Gamma - \frac{\xi}{2\tau} \int_{\Gamma_1} (z^2(x, 1) - v^2(x))d\Gamma. \]  
(2.27)
Using the Cauchy–Schwarz and the Young inequalities, we find
\[
\langle A_1 U, U \rangle_{\mathcal{H}_1} \leq -a_1(v, v) + \left( \frac{|k|}{2} + \frac{\xi}{2\tau} \right) \int_{\Gamma_1} v^2(x) d\Gamma + \left( \frac{|k|}{2} - \frac{\xi}{2\tau} \right) \int_{\Gamma_1} z^2(x, 1) d\Gamma. \tag{2.28}
\]
Using the definition (2.4) of $C_p$, we deduce that
\[
\langle AU, U \rangle_{\mathcal{H}_1} \leq \left( -1 + C_p \left( \frac{|k|}{2} + \frac{\xi}{2\tau} \right) \right) a_1(v, v) + \left( \frac{|k|}{2} - \frac{\xi}{2\tau} \right) \int_{\Gamma_1} z^2(x, 1) d\Gamma. \tag{2.29}
\]
Now, observing that from (2.25),
\[
-1 + C_p \left( \frac{|k|}{2} + \frac{\xi}{2\tau} \right) \leq 0, \quad \frac{|k|}{2} - \frac{\xi}{2\tau} \leq 0,
\]
we obtain $\langle A_1 U, U \rangle_{\mathcal{H}_1} \leq 0$, which means that the operator $A_1$ is dissipative.

Now, we will show that $\lambda I - A_1$ is surjective for a fixed $\lambda > 0$. Given $(f, g, g_1, h)^T \in \mathcal{H}_1$, we seek a $U = (u, v, v_1, z)^T \in \mathcal{D}(A_1)$ solution of
\[
(\lambda I - A_1) \begin{pmatrix} u \\ v \\ v_1 \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ g_1 \\ h \end{pmatrix},
\]
i.e. verifying
\[
\begin{align*}
\lambda u - v &= f \quad \text{in } \Omega, \\
\lambda v + A_0 u + A_1 v &= g \quad \text{in } \Omega, \\
\lambda v_1 + \mu^{-1}(kz(\cdot, 1) - T_{00}u - T_{10}v) &= g_1 \quad \text{on } \Gamma_1, \\
\lambda z + \tau^{-1}z_\rho &= h \quad \text{on } \Gamma_1 \times (0, 1).
\end{align*}
\tag{2.30}
\]
Suppose that we have found $u$ with the appropriate regularity. Then, we have
\[
v := \lambda u - f \quad \text{in } \Omega \tag{2.31}
\]
as well as
\[
v_1 = \gamma_1 v = z(x, 0) = \lambda u - f \quad \text{on } \Gamma_1 \tag{2.32}
\]
and, as in the proof of Theorem 2.1 of Nicaise & Pignotti (2006), we find $z$ in the form
\[
z(x, \rho) = \lambda u(x) e^{-\lambda \rho \tau} - f(x) e^{-\lambda \rho \tau} + \tau e^{-\lambda \rho \tau} \int_0^\rho h(x, \sigma) e^{\lambda \sigma \tau} d\sigma \quad \text{on } \Gamma_1 \times (0, 1), \tag{2.33}
\]
and, in particular,
\[
z(x, 1) = \lambda u(x) e^{-\lambda \tau} + z_0(x), \quad x \in \Gamma_1, \tag{2.34}
\]
with $z_0 \in L^2(\Gamma_1)$ defined by
\[
z_0(x) = -f(x) e^{-\lambda \tau} + \tau e^{-\lambda \tau} \int_0^1 h(x, \sigma) e^{\lambda \sigma \tau} d\sigma, \quad x \in \Gamma_1. \tag{2.35}
\]
By (2.31), (2.32), the function $u$ verifies
\[ \lambda^2 u + A_0 u + \lambda A_1 u = g + \lambda f + A_1 f \quad \text{in } \Omega, \] (2.36)
with the boundary condition (at least formally):
\[ \mu \lambda^2 u - T_{00} u - \lambda T_{10} u = \mu (g_1 + \lambda f) - k z(\cdot, 1) - T_{10} f \quad \text{on } \Gamma_1. \] (2.37)

Multiplying the equation (2.36) by $w \in V$ and integrating in $\Omega$, we find
\[ \lambda^2 \int_{\Omega} u w \, dx + \langle A_0 u, w \rangle + \lambda \langle A_1 u, w \rangle = \int_{\Omega} (g + \lambda f) w \, dx + \langle A_1 f, w \rangle, \quad \forall w \in V. \]

Hence, a formal application of Green's formula (1.1) leads to
\[ \lambda^2 \int_{\Omega} u w \, dx + a_0(u, w) + \lambda a_1(u, w) + \int_{\Gamma_1} (T_{00} u + \lambda T_{10} u) w \, d\Gamma \]
\[ = \int_{\Omega} (g + \lambda f) w \, dx + a_1(f, w) + \int_{\Gamma_1} T_{10} f w \, d\Gamma, \quad \forall w \in V. \]

By taking into account (2.37), we get
\[ \lambda^2 \int_{\Omega} u w \, dx + a_0(u, w) + \lambda a_1(u, w) = \int_{\Omega} (g + \lambda f) w \, dx + a_1(f, w) \]
\[ + \int_{\Gamma_1} [\mu (g_1 + \lambda f) - k z(\cdot, 1) - \mu \lambda^2 u] w \, d\Gamma, \quad \forall w \in V. \]

Finally, using (2.34), we arrive at
\[ a_0(u, w) + \lambda a_1(u, w) + \lambda^2 \int_{\Omega} u w \, dx + \int_{\Gamma_1} [\lambda k e^{-\lambda \tau} u + \mu \lambda^2 u] w \, d\Gamma \]
\[ = \int_{\Omega} (g + \lambda f) w \, dx + a_1(f, w) + \int_{\Gamma_1} [\mu (g_1 + \lambda f) - k z_0] w \, d\Gamma, \quad \forall w \in V. \] (2.38)

As the left-hand side of (2.38) is coercive on $V$, and the right-hand side defines a continuous linear form on $V$, the Lax–Milgram lemma guarantees the existence and uniqueness of a solution $u \in V$ of (2.38). Once we have obtained $u \in V$, we define $v$ by (2.31) and $z$ by (2.33). We can note that $v$ belongs to $V$ since $u$ and $f$ are in $V$, while $z$ belongs to $L^2(\Gamma_1; H^1(0, 1))$ by the regularity of $g$, the fact that $u, v \in V$ and a trace theorem. In particular, $z(\cdot, 1)$ belongs to $L^2(\Gamma_1)$. With these choices, we can equivalently rewrite (2.38) as follows:
\[ a_0(u, w) + a_1(v, w) + \lambda \int_{\Omega} v w \, dx = \int_{\Omega} g w \, dx + \int_{\Gamma_1} [\mu (g_1 - \lambda v) - k z(\cdot, 1)] w \, d\Gamma, \quad \forall w \in V. \] (2.39)

If we consider $w \in \mathcal{D}(\Omega)$ in (2.39), we see that $u, v$ solve in $\mathcal{D}'(\Omega)$
\[ A_0 u + A_1 v + \lambda v = g, \] (2.40)
which is nothing else than the second identity of (2.30). Note that this identity also guarantees that

\[ A_0u + A_1v = g - \lambda v \in L^2(\Omega). \]

It then remains to check the natural boundary conditions, namely the third identity of (2.30) and

\[ T_{0j}u = 0 \quad \text{on} \quad \Gamma_0, \quad \forall \ j = 1, \ldots, m_0 - 1. \quad (2.41) \]

For that purpose, we distinguish the case \( m_0 = 1 \) or \( m_0 \geq 2 \):

(i) If \( m_0 \geq 2 \), we note that \( A_1v \) belongs to \( L^2(\Omega) \) because \( v \) belongs to \( H^2(\Omega) \) (\( V \) being included into \( H^2(\Omega) \)) and therefore \( u \in E(A_0; L^2(\Omega)) \) (with the notations introduced below). Then, in that case, we apply Lemma 2.3 below for the first term of the right-hand side of (2.39), while we can directly apply (1.1) to the second term of the right-hand side of (2.39). This yields

\[
\int_{\Omega} A_0u w \, dx - \sum_{j=0}^{m_0-1} \langle T_{0j}u; D_{0j}w \rangle + \int_{\Omega} A_1v v \, dx - \int_{\Gamma_1} T_{10}v w \, d\Gamma + \lambda \int_{\Omega} v w \, dx
\]

\[
= \int_{\Omega} g w \, dx + \int_{\Gamma_1} \mu (g_1 - \lambda v - k z(\cdot, 1)) w \, d\Gamma, \quad \forall w \in V.
\]

Due to (2.40), this is equivalent to

\[
- \sum_{j=0}^{m_0-1} \langle T_{0j}u; D_{0j}w \rangle - \int_{\Gamma_1} T_{10}v w \, d\Gamma = \int_{\Gamma_1} [\mu (g_1 - \lambda v) - k z(\cdot, 1)] w \, d\Gamma, \quad \forall w \in V.
\]

This proves that the third identity of (2.30) and (2.41) are satisfied because the mapping

\[
w \to (D_{0j}w)_{j=0}^{m_0-1}
\]

is continuous and surjective from \( V \) into \( \prod_{j=0}^{m_0-1} H^{m_0-j-1/2}(\Gamma_1) \).

(ii) If \( m_0 = 1 \), we directly apply Lemma 2.4 below to the pair \((u, v)\). This yields

\[
\int_{\Omega} (A_0 + A_1) u w \, dx - \langle T_{00}u + T_{10}v; w \rangle + \lambda \int_{\Omega} v w \, dx
\]

\[
= \int_{\Omega} g w \, dx + \int_{\Gamma_1} [\mu (g_1 - \lambda v) - k z(\cdot, 1)] w \, d\Gamma, \quad \forall w \in V.
\]

Due to (2.40), this is now equivalent to

\[
- \langle T_{00}u + T_{10}v; w \rangle = \int_{\Gamma_1} [\mu (g_1 - \lambda v) - k z(\cdot, 1)] w \, d\Gamma, \quad \forall w \in V.
\]

This proves that the third identity of (2.30) holds (while (2.41) is meaningless).

The well-posedness result follows from the Lumer–Phillips theorem. \( \square \)

Let us now recall the next technical results proved in Theorem 4.2 of Lions & Magenes (1961) (see also Theorems 2.3 and 3.2 of Lions & Magenes, 1962).
LEMMA 2.3 Let us set
\[ E(A_0; L^2(\Omega)) := \{ u \in H^{m_0}(\Omega); A_0u \in L^2(\Omega) \}. \]
Then for any \( u \in E(A_0; L^2(\Omega)) \) and any \( j = 0, \ldots, m_0 - 1 \), \( T_{0j}u \) belongs to \( H^{m_0-j-1/2}(\Gamma)' \) and the next Green’s formula holds
\[ a_0(u, w) = \int_{\Omega} A_0uw \, dx - \sum_{j=0}^{m_0-1} \langle T_{0j}u; D_{0j}w \rangle, \quad \forall w \in H^{m_0}(\Omega), \]
where \( \langle T_{0j}u; D_{0j}w \rangle \) means the duality bracket between \( H^{m_0-j-1/2}(\Gamma)' = H^{-m_0+j+1/2}(\Gamma) \) and \( H^{m_0-j-1/2}(\Gamma) \).

In a similar manner, we now prove the next technical results.

LEMMA 2.4 Assume that \( m_0 = m_1 = 1 \). Let us set
\[ E(A_0, A_1; L^2(\Omega)) := \{ (u, v) \in H^1(\Omega)^2; A_0u + A_1v \in L^2(\Omega) \}, \]
which is a Hilbert space for the norm
\[ \| (u, v) \|_E^2 = \| u \|^2_{H^1(\Omega)} + \| v \|^2_{H^1(\Omega)} + \| A_0u + A_1v \|^2_{L^2(\Omega)}. \]
Then for any \( (u, v) \in E(A_0, A_1; L^2(\Omega)) \), \( T_{00}u + T_{10}v \) belongs to \( H^{-1/2}(\Gamma)' \) and the next Green’s formula holds
\[ a_0(u, w) + a_1(v, w) = \int_{\Omega} (A_0u + A_1v)w \, dx - \langle T_{00}u + T_{10}v; w \rangle_{\Gamma}, \quad \forall w \in H^1(\Omega), \quad (2.42) \]
where \( \langle \cdot; \cdot \rangle_{\Gamma} \) means the duality bracket between \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \).

Proof. We follow the lines of the proof of Theorem 1.5.3.10 of Grisvard (1985). In a first step, we prove that \( D(\tilde{\Omega})^2 \) is dense in \( E(A_0, A_1; L^2(\Omega)) \). For that purpose, we fix an extension operator \( P \) from \( H^1(\Omega) \) into \( H^1(\mathbb{R}^n) \). Thus, for every continuous linear form \( l \) on \( E(A_0, A_1; L^2(\Omega)) \), there exist \( f_0, f_1 \in H^{-1}(\mathbb{R}^n) \) and \( g \in L^2(\Omega) \) such that
\[ l(u, v) = \langle f_0; Pu \rangle + \langle f_1; P v \rangle + \int_{\Omega} (A_0u + A_1v)g \, dx, \quad \forall (u, v) \in E(A_0, A_1; L^2(\Omega)). \quad (2.43) \]
Moreover, since \( l \) depends only on \( u \) and \( v \) in \( \Omega \) and not on \( Pu \) and \( P v \) in \( \Omega^c \), we deduce that supp \( f_0 \subset \tilde{\Omega} \) and supp \( f_1 \subset \tilde{\Omega} \).

The density result will be proved if we can show that any \( l \) that vanishes on \( D(\tilde{\Omega})^2 \) is identically equal to 0. Hence, consider \( l \) such that
\[ l(u, v) = 0, \quad \forall (u, v) \in D(\tilde{\Omega})^2. \]
From the previous considerations, this is equivalent to have
\[ \langle f_0; U \rangle + \langle f_1; V \rangle + \int_{\mathbb{R}^n} (\tilde{A}_0 U + \tilde{A}_1 V) \tilde{g} \, dx = 0, \quad \forall (U, V) \in \mathcal{D}(\mathbb{R}^n)^2, \]
where \( \mathcal{D}(\mathbb{R}^n)^2 \) denotes the space of \( \mathbb{R}^n \).
where \( \tilde{g} \) means the extension of \( g \) by zero outside \( \Omega \) and for \( i = 1, 2 \), \( \tilde{A}_i \) is an extension of \( A_i \) in \( \Omega^c \) that keeps the same properties than \( A_i \). But this is equivalent to (recall that \( A_i^* = A_i \))

\[
\tilde{A}_0 \tilde{g} = -f_0 \quad \text{and} \quad \tilde{A}_1 \tilde{g} = -f_1 \text{ in } H^{-1}(\mathbb{R}^n).
\]

Consequently, due to the ellipticity assumption, we get that \( \tilde{g} \) belongs to \( H^1(\mathbb{R}^n) \) or equivalently \( g \) belongs to \( H^1_0(\Omega) \). Using these informations in (2.43), we get

\[
I(u, v) = -\langle \tilde{A}_0 \tilde{g}; Pu \rangle - \langle \tilde{A}_1 \tilde{g}; Pv \rangle + \int_{\Omega} (A_0 u + A_1 v) g + \int_{\Gamma_1} \mu (g_1 - \lambda v) x, \quad \forall (u, v) \in E(A_0, A_1; L^2(\Omega)).
\]

Now, we take a sequence \( g_n \in \mathcal{D}(\Omega) \) such that

\[
g_n \to g \quad \text{in } H^1_0(\Omega) \quad \text{as } n \to \infty.
\]

Then, for \( (u, v) \in E(A_0, A_1; L^2(\Omega)) \),

\[
I(u, v) = \lim_{n \to \infty} \left( -\langle \tilde{A}_0 \tilde{g}_n; Pu \rangle - \langle \tilde{A}_1 \tilde{g}_n; Pv \rangle + \int_{\Omega} (A_0 u + A_1 v) g_n \ dx \right).
\]

But since \( g_n \) is smooth, we may write

\[
-\langle \tilde{A}_0 \tilde{g}_n; Pu \rangle - \langle \tilde{A}_1 \tilde{g}_n; Pv \rangle + \int_{\Omega} (A_0 u + A_1 v) g_n \ dx
\]

\[
= -\int_{\Omega} A_0 g_n u \ dx - \int_{\Omega} A_1 g_n v \ dx + \int_{\Omega} (A_0 u + A_1 v) g_n \ dx = 0,
\]

by an application of Green’s formula. These two identities show that \( I \) is zero on \( E(A_0, A_1; L^2(\Omega)) \).

With the density result in hands, we consider the mapping

\[
T : \mathcal{D}(\tilde{\Omega})^2 \to L^2(\Gamma); (u, v) \to T_{00} u + T_{10} v.
\]

Fix for a moment \( (u, v) \in \mathcal{D}(\tilde{\Omega})^2 \). Then by Green’s formula for any \( w \in H^1(\Omega) \), we have

\[
\int_{\Gamma} T(u, v) w \ d\Gamma = \int_{\Omega} (A_0 u + A_1 v) w \ dx - a_0(u, w) - a_1(v, w).
\]

(2.44)

Therefore, by Cauchy–Schwarz’s inequality, we obtain

\[
\left| \int_{\Gamma} T(u, v) w \ d\Gamma \right| \leq C \left\| (u, v) \right\|_E \left\| w \right\|_{H^1(\Omega)},
\]

for some positive constant \( C \) that does not depend on \( u, v, w \). This shows that \( T(u, v) \) belongs to \( H^{-1/2}(\Gamma) \) because the trace operator

\[
w \to w|_{\Gamma}
\]

is continuous and surjective from \( H^1(\Omega) \) onto \( H^{1/2}(\Gamma) \).

Since \( \mathcal{D}(\tilde{\Omega})^2 \) is dense in \( E(A_0, A_1; L^2(\Omega)) \), we can extend \( T \) by density to the whole of \( E(A_0, A_1; L^2(\Omega)) \). Finally, we obtain (2.42) by passing to the limit in the standard Green’s formula (2.44), the left-hand side being transformed into a duality bracket after the limit process. \( \square \)
3. Stability result
In this section, we will prove an exponential stability result for problem (1.2)–(1.7) under the assumption (1.11).

Fix a positive constant $\alpha$ such that
\[ \alpha > |k| \quad \text{and} \quad \alpha C_P < 2 - |k|C_P. \] (3.1)

Note that such a $\alpha$ exists due to the assumption (1.11). Now, let us introduce the energy of the system
\[ E(t) = \frac{1}{2} \left( \int_{\Omega} u_t^2(x,t) \, dx + a_0(u,u) \right) + \frac{\mu}{2} \int_{\Gamma_1} u_t^2(x,t) \, d\Gamma \]
\[ + \frac{\alpha}{2} \int_{t-t}^{t} \int_{\Gamma_1} u_t^2(x,s) \, d\Gamma \, ds, \] (3.2)

which is the standard energy for wave type equation plus an integral term due to the presence of a time delay and, in the case $\mu > 0$, a term due to the dynamical boundary condition.

**Proposition 3.2** Assume (1.11). For any strong solution of problem (1.2)–(1.7), the energy is decreasing and there exists a positive constant $C$ such that
\[ E'(t) \leq -C \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x,t-\tau) \, d\Gamma \right\}. \] (3.3)

**Proof.** Differentiating (3.2), we obtain
\[ E'(t) = \int_{\Omega} u_t u_{tt} \, dx + a_0(u,u_t) + \frac{\alpha}{2} \int_{\Gamma_1} u_t^2(x,t) \, d\Gamma \]
\[ - \frac{\alpha}{2} \int_{\Gamma_1} u_t^2(x,t-\tau) \, d\Gamma + \mu \int_{\Gamma_1} u_t(x,t)u_{tt}(x,t) \, d\Gamma. \]

Using (1.2) and taking into account the regularity of $u$, we get
\[ E'(t) = - \int_{\Omega} u_t (A_0 u + A_1 u_t) \, dx + a_0(u,u_t) + \frac{\alpha}{2} \int_{\Gamma_1} u_t^2(x,t) \, d\Gamma \]
\[ - \frac{\alpha}{2} \int_{\Gamma_1} u_t^2(x,t-\tau) \, d\Gamma + \mu \int_{\Gamma_1} u_t(x,t)u_{tt}(x,t) \, d\Gamma. \]

Applying Lemmas 2.3 and 2.4 and using the boundary condition (1.4), we obtain
\[ E'(t) = -a_1(u_t, u_t) + \langle T_{00} u + T_{10} u_t ; u_t \rangle_{\Gamma_1} + \frac{\alpha}{2} \int_{\Gamma_1} u_t^2(x,t) \, d\Gamma \]
\[ - \frac{\alpha}{2} \int_{\Gamma_1} u_t^2(x,t-\tau) \, d\Gamma + \mu \int_{\Gamma_1} u_t(x,t)u_{tt}(x,t) \, d\Gamma. \]
By the feedback law (1.5), we arrive at
\[ E'(t) = -a_1(u_t, u_t) - k \int_{\Gamma_1} u_t(x, t)u_t(x, t - \tau) \, d\Gamma \]
\[ + \frac{a}{2} \int_{\Gamma_1} u_t^2(x, t) \, d\Gamma - \frac{a}{2} \int_{\Gamma_1} u_t^2(x, t - \tau) \, d\Gamma. \] (3.4)

Then, from Cauchy–Schwarz inequality,
\[ E'(t) \leq -a_1(u_t, u_t) + \frac{|k|}{2} \int_{\Gamma_1} u_t^2(x, t) \, d\Gamma + \frac{|k|}{2} \int_{\Gamma_1} u_t^2(x, t - \tau) \, d\Gamma \]
\[ + \frac{a}{2} \int_{\Gamma_1} u_t^2(x, t) \, d\Gamma - \frac{a}{2} \int_{\Gamma_1} u_t^2(x, t - \tau) \, d\Gamma, \] (3.5)
and so
\[ E'(t) \leq - \left(1 - \frac{|k| + \alpha}{2} C_P \right) a_1(u_t, u_t) - \frac{a - |k|}{2} \int_{\Gamma_1} u_t^2(x, t - \tau) \, d\Gamma, \]
where in this estimate, we have used the Poincaré’s estimate (2.4). Since the constant \( \alpha \) satisfies (3.1), estimate (3.3) is proved. \( \square \)

**Proposition 3.3** Assume (1.11). There exists a time \( \overline{T} > 0 \) such that for all times \( T > \overline{T} \), there exists a positive constant \( C_0 \) (depending on \( T \)) for which
\[ E(T) \leq C_0 \int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x, t - \tau) \, d\Gamma \right\} \, dt \] (3.6)
for any strong solution \( u \) of problem (1.2) – (1.7).

**Proof.** Multiplying (1.2) by \( u \) and integrating in space and time, we have
\[ \int_0^T \int_{\Omega} (u_{tt} + A_0 u + A_1 u_t) u \, dx \, dt = 0. \]

So, integrating by parts, i.e. using Lemmas 2.3 and 2.4, we obtain
\[ \left[ \int_{\Omega} u(t)u(t) \, dx \right]_0^T - \int_0^T \int_{\Omega} u_t^2(t) \, dx \, dt + \int_0^T a_0(u, u) \, dt \]
\[ + \int_0^T a_1(u_t, u) \, dt + \int_0^T \langle T_0 u + T_1 u_t; u \rangle_{\Gamma_1} \, dt = 0, \] (3.7)
where we used also the boundary conditions (1.3) and (1.4).

From (3.7) using the boundary condition (1.5), we have
\[ \int_0^T a_0(u, u) \, dt = - \left[ \int_{\Omega} u(t)u(t) \, dx \right]_0^T + \int_0^T \int_{\Omega} u_t^2(t) \, dx \, dt \]
\[ - \int_0^T \int_{\Gamma_1} u(t)(\mu u_{tt}(t) + ku_t(t - \tau)) \, d\Gamma \, dt - \int_0^T a_1(u_t, u) \, dt. \] (3.8)
Then, integrating by parts in time,

\[
\int_0^T a_0(u, u) dt = - \left[ \int_\Omega u_r(t) u(t) dx \right]_0^T + \mu \left[ \int_{\Gamma_1} u(t) u_r(t) d\Gamma \right]_0^T + \mu \int_0^T \int_{\Gamma_1} u_r^2(t) d\Gamma dt + \int_0^T \int_\Omega u_r^2(t) dx dt + \mu \int_0^T \int_{\Gamma_1} u_r^2(t) d\Gamma dt - k \int_0^T \int_{\Gamma_1} u(t) u_r(t - \tau) d\Gamma dt - \int_0^T a_1(u_t, u) dt.
\]

(3.9)

Since (2.3) guarantees that \( a_1(u, v) \) is an inner product on \( H^1 \) by Cauchy–Schwarz inequality, we obtain

\[
|a_1(u_t, u)| \leq a_1(u_t, u)^{1/2} a_1(u, u)^{1/2}.
\]

Now, by the continuity of \( a_1 \) and (2.2), we find a constant \( C > 0 \) such that

\[
|a_1(u_t, u)| \leq C a_1(u_t, u)^{1/2} a_0(u, u)^{1/2},
\]

and finally, by Young’s inequality, we obtain

\[
|a_1(u_t, u)| \leq C^2 a_1(u_t, u) + \frac{1}{4} a_0(u, u).
\]

On the other hand, using a trace result, there exists \( C_1 > 0 \) such that

\[
\left| \int_{\Gamma_1} u(t) u_r(t - \tau) d\Gamma \right| \leq C_1 \| u \|_{H^1(\Omega)} \left( \int_{\Gamma_1} u_r^2(t - \tau) d\Gamma \right)^{1/2}.
\]

Hence, using again (2.2) and Young’s inequality, we get

\[
\left| \int_{\Gamma_1} u(t) u_r(t - \tau) d\Gamma \right| \leq C_1^2 \int_{\Gamma_1} u_r^2(t - \tau) d\Gamma + \frac{1}{4} a_0(u, u).
\]

(3.11)

From (3.9), (3.10), (3.11) and (2.2) and (2.4) we deduce the following inequality:

\[
\frac{1}{2} \int_0^T a_0(u, u) dt \leq \tilde{C} (E(T) + E(0)) + C \int_0^T a_1(u_t, u_t) dt + C \int_0^T \int_{\Gamma_1} u_r^2(t - \tau) d\Gamma dt
\]

(3.12)

for suitable positive constants \( C, \tilde{C} \). In the following, we will denote by \( C \) any suitable positive constant, while \( \tilde{C} \) is the constant in (3.12). Note that the constant \( \tilde{C} \) is independent of \( T \).

By adding \( \frac{1}{2} \int_0^T \int_\Omega u_r^2(t) dx dt \) and \( \frac{\mu}{2} \int_0^T \int_{\Gamma_1} u_r^2(t) d\Gamma dt \) to both sides of (3.12) and using once more (2.2), we obtain

\[
\int_0^T E_S(t) dt + \frac{\mu}{2} \int_0^T \int_{\Gamma_1} u_r^2(t) d\Gamma dt \leq \tilde{C} (E(T) + E(0)) + C \int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_r^2(t - \tau) d\Gamma \right\} dt,
\]

(3.13)

where \( E_S(\cdot) \) denotes the standard energy for our system, i.e. \( E_S(t) := \frac{1}{2} \int_\Omega u_r^2 dx + a_0(u, u) \).
Now observe that, for $T > \tau$, we have
\[
\int_0^T \int_{t-\tau}^t \int_{\Gamma_1} u_t^2(x, s) \, d\Gamma \, ds \, dt \leq \int_0^T \int_{t-\tau}^T \int_{\Gamma_1} u_t^2(x, s) \, d\Gamma \, ds \, dt
\]
\[
= \int_0^T \int_0^{T+\tau} \int_{\Gamma_1} u_t^2(x, s-\tau) \, d\Gamma \, ds \, dt \leq T \int_0^{T+\tau} \int_{\Gamma_1} u_t^2(x, t-\tau) \, d\Gamma \, dt.
\]
Adding $\frac{\alpha}{2} \int_0^T \int_{t-\tau}^T \int_{\Gamma_1} u_t^2(x, s) \, d\Gamma \, ds \, dt$ to both sides of (3.13) and using the above estimate, we deduce
\[
\int_0^T E(t) \, dt \leq \tilde{C} (E(T) + E(0)) + C \int_0^{T+\tau} \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(t-\tau) \, d\Gamma \right\} \, dt. \tag{3.14}
\]
From (3.4), using a trace result and (2.3), we have
\[
E(0) \leq E(T) + C \int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(t-\tau) \, d\Gamma \right\} \, dt,
\]
and so, using this estimate in (3.14) and the fact that the energy $E(\cdot)$ is decreasing,
\[
TE(T) \leq \int_0^T E(t) \, dt \leq 2\tilde{C} E(T) + C \int_0^{T+\tau} \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(t-\tau) \, d\Gamma \right\} \, dt. \tag{3.15}
\]
Therefore, we can estimate
\[
(T - 2\tilde{C}) E(T + \tau) \leq (T - 2\tilde{C}) E(T) \leq C \int_0^{T+\tau} \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(t-\tau) \, d\Gamma \right\} \, dt.
\]
Since the constant $\tilde{C}$ is independent of $T$ (while $C$ depends on $T$), then for $T$ sufficiently large estimate, (3.6) is proved. \qed

**Theorem 3.3** Assume (1.11) and let $E(\cdot)$ be the energy functional defined in (3.2). Then, for any strong solution of problem (1.2)–(1.7)
\[
E(t) \leq C_1 E(0) e^{-C_2 t}, \quad t > 0, \tag{3.16}
\]
with constants $C_1, C_2$ independent of the initial data and of the constant $\mu$ in the boundary condition (1.5).

**Proof.** From (3.3), we have
\[
E(T) - E(0) \leq -C \int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x, t-\tau) \, d\Gamma \right\} \, dt
\]
or equivalently
\[
\int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x, t-\tau) \, d\Gamma \right\} \, dt \leq C^{-1} (E(0) - E(T)). \tag{3.17}
\]
By (3.17) and the estimate (3.6), we obtain
\[
E(T) \leq C_0 \int_0^T \left\{ a_1(u_t, u_t) + \int_{\Gamma_1} u_t^2(x, t - \tau) d\Gamma \right\} \leq C_0 C^{-1}(E(0) - E(T)),
\]
then
\[
E(T) \leq \tilde{C} E(0), \tag{3.18}
\]
with \( \tilde{C} < 1 \). This easily implies the exponential stability estimate since our system (1.2)–(1.7) is invariant by translation and the energy \( E(\cdot) \) is decreasing. \( \Box \)

**Remark 3.4** Note that, as can be deduced by the above proof, (3.18) is satisfied for
\[
T := T(\tau) = T^* + \tau,
\]
where \( T^* \) is a positive time independent of the delay \( \tau \). Then, even if the constant \( \tilde{C} \) in (3.18) is independent of \( \tau \), the constant \( C_2 \) in the exponential estimate (3.16) depends on \( \tau \). Indeed, it results
\[
C_2 = \frac{1}{T + \tau} \ln \frac{1}{\tilde{C}}.
\]
On the contrary, \( C_1 = \tilde{C}^{-1} \) is independent of \( \tau \).

**Remark 3.5** Analogous arguments apply if we have more than one delay term in the boundary feedback, i.e. if condition (1.5) is substituted by
\[
\mu u_{tt}(x, t) + \sum_{i=1}^{r} k_i u_t(x, t - \tau_i) = T_{00} u(x, t) + T_{10} u_t(x, t) \quad \text{on} \quad \Gamma_1 \times (0, +\infty),
\]
with \( \tau_i, i = 1, \ldots, r \), positive parameters and real numbers \( k_i, i = 1, \ldots, r \). In this case, the right energy for our problem is
\[
E(t) := \frac{1}{2} \int_{\Omega} u_t^2(x, t) dx + \frac{1}{2} a_1(u, u) + \mu \int_{\Gamma_1} u_t^2(x, t) d\Gamma + \sum_{i=1}^{r} \frac{\alpha_i}{2} \int_{t-\tau_i}^{t} \int_{\Gamma_1} u_t^2(x, s) d\Gamma ds,
\]
with suitable positive constants \( \alpha_i, i = 1, \ldots, r \). Indeed, if
\[
1 > C_P \sum_{i=1}^{r} |k_i|,
\]
choosing \( \alpha_i \) such that
\[
\alpha_i > |k_i|, \quad i = 1, \ldots, r \quad \text{and} \quad C_P \sum_{i=1}^{r} \alpha_i < 2 - C_P \sum_{i=1}^{r} |k_i|,
\]
we can prove that the energy is exponentially decaying to zero.
4. Some examples

In this section, we first give some explicit examples of our abstract model. In particular, the first example is the wave equation with structural damping. In this case, we can characterize condition (1.11) explicitly (see (4.8) below) and show that if the damping (which is determined by a damping parameter $a$) goes to zero, then the length of the interval of feedback parameters for which the system is exponentially stable also goes to zero with the same rate. Similar analysis holds for the second example, i.e. the damped Kirchoff model in dimension 1 or 2.

Finally, we also give some instability examples which illustrate that if condition (4.8) is not satisfied, then there exist arbitrarily small (large) delays for which the 1D damped wave equation becomes unstable.

4.1 The damped wave equation

As a first example, we can take $A_0 = -\Delta$ and $A_1 = -a\Delta$ with associated forms

$$a_0(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad a_1(u,v) = aa_0(u,v),$$

where $a$ is a fixed positive parameter. In that case, $V = H^1_{\nu_0}(\Omega)$ and (2.2) and (2.3) hold by the Poincaré–Friedrichs inequality.

With that choice (1.1) holds with $T_{00}u = -\frac{\partial u}{\partial v}$, $T_{10}u = -a\frac{\partial u}{\partial v}$,

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial v}$ is the normal derivative.

In this situation problem, (1.2)–(1.7) reduces to

$$u_{tt}(x,t) - \Delta u(x,t) - a\Delta u_t(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \quad (4.1)$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \quad (4.2)$$

$$\mu u_{tt}(x,t) + \frac{\partial u}{\partial v}(x,t) = -a\frac{\partial u_t}{\partial v}(x,t) - ku_t(x,t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \quad (4.3)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega, \quad (4.4)$$

$$u_t(x, t) = f_0(x,t) \quad \text{in} \quad \Gamma_1 \times (-\tau, 0). \quad (4.5)$$

For $\mu = 0$, the above model, with boundary condition on $\Gamma_1$,

$$\frac{\partial u}{\partial v}(x,t) = -ku_t(x,t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \quad (4.6)$$

instead of (4.3), has been introduced and investigated in Datko (1991). It can be shown that system (4.1), (4.2) and (4.6) with initial data in suitable spaces is exponentially stable if $\tau = 0$, i.e. in absence of delay. We refer also to Chen (1981), Lagnese (1983, 1988), Lasiecka & Triggiani (1987), Komornik (1991, 1994), Komornik & Zuazua (1990), Lions (1988), Zuazua (1990) for the more studied case $a = 0$. However, the feedback law (4.6) is not stable with respect to small time delays. Indeed, by direct eigenvalue calculations, in Datko (1991), it is proved that for any $a, k > 0$ and any arbitrarily small
delay, system (4.1), (4.2) and (4.6) admits solutions whose energy is not decaying to zero. Hence, it is important to look for feedback laws that are robust with respect to (small) time delays. Our arguments below show that the boundary condition (4.3) instead of (4.6) allows to obtain exponential decay rate for any delay $\tau > 0$ as far as $k$ is small enough with respect to $a$ (see (4.8) below).

The 1D version of the above model with $\mu = 0$ in the boundary condition (4.3) has been considered by Mörgul (1995) who proposed a class of dynamic boundary controllers to solve the stability robustness problem.

In the case of dynamic boundary condition, i.e. $\mu > 0$ in (4.3), the above model without delay (e.g. $\tau = 0$) has been proposed in one dimension by Pellicer & Sòla-Morales (2004) as an alternative model for the classical spring–mass damper system. It is well known that in absence of delay, the system is exponentially stable. Then, we are interested in conditions ensuring the robustness with respect to small delays in the boundary feedback.

$$\int_{\Gamma} |v|^2 d\Gamma \leq \frac{\tilde{C}_P}{a} \int_{\Omega} \nabla v \cdot \nabla v \, dx, \quad \forall v \in H^{1/2}_{0}\Omega, \quad (4.7)$$

where $\tilde{C}_P$ is the smallest positive constant such that

$$\int_{\Gamma} |v|^2 d\Gamma \leq C \int_{\Omega} \nabla v \cdot \nabla v \, dx, \quad \forall v \in H^{1/2}_{0}\Omega, \quad (4.7)$$

holds. This means that $C_P = \frac{\tilde{C}_P}{a}$ and (1.11) is equivalent to

$$a > |k|\tilde{C}_P. \quad (4.8)$$

Therefore, Theorem 3.3 implies exponential stability of system (4.1)–(4.5) under the assumption (4.8). Some counterexamples when this condition is not satisfied are illustrated in the following.

Note finally that without internal damping (i.e. if $a = 0$), the previous model is destabilized for arbitrarily small delays for every value $k > 0$ (cf. Datko, 1997). Thus, the internal damping $-a \Delta u_t$ makes the system robust with respect to time delays in the boundary condition if the coefficient $a$ is sufficiently large with respect to $k$.

4.2 The Damped Kirchoff Model

Here, we reduce to the 1D or 2D case, i.e. we let $\Omega \subset \mathbb{R}^n$, with $n = 1$ or 2. As operator $A_0$ we take the biharmonic operator

$$A_0 = \Delta^2,$$

with associated bilinear form

$$a_0(u, v) = \int_{\Omega} u'' v'' \, dx, \quad (4.9)$$

in dimension 1, while in dimension 2, we take

$$a_0(u, v) = \int_{\Omega} \left\{ \Delta u \Delta v - (1 - \tilde{v}) \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \right) \right\} \, dx \, dy, \quad (4.10)$$
where $\tilde{\nu}$ is the so-called Poisson coefficient that depends on the constitutive material of the plate $\Omega$ and is a real parameter that belongs to $(-1, \frac{1}{2})$. As operator $D_{0j}$, $j = 0, 1$, we take
\[ D_{00} = I_d \quad \text{and} \quad D_{01} = \frac{\partial}{\partial \nu}. \]

With that choice, we have
\[ V = \left\{ u \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \right\}, \]
and (2.2) holds still by the Poincaré–Friedrichs inequality.

With this choice, we know that (1.1) holds for $i = 0$ with
\[ T_{00}u = Nu = \frac{\partial u''}{\partial \nu}, \quad T_{01}u = -Mu = -u'' \]
in dimension 1, while
\[ T_{00}u = Nu = \frac{\partial \Delta u}{\partial \nu} + (1 - \tilde{\nu})\frac{\partial^3 u}{\partial \nu \partial \tau^2}, \quad T_{01}u = -Mu = -\tilde{\nu}\Delta u + (1 - \tilde{\nu})\frac{\partial^2 u}{\partial \nu^2}. \]
in dimension 2.

We make the same choice as before for the operator $A_1$ and the form $a_1$.

For that choice, problem (1.2)–(1.7) reduces to
\begin{align*}
u_{tt}(x, t) + \Delta^2 u(x, t) - a \Delta u_t(x, t) &= 0 \quad \text{in} \quad \Omega \times (0, +\infty), \\
u(x, t) &= \frac{\partial u}{\partial \nu}(x, t) = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \\
M u(x, t) &= 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \\
\mu \nu_{tt}(x, t) &= Nu(x, t) - a \frac{\partial u_t}{\partial \nu}(x, t) - ku_t(x, t - \tau) \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \\
u(x, 0) &= u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \\
u_t(x, t) &= f_0(x, t) \quad \text{in} \quad \Gamma_1 \times (-\tau, 0). \\
\end{align*}

This system with $a = \mu = \tau = 0$ has been studied in Section 9.4 of Komornik (1994) where it is proved that the system is exponentially stable under some standard geometrical conditions on $\Gamma_0$ and $\Gamma_1$ (for the use of two feedbacks, we refer to Lagnese, 1989; Rao, 1993; Guo & Yao, 2006). According to Datko (1988, 1997), instability phenomena occur when $a = \mu = 0$ and $\tau > 0$.

The system with $a = \tau = 0$ is extensively studied in the literature and corresponds to some SCOLE models, where some exponential or polynomial stability results are proved in Chentouf (2003), Chentouf & Wang (2007), Conrad & Morgül (1998), Conrad & Saouri (2002), Littman & Markus (1988), Liu et al. (2005), Rao (1995, 1998) (for non-linear problems, see, e.g. Gerbi & Said-Houari, 2008; Grobbelaar-Van Dalsen & van der Merwe, 1999). As in the previous section, the term $a \Delta u_t$ with $a > 0$ is introduced in order to restitute a stability result independently of the delay.

Since the operator $A_1$ has not changed with respect to the previous subsection, Theorem 3.3 implies exponential stability of system (4.11)–(4.16) under the assumption (4.8).
4.3 Some instability examples

Now, we will give some instability examples, in dimension 1, for problem (4.1)–(4.5) if \( k > 0 \) and \( a \) is close enough to zero and hence if condition (4.8) is no more valid.

For that purpose, let us consider the spectral problem for the system (4.1)–(4.5). Namely, we look for a solution \( u \) of this system in the form

\[
u(x, t) = e^{\lambda t} \varphi(x), \quad \lambda \in \mathbb{C}.
\]

Then, \( \varphi \) has to be a solution of the eigenvalue problem

\[
\begin{aligned}
\lambda^2 \varphi - (1 + a \lambda) \Delta \varphi &= 0 \quad \text{in } \Omega, \\
\varphi &= 0 \quad \text{on } \Gamma_0, \\
(1 + a \lambda) \frac{\partial \varphi}{\partial \nu} + (\mu \lambda^2 + k \lambda e^{-\lambda t}) \varphi &= 0 \quad \text{on } \Gamma_1.
\end{aligned}
\]

(4.17)

4.3.1 The case \( \mu = 0 \). Take \( \Omega = (0, 1), \Gamma_0 = \{0\} \) and \( \Gamma_1 = \{1\} \). In that situation, the constant \( \tilde{C}_P = 1 \). Indeed by the identity,

\[
v(1) = \int_0^1 v'(x) dx,
\]

valid for all \( v \in H^1_{\Gamma_0}(\Omega) \), we obtain by Cauchy–Schwarz’s inequality that

\[
|v(1)|^2 \leq \int_0^1 |v'|^2 dx, \quad \forall v \in H^1_{\Gamma_0}(\Omega),
\]

which shows that (2.4) holds with \( \tilde{C}_P \leq 1 \). But the choice \( v(x) = x \) implies that \( \tilde{C}_P = 1 \) because in that case the above inequality becomes an equality.

Now with the above choices, the eigenvalue problem (4.17) reduces to find \( \lambda \in \mathbb{C} \) and \( \varphi \in H^2(0, 1) \) solution of

\[
\begin{aligned}
\lambda^2 \varphi - (1 + a \lambda) \varphi'' &= 0 \quad \text{in } (0, 1), \\
\varphi(0) &= 0, \\
(1 + a \lambda) \varphi'(1) + k \lambda e^{-\lambda t} \varphi(1) &= 0.
\end{aligned}
\]

(4.18)

Hence, for \( \Re \lambda \geq 0 \) and \( \lambda \neq 0 \), \( \varphi \) takes the form

\[
\varphi(x) = \alpha \sinh \left( \frac{\lambda x}{\sqrt{1 + a \lambda}} \right)
\]

for some constant \( \alpha \). The boundary condition at 1 is then equivalent to

\[
\alpha \left( \sqrt{1 + a \lambda} \cosh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) + k e^{-\lambda t} \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) \right) = 0.
\]

Therefore, any non-zero eigenvalue \( \lambda \) of problem (4.18) such that \( \Re \lambda \geq 0 \) is a solution of the equation

\[
\cosh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) + \frac{k e^{-\lambda t}}{\sqrt{1 + a \lambda}} \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) = 0.
\]

(4.19)
This characteristic equation is similar to the equation (3.15) of Datko (1991) but unfortunately, the analysis performed in that paper to find solutions \( \lambda \) with a non-negative real part cannot be adapted to our situation. Nevertheless, some numerical results for \( k = 1 \) presented in Figs 1–4 for different values of \( \tau \) show that (4.19) has indeed solution \( \lambda \) such that \( \Re \lambda \geq 0 \) if \( a \) is small enough. For \( \tau = 6 \), Fig. 2 shows even that for all \( a \in [0, 1] \), there exists a solution \( \lambda \) such that \( \Re \lambda \geq 0 \) and therefore, the system is not stable for \( \tau = 6 \).

In order to obtain analytic results, we use a perturbation argument. Indeed, using Theorem 4.1 below, we know that if (4.19) with \( a = 0 \) has a solution \( \lambda_0 \) such that \( \Re \lambda_0 > 0 \), then (4.19) has a solution \( \lambda_a \) such that \( \Re \lambda_a > 0 \) if \( a > 0 \) is small enough. Hence, we are reduced to study (4.19) with \( a = 0 \), namely

\[
\cosh \lambda + k e^{-\lambda \tau} \sinh \lambda = 0. \tag{4.20}
\]

In a first attempt, we look at \( \tau \) in the form

\[
\tau = 2n,
\]

with \( n \in \mathbb{N} \). In that case, (4.20) becomes

\[
(e^\lambda + e^{-\lambda}) + k e^{2n\lambda} (e^\lambda - e^{-\lambda}) = 0.
\]

Hence, by setting \( y = e^{-2\lambda} \), we find the polynomial equation

\[
k y^{n+1} - ky^n - y - 1 = 0. \tag{4.21}
\]

Now setting \( p(y) = ky^{n+1} - ky^n - y - 1 \), we note that \( p(0) = -1 \) and \( p(-1) = 2k \) if \( n \) is odd. Hence, for \( n \) odd, we deduce that \( p \) has a root \( y_0 \in (-1, 0) \) and coming back to \( \lambda \), we find a solution \( \lambda_0 \) of (4.20) given by

\[
-2\lambda_0 = \ln(-y_0) \pm i \pi,
\]

or equivalently

\[
\lambda_0 = -\frac{1}{2} \ln(-y_0) \pm \frac{i \pi}{2}.
\]

Since \( y_0 \in (-1, 0) \), \( \ln(-y_0) \) is negative and therefore, the real part of \( \lambda_0 \) is positive. Figures 1 and 2 show the real part of the roots of (4.19) for the values \( \tau = 2 \) and \( \tau = 6 \), respectively, and for \( a \in (0, 2) \). Red lines correspond to real roots, while blue lines give the real part of complex roots. These figures confirm the above considerations.

To get similar results for small values of delays, we now take \( \tau \) in the form

\[
\tau = \frac{2}{n},
\]

with \( n \in \mathbb{N} \) odd. In that case, (4.20) becomes

\[
(e^\lambda + e^{-\lambda}) + k e^{\frac{2\lambda}{n}} (e^\lambda - e^{-\lambda}) = 0.
\]

Hence, by setting \( y = e^{-\frac{2\lambda}{n}} \), we find again the polynomial equation (4.21). Therefore, we find a solution \( \lambda_0 \) of (4.20) given by

\[
-\frac{2\lambda_0}{n} = \ln(-y_0) \pm i \pi,
\]
Fig. 1. The case $\tau = 2$ and $k = 1$, $a \in [0, 2]$.

Fig. 2. The case $\tau = 6$ and $k = 1$, $a \in [0, 2]$.

or equivalently

$$\lambda_0 = -\frac{n}{2} \ln(-y_0) \pm i \frac{n\pi}{2}.$$  

Since $\ln(-y_0)$ is negative and the real part of $\lambda_0$ is positive. Figures 3 and 4 show the real part of the roots of (4.19) for the values $\tau = 0.4$ and $\tau = 2/3$, respectively, and for $a \in (0, 2)$. As before these figures confirm the above considerations.

In summary, we have shown that there exist arbitrarily small or large delays for which the system (4.1)–(4.5) in 1D becomes unstable if $a$ is too small.

Note also that we can repeat the above argument with the substitution $y = e^{-\frac{2j}{m}}$, $m \in \mathbb{N}$, for both sequences $\tau = \frac{2}{m}$ and $\tau = 2n$ obtaining, for $m, n$ odd, for arbitrarily small or large delays, solutions with arbitrarily large real part (cf. Datko, 1991).

Note that (4.21) is analogous to the equation

$$z^{2j+1} + z^{2j} - f z + f = 0,$$  

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In summary, we have shown that there exist arbitrarily small or large delays for which the system (4.1)–(4.5) in 1D becomes unstable if $a$ is too small.

Note also that we can repeat the above argument with the substitution $y = e^{-\frac{2j}{m}}$, $m \in \mathbb{N}$, for both sequences $\tau = \frac{2}{m}$ and $\tau = 2n$ obtaining, for $m, n$ odd, for arbitrarily small or large delays, solutions with arbitrarily large real part (cf. Datko, 1991).

Note that (4.21) is analogous to the equation

$$z^{2j+1} + z^{2j} - f z + f = 0,$$  

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In summary, we have shown that there exist arbitrarily small or large delays for which the system (4.1)–(4.5) in 1D becomes unstable if $a$ is too small.
studied in Gugat & Tucsnak (2011), where the 1D case without internal damping has been considered and it has been shown that the system without internal damping is stable if the delay $\tau$ is a multiple of four and $k > 0$ is sufficiently small. In the notation of the present paper, this means that $n = 2j$ is even.

4.3.2 A perturbation result. In the previous section, we have seen that we need to show that the eigenvalues of problem (4.17) approach the ones corresponding to $a = 0$ as $a \to 0$ but going to zero. This is a natural perturbation result that we shall show by using some classical perturbation results from Kato. For that purpose, we note that $\varphi$ is a solution of (4.17) if and only if the vector

$$U := (\varphi, \lambda \varphi, \lambda e^{-\lambda \tau \rho} \varphi)^T$$

belongs to $D(\mathcal{A})$ and satisfies

$$\mathcal{A}U = \lambda U.$$

Hence, we only need to study the dependence of the eigenvalue of the operator $\mathcal{A}$ with respect to $a$. For convenience we need to specify the dependency of $\mathcal{A}$ with respect to $a$, i.e. denoted by $\mathcal{A}_a$. 
We now state and prove the following result.

**Theorem 4.1** The operator \( A_a \) tends to \( A_0 \) in the generalized sense of Kato (cf. Kato, 1976, Section IV.2.6) or equivalently

\[
(\lambda - A_a)^{-1} \rightarrow (\lambda - A_0)^{-1}
\]

in norm as \( a \rightarrow 0 \), \( \forall \lambda \in \rho(A_0) \).

Consequently, if \( \lambda_0 \) is an eigenvalue of \( A_0 \) of algebraic multiplicity \( k \), then for all \( \epsilon > 0 \), there exists \( \delta_\epsilon > 0 \) such that for all \( a \in (0, \delta_\epsilon) \), \( A_a \) has \( k \) eigenvalues in the open ball \( B(\lambda_0, \epsilon) \).

**Proof.** Fix a positive real number \( \lambda \). By Theorem 2.1 for all \( F := (f, g, h)^\top \in \mathcal{H} \), there exists a unique \( U_a = (u_a, v_a, z_a)^\top \in \mathcal{D}(A_a) \) solution of

\[
(\lambda I - A_a) \begin{pmatrix} u_a \\ v_a \\ z_a \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}.
\]

According to the proof of Theorem 2.2, they are given by

\[
z_a(x, \rho) = \lambda u_a(x) e^{-\lambda \rho \tau} - f(x) e^{-\lambda \rho \tau} + \tau e^{-\lambda \rho \tau} \int_0^\rho h(x, \sigma) e^{\lambda \sigma \tau} d\sigma \quad \text{on} \quad \Gamma_1 \times (0, 1), \tag{4.22}
\]

\[
v_a = \frac{\lambda}{1 + \lambda a} s_a - \frac{1}{1 + \lambda a} f, \tag{4.23}
\]

\[
u_a = \frac{1}{1 + \lambda a} s_a + \frac{a}{1 + \lambda a} f, \tag{4.24}
\]

where \( s_a \in H^{1}_{\Gamma_0}(\Omega) \) is the unique solution of

\[
b_a(s_a, w) = \int_{\Omega} \left( g + \frac{\lambda}{1 + \lambda a} f \right) w \mathrm{d}x - \int_{\Gamma_1} \left( \frac{k \lambda}{1 + \lambda a} e^{-\lambda \tau f} + k z_0 \right) w \mathrm{d}\Gamma, \quad \forall w \in H^{1}_{\Gamma_0}(\Omega), \tag{4.25}
\]

with

\[
b_a(s, w) = \int_{\Omega} \left( \frac{\lambda^2}{1 + \lambda a} s w + \nabla s \cdot \nabla w \right) \mathrm{d}x + \int_{\Gamma_1} \frac{k \lambda}{1 + \lambda a} e^{-\lambda \tau} s w \mathrm{d}\Gamma.
\]

Since

\[
b_a(s, s) = \int_{\Omega} \left( \frac{\lambda^2}{1 + \lambda a} s^2 + |\nabla s|^2 \right) \mathrm{d}x + \int_{\Gamma_1} \frac{k \lambda}{1 + \lambda a} e^{-\lambda \tau} s^2 \mathrm{d}\Gamma,
\]

we deduce that there exists \( a_0 > 0 \) small enough and a positive constant \( a_0 \) (independent of \( a \)) such that for all \( a \in [0, a_0] \), one has

\[
b_a(s, s) \geq a_0 \left( \|s\|_{H^1(\Omega)}^2 + \|s\|_{L^2(\Gamma_1)}^2 \right), \quad \forall s \in H^1_{\Gamma_0}(\Omega). \tag{4.26}
\]

For the remainder of the proof \( a \) is arbitrary in \( [0, a_0] \), and \( \beta_0 > 0 \) is a positive constant independent of \( a \) (that may depend on \( \lambda \)) and that varies from place to place.
The identity (4.25), the estimate (4.26) and Cauchy–Schwarz’s inequality yield
\[
\|s_a\|_{H^1(\Omega)} + \|s_a\|_{L^2(\Gamma_1)} \leq \beta_0 \|F\|_{\mathcal{H}},
\]
(4.27)
where \(\cdot\) \(\|\cdot\|_{\mathcal{H}}\) is the natural norm of \(\mathcal{H}\), i.e.
\[
\|(u, v, z)^\top\|^2_{\mathcal{H}} = \int_{\Omega} \{\|\nabla u(x)\|^2 + v(x)^2\} \, dx + \int_{\Gamma_1} \int_0^1 (z(x, \rho)^2 \, d\rho \, d\Gamma.
\]

Now for an arbitrary \(w \in H^1_{I_0}(\Omega)\), by (4.25) and the definition of \(b_a\), we may write
\[
b_0(s_a - s_0, w) = b_a(s_a, w) - b_0(s_0, w) + (b_0 - b_a)(s_a, w)
\]
(4.28)
\[
= \frac{\lambda^2 a}{1 + \lambda a} \int_{\Omega} f w \, dx + \frac{k a \lambda}{1 + \lambda a} e^{-\lambda t} \int_{\Gamma_1} f w \, d\Gamma \quad (4.29)
\]
\[
+ \frac{\lambda^2 a}{1 + \lambda a} \int_{\Omega} s_a w \, dx + \frac{k a \lambda^2}{1 + \lambda a} e^{-\lambda t} \int_{\Gamma_1} s_a w \, d\Gamma. \quad (4.30)
\]
Taking \(w = s_a - s_0\) and using (4.26), we obtain for all \(a \in [0, a_0]\),
\[
\|s_a - s_0\|_{H^1(\Omega)} + \|s_a - s_0\|_{L^2(\Gamma_1)} \leq a \beta_0 \left( \|f\|_{0, \Omega} + \|f\|_{L^2(\Gamma_1)} + \|s_a\|_{0, \Omega} + \|s_a\|_{L^2(\Gamma_1)} \right).
\]
Hence, (4.27) and a trace theorem yield
\[
\|s_a - s_0\|_{H^1(\Omega)} + \|s_a - s_0\|_{L^2(\Gamma_1)} \leq a \beta_0 \|F\|_{\mathcal{H}_0}. \quad (4.31)
\]
Now using (4.23), we have
\[
\|v_a - v_0\|_{0, \Omega} \leq \lambda \|s_a - s_0\|_{0, \Omega} + \frac{\lambda^2 a}{1 + \lambda a} \|s_a\|_{0, \Omega} + \frac{\lambda a}{1 + \lambda a} \|f\|_{0, \Omega}.
\]
By (4.31) and (4.27), we deduce that
\[
\|v_a - v_0\|_{0, \Omega} \leq a \beta_0 \|F\|_{\mathcal{H}_0}. \quad (4.32)
\]
Similarly, using (4.24), (4.31) and (4.27), we get
\[
\|u_a - u_0\|_{H^1(\Omega)} \leq a \beta_0 \|F\|_{\mathcal{H}_0}. \quad (4.33)
\]
Finally, thanks to (4.22), we have
\[
\|z_a - z_0\|_{L^2(\Gamma_1 \times (0, 1))} \leq \beta_0 \|u_a - u_0\|_{L^2(\Gamma_1)}
\]
and by a trace theorem and (4.33), we deduce that
\[
\|z_a - z_0\|_{L^2(\Gamma_1 \times (0, 1))} \leq a \beta_0 \|F\|_{\mathcal{H}_0}. \quad (4.34)
\]
The estimates (4.32), (4.33) and (4.34) show that
\[
\| (\lambda - A_a)^{-1} F - (\lambda - A_0)^{-1} F \|_{\mathcal{H}_0} \leq a \beta_0 \|F\|_{\mathcal{H}_0},
\]
which implies that
\[
(\lambda - A_a)^{-1} \to (\lambda - A_0)^{-1} \text{ in norm as } a \to 0.
\]
The reminder of the statements of the Theorem follow from Theorems IV.2.25 and IV.3.16 of Kato (1976). \(\square\)
4.3.3 *The case* \( \mu > 0 \). In the case \( \mu > 0 \), with the above choices, the eigenvalue problem (4.17) reduces to find \( \lambda \in \mathbb{C} \) and \( \varphi \in H^2(0, 1) \) solution of

\[
\begin{cases}
\lambda^2 \varphi - (1 + a \lambda) \varphi'' = 0 \quad \text{in} \ (0, 1), \\
\varphi(0) = 0, \\
(1 + a \lambda) \varphi'(1) + (\lambda e^{-\lambda \tau} + \lambda^2 \mu) \varphi(1) = 0.
\end{cases}
\] (4.35)

Hence, for \( \Re \lambda \geq 0 \) and \( \lambda \neq 0 \), \( \varphi \) takes the form

\[ \varphi(x) = \alpha \sinh \left( \frac{\lambda x}{\sqrt{1 + a \lambda}} \right) \]

for some constant \( \alpha \). The boundary condition at 1 is then equivalent to

\[ a \left( \lambda \sqrt{1 + a \lambda} \cosh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) + (\lambda e^{-\lambda \tau} + \lambda^2 \mu) \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) \right) = 0. \]

Hence, any non-zero eigenvalue \( \lambda \) of problem (4.35) such that \( \Re \lambda \geq 0 \) is a solution of the equation

\[ \cosh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) + \frac{e^{-\lambda \tau} + \lambda \mu}{\sqrt{1 + a \lambda}} \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) = 0. \] (4.36)

We rewrite (4.36) as

\[ F(\lambda) + G(\lambda) = 0, \] (4.37)

where

\[ F(\lambda) = \cosh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) + \frac{e^{-\lambda \tau}}{\sqrt{1 + a \lambda}} \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right), \]

\[ G(\lambda) = \frac{\lambda \mu}{\sqrt{1 + a \lambda}} \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right). \]

We know from the previous analysis that, for small \( a \) and for arbitrarily small (large) delays, the equation \( F(\lambda) = 0 \) admits a solution \( \bar{\lambda} \) with positive real part. Now, consider a ball \( B \subset \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \} \) centered at \( \bar{\lambda} \) and that does not contain other zeroes of \( F \). Then, we have

\[ |F(\lambda)| \geq \epsilon > 0, \quad \forall \lambda \in \partial B, \] (4.38)

and

\[ |G(\lambda)| \leq \mu \max_{\lambda \in \partial B} \left| \frac{\lambda}{\sqrt{1 + a \lambda}} \sinh \left( \frac{\lambda}{\sqrt{1 + a \lambda}} \right) \right|, \quad \forall \lambda \in \partial B. \] (4.39)

So, for \( \mu \) sufficiently small,

\[ |G(\lambda)| < |F(\lambda)|, \quad \forall \lambda \in \partial B, \]

and therefore, from Rouché’s theorem, (4.37) has a zero in the ball \( B \).

In conclusion, we have also found unstable solutions of problem (4.1)–(4.5) if \( a > 0 \) and \( \mu > 0 \) are small.
REFERENCES


DATKO, R. (1988) Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. SIAM J. Control Optim., 26, 697–713.


