A Priori and A Posteriori Error Estimations for the Dual Mixed Finite Element Method of the Navier-Stokes Problem

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This article is concerned with a dual mixed formulation of the Navier-Stokes system in a polygonal domain of the plane with Dirichlet boundary conditions and its numerical approximation. The gradient tensor, a quantity of practical interest, is introduced as a new unknown. The problem is then approximated by a mixed finite element method. Quasi-optimal a priori error estimates are obtained. These a priori error estimates, an abstract nonlinear theory (similar to (Verfürth, RAIRO Model Math Anal Numer 32 (1998), 817–842)) and a posteriori estimates for the Stokes system from (Farhloul et al., Numer Funct Anal Optim 27 (2006), 831–846) lead to an a posteriori error estimate for the Navier-Stokes system. © 2008 Wiley Periodicals, Inc.

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I. INTRODUCTION

Any solution of the Navier-Stokes equations in polygonal domains has in general corner singularities [1–3]. Hence standard numerical methods lose accuracy on quasi-uniform meshes, and locally refined meshes are necessary to restore the optimal order of convergence. Standard finite element methods for second order elliptic operators, the Stokes or the Navier-Stokes system with corner singularities (and mixed boundary conditions) have been analyzed in [2, 4–11], where it is shown that the use of appropriate a priori refined meshes near the singular points allows to restore optimal order of convergence. Mixed methods for the Stokes and Navier-Stokes with Dirichlet boundary conditions were initiated in [12, 13]. Similarly a mixed method for the Boussinesq equations (coupling between the Navier-Stokes equations and the heat equation) with Dirichlet

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boundary condition on the velocity field and corner singularities is analyzed in [3] (some geometrical restrictions were made due to the mixed boundary conditions on the temperature). More recently a posteriori error analyses were developed for standard variational formulation of the abovementioned problems [14–20] where some estimators are introduced and proved to be reliable and efficient. Similar results for the dual mixed formulation of second order operators are obtained in [21–25], while for the Stokes system we may cite [26].

Our goal is here to consider the stationary Navier-Stokes equations with Dirichlet boundary condition in a two-dimensional polygonal domain and to approximate them by a dual mixed finite element method. Our method introduces as a new unknown (of physical interest) the gradient tensor \( \nabla u \). The poor regularity of any solution forces us to use appropriate Banach spaces and then to introduce an appropriate mixed formulation of the problem. We next consider some discretization of our mixed formulation by using some mixed finite elements developed in [27]. Namely the approximation spaces are piecewise constant for the velocity, piecewise constant for the pressure, piecewise \( RT_0 \) for each line of the gradient tensor. We then prove that the discrete mixed formulation has at least one solution near any nonsingular solution of the Navier-Stokes equations. Furthermore using some interpolation error estimates we show quasi-optimal a priori error estimates. Finally our a posteriori error analysis relies on these a priori error estimates, a modified version of an abstract nonlinear theory developed by Verfürth [20] and on a posteriori estimate for the Stokes system that we obtained in [26].

Let us mention that in [28, 29] adaptive experiments have been performed for a symmetric dual mixed method for the Lamé system in an \( L \)-shaped domain for large Lamé coefficients \( \lambda \). There it was shown that the estimator (of residual type, as here) is efficient and reliable independently of the Lamé coefficient \( \lambda \). These experiments suggest that similar results should hold for the Stokes equations (because as the Lamé coefficient \( \lambda \) becomes large the Lamé system tends to the Stokes one). Of course, for the Navier-Stokes equations, we have the additional difficulty due to the nonlinear convection term, but we think that we could treat it by using a Newton-Galerkin scheme as in [30]. We plan to make such numerical experiments in the future.

The outline of the article is as follows: in Section II we state the incompressible Navier-Stokes equations with Dirichlet boundary conditions, introduce its classical variational formulation, and recall some regularity results in terms of weighted Sobolev spaces that will be useful for our further analysis. Section III is devoted to the dual mixed formulation of the problem, its equivalence with the variational formulation and a uniform invertibility property. In Section IV we describe the discretization of the problem and prove quasi-optimal a priori error estimates. Finally in Section V we present an abstract nonlinear result and give its consequence, namely our a posteriori error analysis.

II. THE CONTINUOUS PROBLEM AND REGULARITY RESULTS

Let \( \Omega \) be a plane domain with a polygonal boundary. More precisely, we assume that \( \Omega \) is a simply connected domain and that its boundary \( \Gamma \) is the union of a finite number of linear segments \( \Gamma_j, 1 \leq j \leq n_e \) (\( \Gamma_j \) is assumed to be open).

In \( \Omega \), we consider the stationary Navier-Stokes system with Dirichlet boundary conditions for the velocity \( u \):

\[
\begin{aligned}
-\nu \Delta u + (u \cdot \nabla)u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u \cdot u &= 0 \quad \text{on } \Gamma.
\end{aligned}
\]
In these equations $p$ denotes the (kinematic) pressure and
\[
(u \cdot \nabla) u \left( \sum_{j=1}^{2} u_{j} \frac{\partial u_{1}}{\partial x_{j}}, \sum_{j=1}^{2} u_{j} \frac{\partial u_{2}}{\partial x_{j}} \right), \quad \nabla \cdot u = \text{div } u = \frac{\partial u_{1}}{\partial x_{1}} + \frac{\partial u_{2}}{\partial x_{2}}.
\]

The dual mixed formulation of this problem and similar ones were recently studied in [27, 31], where some existence and approximation results were obtained.

As in [27], we consider the mixed formulation of problem (2.1), where the gradient of the velocity is introduced as a new unknown. This means that problem (2.1) is reformulated as
\[
\begin{cases}
\sigma = \nu \nabla u & \text{in } \Omega, \\
\nabla \cdot (\sigma - p\delta) - \frac{1}{\nu} \sigma \cdot u + f = 0 & \text{in } \Omega, \\
\nabla \cdot u = 0 & \text{in } \Omega, \\
\sigma = 0 & \text{on } \Gamma,
\end{cases}
\]

where $\delta$ is the identity tensor,
\[
\nabla u = \left( \frac{\partial u_{i}}{\partial x_{j}} \right)_{1 \leq i,j \leq 2}, \quad \sigma \cdot u = \left( \sum_{j=1}^{2} \sigma_{i j} u_{j}, \sum_{j=1}^{2} \sigma_{2 j} u_{j} \right),
\]
and for a tensor $\tau$,
\[
\nabla \cdot \tau = \text{div } \tau = \left( \frac{\partial \tau_{11}}{\partial x_{1}} + \frac{\partial \tau_{12}}{\partial x_{2}}, \frac{\partial \tau_{21}}{\partial x_{1}} + \frac{\partial \tau_{22}}{\partial x_{2}} \right).
\]

In general the original formulation (2.1) is preferred because it is simpler, it uses few variables and many efficient methods are developed for its approximation. However in some applications like turbulent or non-Newtonian flows, the auxiliary unknown $\sigma$ is more relevant and is the data that has to be transferred into other equations coupled with (2.1). In such cases, the use of the formulation (2.2) might be preferred, as long as it provides a better accuracy for $\sigma$.

The mixed conforming finite element method that we will consider in this article to compute approximations of the solutions of problem (2.2) is the same element than in [27], namely $u, p$ are respectively approximated by piecewise constant fields $u_{h}, p_{h}$ on the triangulation $T_{h}$, while each line of $\sigma$ is approximated by Raviart-Thomas elements of degree 0 (see Section IV for the details).

Let us now recall the classical variational formulation of the Navier-Stokes equations [32]. Find $u \in (H_{0}^{1}(\Omega))^{2}$ and $p \in L_{0}^{2}(\Omega)$ such that
\[
\begin{align*}
\nu \int_{\Omega} \nabla u : \nabla v dx + \int_{\Omega} (u \cdot \nabla) u \cdot v dx - \int_{\Omega} p \text{div } v dx &= (f, v), \forall v \in (H_{0}^{1}(\Omega))^{2}; \\
\int_{\Omega} q \text{div } u dx &= 0, \forall q \in L_{0}^{2}(\Omega),
\end{align*}
\]

where the datum $f$ belongs to $(L^{2}(\Omega))^{2}$. The notation $(\cdot, \cdot)$ means here and below the usual inner product in $L^{2}(\Omega)$ or in $(L^{2}(\Omega))^{2}$ according to the context and we recall that $L_{0}^{2}(\Omega) = \{ q \in L^{2}(\Omega) : \int_{\Omega} q(x) dx = 0 \}$.

For existence results of $(u, p)$, we refer to [32]. We now shortly describe the regularity of any solution $(u, p) \in (H_{0}^{1}(\Omega))^{2} \times L_{0}^{2}(\Omega)$ of the above problem (2.3). In our days, it is well known [1, 2, 33] that this regularity is related to the singularities of the solution of the Stokes problem with Dirichlet boundary conditions. Here we prefer regularity results in terms of weighted
Sobolev spaces that are more convenient for error estimates in the finite element methods. We first recall some notation about the regularity results for this problem obtained in [1, 2, 33–35]. Let \( S_j, j = 1, \ldots, n_e \) denote the set of vertices of \( \Omega \) and let \( \omega_j \) denote the interior opening of \( \Omega \) at \( S_j \). Then the singular exponents of the Stokes problem near \( S_j \) are the roots \( \lambda \in \mathbb{C} \setminus \{0\} \) of (see [1] for more details):

\[
\sin^2(\lambda \omega_j) - \lambda^2 \sin^2 \omega_j = 0. \tag{2.4}
\]

Let us set \( \xi_S(\omega_j) = \min\{\Re \lambda; \lambda \text{ is solution of (2.4) and } \Re \lambda > 0\} \). It is well known that [1]:

\[
\begin{cases}
\xi_S(\omega_j) > 1 & \text{if } \omega_j < \pi, \\
\xi_S(\omega_j) > \frac{1}{2} & \text{if } \pi < \omega_j < 2\pi.
\end{cases}
\]

We further introduce the following weighted Sobolev spaces (see for instance [35]): For any positive integer \( k \) and any nonnegative real number \( \alpha \), we define

\[
H^{k,\alpha} = \{ v \in H^{k-1}; r^\alpha D^\beta v \in L^2, \forall |\beta| = k \},
\]

which is a Hilbert space for the norm

\[
\| v \|_{k,\alpha,\Omega} = \left( \| v \|_{k-1,\Omega}^2 + \| v \|_{k,\alpha,\Omega}^2 \right)^{1/2},
\]

where the seminorm \( | \cdot |_{k,\alpha,\Omega} \) is defined by

\[
| v |_{k,\alpha,\Omega} = \left( \sum_{|\beta| = k} \| r^\alpha D^\beta v \|_{0,\Omega}^2 \right)^{1/2},
\]

\( r(x) \) being the distance from \( x \) to the set of vertices of \( \Omega \).

Now we are able to state the following regularity result:

**Theorem 2.1.** Let \((u, p) \in (H^1_0(\Omega))^2 \times L^2_0(\Omega)\) be a solution of (2.3). Then

\[
(u, p) \in (H^{2,\alpha}(\Omega))^2 \times H^{1,\alpha}(\Omega),
\]

for all nonnegative \( \alpha \) such that \( \alpha > 1 - \min_{j=1,\ldots,n_e} \xi_S(\omega_j) \). Moreover, there exists a positive constant \( C \) independent of \( \nu \) such that

\[
\nu \| u \|_{2,\alpha,\Omega} + p \| _{1,\alpha,\Omega} \leq CC_1(\nu, f), \tag{2.6}
\]

where

\[
C_1(\nu, f) := \| f \|_{0,\Omega} + \frac{1}{\nu^2} \| f \|_{0,\Omega}^2 \left( 1 + \frac{1}{\nu^2} \| f \|_{0,\Omega} \right).
\]

**Proof.** Theorem 3.1 of [27] shows that

\((u \cdot \nabla) u \in (L^2(\Omega))^2\).

Therefore, \((u, p)\) may now be seen as the solution of the Stokes problem with a datum \( f - (u \cdot \nabla) u \) in \((L^2(\Omega))^2\). By Theorem 2.1 of [9] we get the regularities (2.5).
To prove the estimate (2.6), we need to give the dependence of the $L^2$-norm of $(u \cdot \nabla)u$ in terms of $v$ and $f$. First we remark that by Theorem II.1.3 of [36]
\[ \|u\|_{1,\Omega} \leq C \|f\|_{0,\Omega}, \tag{2.7} \]
where here and below $C$ is a positive constant independent of $v$.

Now we use a quite standard linearization argument (see for instance [37, section 6.3] in our setting) except that we take care of the dependence on the parameter $v$. For that purpose, we set $\tilde{u} = v u$, and look the pair $(\tilde{u}, p)$ as the solution of the Stokes equations
\[
\begin{cases}
-\Delta \tilde{u} + \nabla p = F & \text{in } \Omega, \\
\nabla \cdot \tilde{u} = 0 & \text{in } \Omega, \\
\tilde{u} = 0 & \text{on } \partial \Omega, 
\end{cases}
\tag{2.8}
\]
where $F = f - (u \cdot \nabla)u$. By Theorem 1.4.4.2 of [35], $(u \cdot \nabla)u$ belongs to $(H^{-1/2-\epsilon}(\Omega))^2$, for any $\epsilon > 0$ with the estimate
\[ \|(u \cdot \nabla)u\|_{-1/2-\epsilon,\Omega} \leq C \|u\|_{1,\Omega}^2. \tag{2.9} \]

By (2.7), we then obtain
\[ \|F\|_{-1/2-\epsilon,\Omega} \leq C \left( \|f\|_{0,\Omega} + \frac{1}{v^2} \|f\|_{0,\Omega}^2 \right). \tag{2.10} \]

By Theorem 3.6 of [1] applied to (2.8) (see [37, Section 6.3]), we conclude that $(\tilde{u}, p)$ belongs to $(H^{3/2-\epsilon}(\Omega))^2 \times H^{1-\epsilon}(\Omega)$ with the estimate
\[ \|\tilde{u}\|_{3/2-\epsilon,\Omega} + \|p\|_{1/2-\epsilon,\Omega} \leq C \|F\|_{-1/2-\epsilon,\Omega}. \]

By (2.9), $u$ satisfies
\[ \|u\|_{3/2-\epsilon,\Omega} \leq C \frac{1}{v} \|f\|_{0,\Omega} \left( 1 + \frac{1}{v^2} \|f\|_{0,\Omega} \right). \tag{2.11} \]

Now by the Sobolev embedding Theorem we have $H^{3/2-\epsilon}(\Omega) \hookrightarrow C(\bar{\Omega})$, and therefore the last estimate yields
\[ \|u\|_{\infty,\Omega} \leq C \frac{1}{v} \|f\|_{0,\Omega} \left( 1 + \frac{1}{v^2} \|f\|_{0,\Omega} \right). \tag{2.12} \]

This estimate and (2.7) finally lead to
\[ \|(u \cdot \nabla)u\|_{0,\Omega} \leq \frac{C}{v^2} \|f\|_{0,\Omega} \left( 1 + \frac{1}{v^2} \|f\|_{0,\Omega} \right), \]
and therefore
\[ \|F\|_{0,\Omega} \leq CC_1(v, f). \tag{2.13} \]

By Theorem 2.1 of [9] applied to (2.8), $(\tilde{u}, p)$ belongs to $(H^{2\alpha}(\Omega))^2 \times H^{1\alpha}(\Omega)$, with $\alpha$ as in the statement of Theorem 2.1 with the estimate
\[ \|\tilde{u}\|_{2\alpha,\Omega} + \|p\|_{1,\alpha,\Omega} \leq C \|F\|_{0,\Omega}. \]

In term of $(u, p)$ we get $(u, p)$ belongs to $(H^{2\alpha}(\Omega))^2 \times H^{1\alpha}(\Omega)$ and satisfies (2.6).
From the above properties of $\xi_S(\omega_j)$, we remark that in the previous Theorem, we may always choose $\alpha < 1/2$.

III. A MIXED FORMULATION FOR THE NAVIER-STOKES EQUATIONS

In this section, we introduce a mixed variational formulation of problem (2.3). This mixed formulation is a simplified version of the mixed formulation of the Boussinesq equations introduced in [3].

Let us fix $\alpha \in ]0, 1/2]$ satisfying the assumptions of Theorem 2.1. We now choose $r_1 \in ]2, \frac{4}{\alpha}]$ and define $r_2$ and $t$ by the relations $\frac{1}{r_1} + \frac{2}{r_2} = 1$ and $t = \frac{2r_1}{1+r_1}$. Note that these relations imply that $2 < \frac{2}{r_1} < r_2 < 4$. We now define the spaces (compare with [3, 27] where different spaces are used)

$$\Sigma = \{(\tau, q) \in (L^{r_1}(\Omega))^2 \times L_0^{r_1}(\Omega); \nabla \cdot (\tau - q\delta) \in (L^t(\Omega))^2\},$$

$$M = (L^{r_2}(\Omega))^2,$$

equipped with the norms

$$\|(\tau, q)\|_\Sigma = \|\tau\|_{0, r_1, \Omega} + \|q\|_{0, r_1, \Omega} + \|\nabla \cdot (\tau - q\delta)\|_{0, t, \Omega}, \quad \|v\|_M = \|v\|_{0, r_2, \Omega}.$$

Then the mixed formulation of (2.3) reads as follows: Find $(\sigma, p) \in \Sigma$ and $u \in M$ solution of (3.1) to (3.2) hereafter:

$$\frac{1}{\nu} (\sigma, \tau) + (\nabla \cdot (\tau - q\delta), u) = 0 \quad \forall (\tau, q) \in \Sigma, \quad (3.1)$$

$$(\nabla \cdot (\sigma - p\delta), v) - \frac{1}{\nu} (\sigma \cdot u, v) + (f, v) = 0 \quad \forall v \in M. \quad (3.2)$$

Let us observe that all the terms in the above equations have a meaning due to the appropriate choice of the parameters $r_1, r_2$, and $t$.

We have the following equivalence between (2.3) and (3.1)–(3.2). We do not give the details of the proof because it is a simplified version of Theorem 3.1 of [3].

**Theorem 3.1.** The pair $(u, p) \in (H^1(\Omega))^2 \times L^2(\Omega)$ is solution of (2.3) with $f \in (L^2(\Omega))^2$ if and only if $(\sigma, p) \in \Sigma$ and $u \in M$ are solutions of (3.1)–(3.2), with the next relation:

$$\sigma = \nu \nabla u. \quad (3.3)$$

Moreover there exists a positive constant $C$ independent of $\nu$ such that

$$\|\sigma\|_{0, r_1, \Omega} + \nu \|u\|_{0, r_2, \Omega} \leq C C_1(\nu, f), \quad (3.4)$$

where we recall that $C_1(\nu, f)$ was introduced in Theorem 2.1.
**Proof.** As mentioned earlier the proof of the equivalence is a simplified version of Theorem 3.1 of [3]. Let us concentrate on the estimate (3.4). The bound of the first term of the left-hand side of (3.4) follows from the estimate (2.6) and the embedding

\[ H^{1,\alpha}(\Omega) \hookrightarrow W^{1,p}(\Omega) \]

for any \( p > 1 \) such that \( \alpha < \frac{2}{p} - 1 \) and the Rellich-Kondrachov Theorem. The bound of the second term of the left-hand side of (3.4) follows from (2.6) and the embedding

\[ H^{2,\alpha}(\Omega) \hookrightarrow W^{2,p}(\Omega) \]

under the same condition on \( p \) and the embedding \( W^{2,p}(\Omega) \hookrightarrow L^r(\Omega) \), for all \( r > 1 \).

**Remark.** Another mixed formulation introducing as new unknowns \( \sigma = 2\nu\varepsilon(u) = \nu(\nabla u + (\nabla u)^\top) \) and \( \omega = \frac{1}{2}(\nabla u - (\nabla u)^\top) \) has been studied in [38]. This formulation allows to consider mixed boundary conditions but the prize to pay is to use the extra unknown \( \omega \). For Dirichlet boundary conditions, the use of \( \varepsilon(u) \) and \( \omega \) is not necessary.

As usual, we analyze an approximation of nonsingular solutions of the mixed formulation (3.1)–(3.2) (cf. [3, 27, 30–32, 39, 40]). For this purpose, we define the linear (Stokes) operator \( S \) as follows: The operator \( S \) associates with any function \( \tilde{f} \in (L^t(\Omega))^2 \) the solution \( (\tilde{\sigma}, \tilde{p}); \tilde{u}) \in \Sigma \times M \) of the problem

\[
\begin{align*}
\frac{1}{\nu}(\tilde{\sigma}, \tau) + (\nabla \cdot (\tau - q\delta), \tilde{u}) &= 0 \quad \forall (\tau, q) \in \Sigma, \\
(\nabla \cdot (\tilde{\sigma} - \tilde{p}\delta), v) + (\tilde{f}, v) &= 0 \quad \forall v \in M.
\end{align*}
\]  

(3.5)

Problem (3.5) is nothing else than a mixed formulation of the Stokes problem (cf. [41]). Observe that, because \( t < \frac{2}{\sum_{j=\min(3,\xi)}(\xi_j)} \), the solution of the Stokes problem, with datum \( \tilde{f} \in (L^t(\Omega))^2 \), satisfies (cf. [2, 35, 42]): \( \tilde{u} \in (W^{2,t}(\Omega))^2, \tilde{p} \in W^{1,t}(\Omega) \cap L^2_0(\Omega) \) and \( \|\tilde{u}\|_{2,t,\Omega} + \|\tilde{p}\|_{1,t,\Omega} \leq C\|\tilde{f}\|_{0,t,\Omega} \). Thus, using the techniques from [32, 41, 43] and the regularity results of Section II, problem (3.5) has a unique solution in \( \Sigma \times M \) for every \( \tilde{f} \in (L^t(\Omega))^2 \).

Next we define the mapping \( H \) from \( X := \Sigma \times M \) into itself by

\[ H(\tau) = \tau - S\left( f - \frac{1}{\nu}(\tau \cdot v) \right) \]  

(3.6)

where for shortness we write \( \tau = ((\tau, q); v) \).

With these notations the mixed formulation (3.1)–(3.2) of the Navier-Stokes equations takes the form

\[ \text{Find } \sigma \in X \text{ such that } H(\sigma) = 0 \]  

(3.7)

where \( \sigma = ((\sigma, p); u) \).

Let us recall [3, 30, 32] that a solution \( \sigma \in X \) of (3.7) is said to be nonsingular if the Fréchet derivative of \( H \) at the point \( \sigma = ((\sigma, p); u) \):

\[ DH(\sigma) : X \rightarrow X : \tau = ((\tau, q); v) \mapsto \tau + \frac{1}{\nu} S(\sigma \cdot v + \tau \cdot u), \]  

(3.8)

is an isomorphism.

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Equivalently \([3, 30]\) \(\sigma \) is a nonsingular solution of (3.7) if and only if the linearized homogeneous Navier-Stokes problem at the point \(\sigma = ((\sigma, p); u)\)

\[
\begin{cases}
\frac{1}{\nu} (\sigma^\ast, \tau) + (\nabla \cdot (\tau - q\delta), u^\ast) = 0 & \forall (\tau, q) \in \Sigma, \\
(\nabla \cdot (\sigma^\ast - p^\ast \delta), v) - \left( \frac{1}{\nu} (\sigma \cdot u^\ast + \sigma^\ast \cdot u), v \right) = 0 & \forall v \in M,
\end{cases}
\]

has a unique solution \(((\sigma^\ast, p^\ast); u^\ast) = 0\).

We close this section by defining the following operator that we need in the next section:

\[
K((\tau, q); v) := \frac{1}{\nu} S(\sigma \cdot v + \tau \cdot u).
\]

**Lemma 3.2.** Let \(\hat{X} = (L^r(\Omega))^2 \times L^2_0(\Omega) \times (L^2(\Omega))^2\) where \(2 \leq r < r_1 \leq 3\). Assume that \(\sigma \in X\) is a nonsingular solution of (3.7). Then the operator \((I + K)\) is invertible and has a continuous inverse in \(\mathcal{L}(\hat{X}, \hat{X})\). Moreover, its inverse is uniformly bounded wrt \(r\), i.e., there exists \(r_0 < r_1\) and a positive constant \(C\) independent of \(r\) such that

\[
\| (I + K)^{-1} \|_{\mathcal{L}(\hat{X}, \hat{X})} \leq C, \forall r \in [2, r_0).
\]

**Proof.** The invertibility property is proved as in Lemma 3.2 in [3], the case \(r = 2\) is excluded there, but is treated in Lemma 5.3 later. The uniform bound follows by interpolation between 2 and \(r_0 < r_1\). \(\blacksquare\)

**IV. THE DISCRETE PROBLEM AND A PRIORI ERROR ESTIMATES**

Let us now introduce the discrete version of (3.7) by using the mixed finite element methods.

Let \((T_h)_{h>0}\) be a family of triangulations of \(\Omega\) regular in Ciarlet’s sense [44]: there exists a positive constant \(\sigma\) such that

\[
\frac{h_K}{\rho_K} \leq \sigma \quad \forall K \in T_h, \quad \forall h > 0,
\]

where \(h_K\) (resp. \(\rho_K\)) denotes the exterior (resp. interior) diameter of \(K\). For \(K \subset IR^2\), let \(P_k(K)\), \(k \geq 0\), denotes the restrictions of polynomials of total degree \(\leq k\) to \(K\).

For any \(K \in T_h\) and \(x = (x_1, x_2)\), let

\[
RT_0(K) = (P_0(K))^2 \oplus xP_0(K) = \{(a, b) + c(x_1, x_2); a, b, c \in IR\}.
\]

Now let us set

\[
\Sigma_h = \{(\tau_h, q_h) \in \Sigma; \tau_{h|K} \in (RT_0(K))^2, q_{h|K} \in P_0(K), \forall K \in T_h\},
\]

\[
M_h = \{v_h \in M; v_{h|K} \in (P_0(K))^2, \forall K \in T_h\}.
\]

By \(\tau_{h|K} \in (RT_0(K))^2\), we mean that each of the two lines of \(\tau_{h|K}\) belongs to \(RT_0(K)\).
The discrete version of the continuous mixed formulation (3.1)–(3.2) consists in the following problem: Find \((\sigma_h, p_h) \in \Sigma_h\) and \(u_h \in M_h\) solution of:

\[
\frac{1}{\nu} (\sigma_h, \tau_h) + (\nabla \cdot (\tau_h - q_h \delta), u_h) = 0 \quad \forall (\tau_h, q_h) \in \Sigma_h, \tag{4.1}
\]

\[
(\nabla \cdot (\sigma_h - p_h \delta), v_h) - \frac{1}{\nu} (\sigma_h \cdot u_h, v_h) + (f, v_h) = 0 \quad \forall v_h \in M_h. \tag{4.2}
\]

Let us briefly compare this approximation method with the primitive variable formulation of the Navier-Stokes equations. Because in our method \(\sigma\) is approximated by piecewise Raviart-Thomas elements, hence piecewise \(P_1\), in the case of the primitive variable formulation the velocity has to be approximated by piecewise \(P_2\) elements because in this method the tensor gradient is obtained by numerical differentiation. Then consider either the \(P_2-P_0\) finite element (i.e., the velocity is approximated by \(P_2\) continuous elements and the pressure by \(P_0\)) or the Hood-Taylor elements (i.e., the velocity is approximated by \(P_2\) continuous and the pressure by \(P_1\) continuous) \([32, 43]\).

As we are in dimension 2, the \(P_2-P_0\) elements involve 13 degrees of freedom per element, while the Hood-Taylor elements use 15 degrees of freedom per element. On the contrary, after linearization of the problem by the Newton method (see for instance \([30]\), the continuity of the normal trace of \((\sigma - p \delta) n\) across interelements of the triangulation is imposed by introducing a Lagrange multiplier \(\lambda\), which is nothing else than the normal and tangential components of the velocity. The Lagrange multipliers will be approximated by constant functions on each edge. This technique allows to eliminate \(\sigma\) and \(u\) at the element level and then leads to a system which involves only the pressure and the Lagrange multipliers as degrees of freedom. In other words, our method involves only 7 degrees of freedom per element.

Now, as in the continuous problem, we introduce the discrete operator \(S_h\) of \(S\). For \(\tilde{f} \in (L^2(\Omega))^2\), \(S_h\tilde{f} = ((\tilde{\sigma}_h, \tilde{p}_h); \tilde{u}_h)) \in \Sigma_h \times M_h\) is the solution of the problem

\[
\begin{cases}
\frac{1}{\nu} (\tilde{\sigma}_h, \tau_h) + (\nabla \cdot (\tau_h - q_h \delta), \tilde{u}_h) = 0 \quad \forall (\tau_h, q_h) \in \Sigma_h \\
(\nabla \cdot (\tilde{\sigma}_h - \tilde{p}_h \delta), v_h) + (\tilde{f}, v_h) = 0 \quad \forall v_h \in M_h.
\end{cases} \tag{4.3}
\]

Next we define the mapping \(H_h\) from \(X_h := \Sigma_h \times M_h\) into itself by

\[
H_h(\tau_h) = \tau_h - S_h \left( f - \frac{1}{\nu} (\tau_h \cdot v_h) \right) \tag{4.4}
\]

where \(\tau_h = ((\tau_h, q_h); v_h)\).

Then, the discrete problem (4.1)–(4.2) is equivalent to

\[
\text{Find } \sigma_h \in X_h \text{ such that } H_h(\sigma_h) = 0. \tag{4.5}
\]

Our goal now is to prove a priori error estimates for nonsingular solutions of (3.7) under only the regularity assumption of the mesh stated earlier. The techniques of this section are inspired by those of \([3]\). But, due to the fact that the mesh is assumed only regular, they considerably differ in details.

We define

\[
Z = \{ \eta \in (L^r(\Omega))^2; \text{div} \eta \in L^r(\Omega) \},
\]

\[
Z_h = \{ \eta_h \in Z; \eta_h|_K \in RT_0(K), \forall K \in \mathcal{T}_h \}.
\]
We start with the following approximation results:

**Proposition 4.1.** There exist two interpolant operators \( \Pi_h \in \mathcal{L}(\Sigma \cap ((H^{1,\alpha}(\Omega))) \times H^{1,\alpha}(\Omega)), \Sigma_h) \) and \( \Pi^0_h \in \mathcal{L}(Z \cap (H^{1,\alpha}(\Omega))^2, Z_h) \) such that, for all \( s \in [2, 2/\alpha[ \),

\[
\| (\Pi_h - I)(\tau, q) \|_{0, r, s, \Omega} \leq Ch^{2/s - \alpha} \| (\tau, q) \|_{1, r, \Omega},
\]

where \( |(\tau, q)|_{1, r, \Omega} = |\tau|_{1, r, \Omega} + |q|_{1, r, \Omega} \) and \( C \) is, here and below, a positive constant independent of \( h \).

There exists a projection operator \( \mathcal{P}_h \in \mathcal{L}((H^{1,\alpha}(\Omega))^2, M_h) \) such that, for all \( s \in [2, 2/\alpha[ \),

\[
\| \mathcal{P}_h v - v \|_{0, r, s, \Omega} \leq Ch^{2/s - \alpha} \| v \|_{1, r, \Omega},
\]

**Proof.** Let us start with the proof of (4.7). First observe that the finite element space \( Z_h \) is simply the approximation of the space \( Z \) by the lowest-degree Raviart-Thomas element (cf. [45]). Then there exists an operator \( \Pi^0_h \) defined from \( Z \cap (H^{1,\alpha}(\Omega))^2 \) onto \( Z_h \) such that, for all \( K \in T_h \),

\[
\int_{\partial K} (v - \Pi_K v) \cdot n p_0 ds = 0 \quad \forall p_0 \in R_0(\partial K),
\]

where \( \Pi_K v = \Pi^0_h v|_K \), \( n \) denotes the unit outward normal to the boundary \( \partial K \) of \( K \), and \( R_0(\partial K) = \{ \psi; \psi \in P_0(e) \forall e \subset \partial K \} \). Let us point out that the operator \( \Pi^0_h \) is well defined for all \( v \in (H^{1,\alpha}(\Omega))^2 \) [3, 46].

Scaling arguments and the use of Piola’s transformation yield (see Proposition 1.12 of [46] or the proof of Proposition 4.2 of [3])

\[
\| v - \Pi^0_h v \|_{0, s, K} \leq Ch^{2/s - \alpha} \| v \|_{1, r, K}, \quad \forall K \in T_h,
\]

for all \( v \in (H^{1,\alpha}(\Omega))^2 \) and all \( s \in [2, 2/\alpha[ \). 

The estimate (4.7) directly follows from this estimate since \( 2/s - \alpha > 0 \).

Applying the estimate (3.1.34) of [44] to the mean operator on \( K \) for each component of the vector field \( v \), and using the estimate

\[
|\tilde{\psi}|_{1, r, K} \leq Ch^{2-\alpha} |\psi|_{1, r, K}, \quad \text{for any function } \psi \in H^{1,\alpha}(\Omega),
\]

we get the estimate (4.8). Finally, by using the definition of the operator \( \Pi_h \) in [47, p.86], the arguments to obtain (4.7) and the estimate (4.8), we obtain the estimate (4.6).

We have also the following estimates (see Remark 4.3 in [3]).

**Remark.** \( \forall \tilde{t} \in ]1, 2[ \) and \( \forall s \) such that \( 2 \leq s < 2\tilde{t}/(2 - \tilde{t}) \)

i) \( \Pi_h \in \mathcal{L}(\Sigma \cap ((W^{1,\tilde{t}}(\Omega))^2 \times W^{1,\tilde{t}}(\Omega)), \Sigma_h) \) and

\[
\| (\Pi_h - I)(\tau, q) \|_{0, r, s, \Omega} \leq Ch^{1+2/s - 2/\tilde{t}} \| (\tau, q) \|_{1, r, \Omega},
\]

where \( |(\tau, q)|_{1, r, \Omega} = |\tau|_{1, r, \Omega} + |q|_{1, r, \Omega} \) and \( C \) is a positive constant independent of \( h \).
ii) \( \mathcal{P}_h \in \mathcal{L}((W^{1/2}(\Omega))^2, M_h) \) and

\[
\| \mathcal{P}_h \mathbf{v} - \mathbf{v} \|_{0, r, \Omega} \leq C h^{1/2 - 1/2i} \| \mathbf{v} \|_{1, i, \Omega},
\]

(4.10)

Now, we introduce the discrete operator \( K_h \) of \( K \). Let \((\sigma, p); u) \in X \) and set \((\sigma_h^*, p_h^*); u_h^*) = (\Pi_h(\sigma, p); \mathcal{P}_h u) \). The operator \( K_h \in \mathcal{L}(X_h, X_h) \) is defined by

\[
K_h(\tau_h) = \frac{1}{\nu} S_h(\sigma_h^* \cdot \mathbf{v}_h + \tau_h \cdot u_h^*),
\]

(4.11)

when \( \tau_h = ((\tau_h, q_h); \mathbf{v}_h) \in X_h \).

We also introduce the following space

\[
X_+ := (L'(\Omega))^2 \times L^2_0(\Omega) \times (L^2(\Omega))^2,
\]

where \( r \) is fixed such that \( 2 < r < \min(r_1, r_2) \).

For our future purposes, we further fix \( s \) such that \( r < s < \min(r_1, \frac{4}{a^2 + \tau}) \) (such a \( s \) exists because by the assumption \( r_1 < \frac{1}{a} \), we deduce that \( r < \frac{1}{a} < \frac{2}{a} \) which implies that \( r < \frac{4}{a + \tau} \)) and assume that the triangulation \( T_h \) satisfies the inverse inequality

\[
h_K \geq \tilde{\sigma} h^\beta, \forall K \in T_h,
\]

(4.12)

for some positive constant \( \tilde{\sigma} \) independent of \( h \) and for \( \beta \) satisfying

\[
1 < \beta < \min \left\{ \frac{1}{2}, \frac{2}{1 - \frac{2}{r}}, \frac{2}{1 - \frac{2}{r}} - \frac{2}{r} \right\}.
\]

Observe that the assumption on \( s \) guarantees that

\[
\frac{1}{1 - \frac{2}{s}} > 1 \quad \text{and} \quad \frac{2}{r} - \frac{2}{s} > 1.
\]

Note that the constraint (4.12) is very weak since in standard a priori or a posteriori refinement procedures [9, 19] this condition is satisfied for some \( \beta > 1 \).

We now prove some technical lemmas.

**Lemma 4.2.** Let \( \tilde{\mathbf{f}} \in (L^2(\Omega))^2 \). Then problem (4.3) has a unique solution and

\[
\|(S_h - S)\tilde{\mathbf{f}}\|_{X_+} \leq C h^{2/3 - a} \| \tilde{\mathbf{f}} \|_{1,\alpha, \Omega}.
\]

(4.13)

**Proof.** For shortness we set \( \tilde{S}h = ((\tilde{\sigma}, \tilde{p}); \tilde{u}), S_h \tilde{\mathbf{f}} = ((\tilde{\sigma}_h, \tilde{p}_h); \tilde{u}_h), \) and \((\tilde{\sigma}_h^*, \tilde{p}_h^*); \tilde{u}_h^*) = (\Pi_h(\tilde{\sigma}, \tilde{p}); \mathcal{P}_h \tilde{u}) \). Owing to the uniform inf-sup condition (see Lemma 4.4 of [3] and the proof of Theorem 3.2 of [12]), the problem (4.3) has a unique solution and

\[
\| \tilde{\sigma}_h^* - \tilde{\sigma}_h \|_{0, \Omega} + \| \tilde{p}_h^* - \tilde{p}_h \|_{0, \Omega} + \| \tilde{u}_h^* - \tilde{u}_h \|_{0, \Omega} \leq C \| \tilde{\sigma} - \tilde{\sigma}_h \|_{0, \Omega}.
\]

Thus, owing to (4.6), we have

\[
\| \tilde{\sigma}_h^* - \tilde{\sigma}_h \|_{0, \Omega} + \| \tilde{p}_h^* - \tilde{p}_h \|_{0, \Omega} + \| \tilde{u}_h^* - \tilde{u}_h \|_{0, \Omega} \leq C h^{1/3 - a} (| \tilde{\sigma} |_{1,\alpha, \Omega} + | \tilde{p} |_{1,\alpha, \Omega}).
\]

(4.14)
On the other hand, a standard inverse estimate yields
\[ \|\tilde{\sigma}^*_h - \sigma_h\|_{0,r,K} \leq C|K|^\frac{1}{2} \|\tilde{\sigma}^*_h - \sigma_h\|_{0,K} \leq Ch^\frac{1}{2} \|\tilde{\sigma}^*_h - \sigma_h\|_{0,K}. \]
By the condition (4.12) we have
\[ \|\tilde{\sigma}^*_h - \sigma_h\|_{0,r,\Omega} \leq Ch^{\frac{1}{2} - 1} \|\tilde{\sigma}^*_h - \sigma_h\|_{0,\Omega}, \]
and by the estimate (4.14), we obtain
\[ \|\tilde{\sigma}^*_h - \sigma_h\|_{0,r,\Omega} + \|\tilde{p}^*_h - \tilde{p}_h\|_{0,\Omega} + \|\tilde{u}^*_h - \tilde{u}_h\|_{0,\Omega} \leq Ch^{\frac{1}{2} - 1} (|\tilde{\sigma}|_{1,\omega,\Omega} + |\tilde{p}|_{1,\omega,\Omega}). \]
Since the assumption on \( \beta \) yields
\[ \beta \left( \frac{2}{r} - 1 \right) + 1 - \alpha \geq \frac{2}{s} - \alpha, \]
we get
\[ \|\tilde{\sigma}^*_h - \sigma_h\|_{0,r,\Omega} + \|\tilde{p}^*_h - \tilde{p}_h\|_{0,\Omega} + \|\tilde{u}^*_h - \tilde{u}_h\|_{0,\Omega} \leq Ch^{\frac{1}{2} - s - \alpha} (|\tilde{\sigma}|_{1,\omega,\Omega} + |\tilde{p}|_{1,\omega,\Omega}). \]
Hölder’s inequality yields
\[ \|\tilde{\sigma}^*_h - \tilde{\sigma}\|_{0,r,\Omega} \leq |\Omega|^{-\frac{1}{2} - \frac{1}{2}} \|\tilde{\sigma}^*_h - \tilde{\sigma}\|_{0,r,\Omega} \leq C \|\tilde{\sigma}^*_h - \tilde{\sigma}\|_{0,r,\Omega}. \]
These two inequalities with (4.6) and (4.8) give us
\[ \|\tilde{\sigma} - \sigma_h\|_{0,r,\Omega} + \|\tilde{p} - \tilde{p}_h\|_{0,\Omega} + \|\tilde{u} - \tilde{u}_h\|_{0,\Omega} \leq Ch^{\frac{1}{2} - s - \alpha} (|\tilde{\sigma}|_{1,\omega,\Omega} + |\tilde{p}|_{1,\omega,\Omega} + |\tilde{u}|_{1,\omega,\Omega}), \]
which is nothing else than (4.13).

Remark. From (4.13) one can deduce the following estimate
\[ \|(S_h - S)\tilde{f}\|_{X^+} \leq Ch^{2/s} \|\tilde{f}\|_{0,\Omega}, \quad \forall \tilde{f} \in (L^2(\Omega))^2. \] (4.15)

Lemma 4.3. Let \( \tilde{f} \in (L^2(\Omega))^2 \) where \( \frac{2s}{2+s} \leq \tilde{r} < \frac{2}{2-\min_j \xi_j(\omega_j)} \). Then we have the following estimate
\[ \|(S_h - S)\tilde{f}\|_{X^+} \leq Ch^{2/s+1-2/\tilde{r}} \|\tilde{f}\|_{0,\tilde{r},\Omega}. \] (4.16)

Proof. Let us recall that (cf. [2, 35, 42]) for \( \tilde{f} \in (L^2(\Omega))^2 \), with \( \tilde{r} < \frac{2}{2-\min_j \xi_j(\omega_j)} \), the solution of the Stokes problem (3.5) satisfies: \( \tilde{u} \in (W^{2,\tilde{r}}(\Omega))^2 \), \( \tilde{p} \in W^{1,\tilde{r}}(\Omega) \cap L^{2,\tilde{r}}(\Omega) \), and
\[ \|\tilde{u}\|_{2,\tilde{r},\Omega} + \|\tilde{p}\|_{1,\tilde{r},\Omega} \leq C \|\tilde{f}\|_{0,\tilde{r},\Omega}. \]
Now, as in the proof of Lemma 4.2, using the estimate (4.9), we get
\[ \|\tilde{\sigma}^*_h - \sigma_h\|_{0,r,\Omega} + \|\tilde{p}^*_h - \tilde{p}_h\|_{0,\Omega} + \|\tilde{u}^*_h - \tilde{u}_h\|_{0,\Omega} \leq Ch^{2/s+1-2/\tilde{r}} (|\tilde{\sigma}|_{1,\tilde{r},\Omega} + |\tilde{p}|_{1,\tilde{r},\Omega}), \]
Lemma 4.4. Assume that $\sigma$ is a nonsingular solution of (3.7). Then we have

$$\lim_{h \to 0} \|K - K_h\|_{L(X_+,X_+)} = 0. \quad (4.17)$$

Proof. Let $\tau \in X_+$ and set $((\sigma_h^*, p_h^*); u_h^*) = (\Pi_h(\sigma, p); P_h u)$. Then we may write

$$(K - K_h)(\tau) = \frac{1}{2} \langle S(\sigma \cdot v + \tau \cdot u) - S_h(\sigma_h^* \cdot v + \tau \cdot u_h^*) \rangle. \quad (4.18)$$

To estimate $(K - K_h)(\tau)$, we transform the term $S(\sigma \cdot v + \tau \cdot u) - S_h(\sigma_h^* \cdot v + \tau \cdot u_h^*)$ as follows

$$S(\sigma \cdot v + \tau \cdot u) - S_h(\sigma_h^* \cdot v + \tau \cdot u_h^*) = S((\sigma - \sigma_h^*) \cdot v) + S(\tau \cdot (u - u_h^*))$$

$$+ (S - S_h)(\sigma_h^* \cdot v) + (S - S_h)(\tau \cdot u_h^*). \quad (4.19)$$

For the first term, using the fact that $\|S\tilde{f}\|_{X_+} \leq C\|\tilde{f}\|_{0,t,\Omega}$ for all $\tilde{f} \in (L^j(\Omega))^2$ with $\tilde{j} < \frac{2}{1 + \alpha}$ (observe that $\frac{2}{1 + \alpha} < \frac{2}{2 - \min j \xi \omega_j}$), we have

$$\|S((\sigma - \sigma_h^*) \cdot v)\|_{X_+} \leq C\|\sigma - \sigma_h^*\|_{0,t,\Omega} \leq C\|\sigma - \sigma_h^*\|_{0,\tau,\Omega} \|v\|_{0,\Omega},$$

where $\frac{1}{t} = \frac{1}{\tau} - \frac{1}{2}$. Thus, using the fact that $2 < s < 2/\alpha$ and (4.6), we get

$$\|S((\sigma - \sigma_h^*) \cdot v)\|_{X_+} \leq C h^{2/s - a} \|\sigma, p\|_{1,\tau,\Omega} \|v\|_{0,\Omega}.$$  

Using similar arguments as earlier and (4.8), we also have

$$\|S(\tau \cdot (u - u_h^*))\|_{X_+} \leq C h^{2/s - a} \|u\|_{1,\tau,\Omega} \|\tau\|_{0,\tau,\Omega}.$$  

Now, using (4.16) and Hölder’s inequality, we have

$$\|(S - S_h)(\sigma_h^* \cdot v)\|_{X_+} \leq C h^{2/s + 1 - 2/\tilde{j}} \|\sigma_h^* \cdot v\|_{0,\tau,\Omega} \leq C h^{2/s + 1 - 2/\tilde{j}} \|\sigma_h^*\|_{0,r,\Omega} \|v\|_{0,\Omega},$$

where $\frac{1}{\tilde{j}} = \frac{1}{r_1} + \frac{1}{2}$ which satisfies $\frac{2s}{2 + s} \leq \tilde{j} < \frac{2}{2 - \min j \xi \omega_j}$.

Using Proposition 4.1, we see that

$$\|\sigma_h^*\|_{0,r,\Omega} \leq \|\sigma_h^* - \sigma\|_{0,r,\Omega} + \|\sigma\|_{0,r,\Omega} \leq C(1 + h^{2/t - a}) \|(\sigma, p)\|_{1,\tau,\Omega}.$$
This estimate in the previous one leads to
\[
\left\| (S - S_h) (\sigma_h^* \cdot v) \right\|_{X_+} \leq C h^{2/\alpha + 1 - 2/\tilde{t}} \| (\sigma, p) \|_{1, \alpha, \Omega} \| v \|_{0, \Omega}.
\]
Similarly we may estimate successively
\[
\left\| (S - S_h) (\tau \cdot u_h^*) \right\|_{X_+} \leq C h^{2/\alpha + 1 - 2/\tilde{t}} \| u_h^* \|_{0, r, \Omega} \| \tau \|_{0, r, \Omega}.
\]
Therefore, using these estimates and (4.19), we deduce that
\[
\left\| S(\sigma \cdot v + \tau \cdot u) - S_h (\sigma_h^* \cdot v + \tau \cdot u_h^*) \right\|_{X_+} \leq C h^\gamma \left( \| \tau \|_{0, r, \Omega} + \| v \|_{0, \Omega} \right),
\]
for some $\gamma > 0$. This ends the proof.

Lemmas 3.2, 4.4 and a classical perturbation argument (cf. [32]) lead to the following result:

**Lemma 4.5.** Under the assumptions of Lemma 4.2, for $h$ small enough, the operator $(I + K_h)$ is an isomorphism from $X_+$ into $X_+$. Moreover $(I + K_h)^{-1}$ maps $X_h$ into $X_h$ with a norm bounded independently of $h$.

**Proof.** By Lemma 3.2 we may write
\[
I + K_h = (I + \mathcal{K})(I + (I + \mathcal{K})^{-1}(\mathcal{K}_h - \mathcal{K})),
\]
Now we may write
\[
\|(I + \mathcal{K})^{-1}(\mathcal{K}_h - \mathcal{K})\|_{\mathcal{L}(X_+, X_+)} \leq \|(I + \mathcal{K})^{-1}\|_{\mathcal{L}(X_+, X_+)} \| \mathcal{K}_h - \mathcal{K} \|_{\mathcal{L}(X_+, X_+)},
\]
and again applying Lemma 3.2, we get
\[
\|(I + \mathcal{K})^{-1}(\mathcal{K}_h - \mathcal{K})\|_{\mathcal{L}(X_+, X_+)} \leq C \| \mathcal{K}_h - \mathcal{K} \|_{\mathcal{L}(X_+, X_+)},
\]
By Lemma 4.4, for $h$ small enough, we obtain
\[
\|(I + \mathcal{K})^{-1}(\mathcal{K}_h - \mathcal{K})\|_{\mathcal{L}(X_+, X_+)} \leq 1/2,
\]
and using the Neumann series $(I + (I + \mathcal{K})^{-1}(\mathcal{K}_h - \mathcal{K}))$ is invertible with a norm lower than 2. This yields the conclusion.

**Lemma 4.6.** If $\sigma$ is a nonsingular solution of (3.7), then there exists a constant $C > 0$ such that
\[
\| H_h(\sigma_h^*) \|_{X_+} \leq C h^{\frac{2}{\alpha} - \cdot}
\]
where $\sigma_h^* = ((\sigma_h^*, p_h^*); u_h^*) = (\Pi_h(\sigma, p); \mathcal{P}_h u)$.  

Theorem 4.7. If \( \| \circ \) owing to (4.16), we have

From (4.6), (4.8), and (4.15), we have

Therefore using (4.6) and (4.8), we get

By Lemma 4.5, we may write

These last estimates, with (4.23), in (4.22) lead to the conclusion.

We are now able to prove the error estimate for nonsingular solutions of (3.7).

Theorem 4.7. If \( \circ \) is a nonsingular solution of (3.7), then for \( h \) small enough, problem (4.5) has at least one solution \( \circ \) such that

Proof. We define the following map \( S \) from \( X_h \) into itself

and prove that it has a fixed point in a neighborhood of \( \circ \), where \( \circ \) = \( (\circ, p) ; \circ \) = \( (\Pi_h (\circ, p) ; \circ) \). For that purpose, we start by estimating \( \| S(\circ) - \circ \|_{X_h} \) in term of \( \| \circ \|_{X_h} - \circ \). By Lemma 4.5, we may write

so that (still by Lemma 4.5)
\[ \| S(\tau_h^*) - \sigma_h^* \|_{X_+} \leq C \| (I + \mathbb{K}_h)(S(\tau_h^*) - \sigma_h^*) \|_{X_+}. \] (4.26)

On the other hand, by the definition of \( \mathbb{K}_h \) we have
\[ (I + \mathbb{K}_h)(S(\tau_h^*) - \sigma_h^*) = -\frac{1}{\nu} S_h((\sigma_h^* - \tau_h) \cdot (u_h^* - v_h)) - H_h(\sigma_h^*). \] (4.27)

Now, by Lemma 4.3 with \( \bar{r} \in (1, 2) \) such that \( \frac{1}{\bar{r}} = \frac{1}{r} + \frac{1}{2} \) (observe that \( \bar{r} = \frac{2r}{r+2} \) and \( \frac{1}{s} + \frac{1}{2} > \frac{a+1}{2} > \frac{2-\min \xi \bar{g}(\gamma')} {2} \)) we have \( \| S_h\bar{f} \|_{X_+} \leq C \| \bar{f} \|_{0,\bar{r},\Omega} \) for all \( \bar{f} \in (L^\bar{r}(\Omega))^2 \). Using this fact and the inverse inequality \( \| v_h \|_{0,\bar{r},\Omega} \leq C h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} \| v_h \|_{0,\Omega} \), we have
\[ \| S_h((\sigma_h^* - \tau_h) \cdot (u_h^* - v_h)) \|_{X_+} \leq C \| (\sigma_h^* - \tau_h) \cdot (u_h^* - v_h) \|_{0,\bar{r},\Omega} \]
\[ \leq C \| \sigma_h^* - \tau_h \|_{0,\bar{r},\Omega} \| u_h^* - v_h \|_{0,\Omega} \]
\[ \leq C h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} \| \sigma_h^* - \tau_h \|_{0,\bar{r},\Omega} \| u_h^* - v_h \|_{0,\Omega} \]
\[ \leq C h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} \| \tau_h - \sigma_h^* \|_{X_+}^2. \]

Therefore, by (4.26), (4.27), (4.21), we have
\[ \| S(\tau_h^*) - \sigma_h^* \|_{X_+} \leq C_1 h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} \| \tau_h - \sigma_h^* \|_{X_+}^2 + C_2 h^{\frac{2}{r} - \alpha}. \] (4.28)

Now for \( h \) small enough we consider the smallest root \( \rho_- \) of the second order equation
\[ C_1 h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} \rho^2 - \rho + C_2 h^{\frac{2}{r} - \alpha} = 0. \]

We see that
\[ \rho_- = \frac{1 - \sqrt{1 - 4C_1 C_2 h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} + \frac{2}{r} - \alpha}}{2 C_1 h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})}} = \frac{2 C_2 h^{\frac{2}{r} - \alpha}}{1 + \sqrt{1 - 4C_1 C_2 h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} + \frac{2}{r} - \alpha}} \leq C_3 h^{\frac{2}{r} - \alpha}, \]
for some \( C_3 > 0 \) (independent of \( h \)) since the assumption on \( \beta \) guarantees that \( \beta(\frac{2}{\bar{r}} - \frac{2}{r}) + \frac{2}{r} - \alpha > 0 \). Thus, if
\[ \| \tau_h - \sigma_h^* \|_{X_+} \leq \rho_- \]
then the estimate (4.28) implies that
\[ \| S(\tau_h^*) - \sigma_h^* \|_{X_+} \leq C_1 h^{\beta(\frac{2}{\bar{r}} - \frac{2}{r})} \rho_-^2 + C_2 h^{\frac{2}{r} - \alpha} = \rho_- \]

Consequently \( S \) maps the ball
\[ B_h = \{ \tau_h \in X_h; \| \tau_h - \sigma_h^* \|_{X_+} \leq \rho_- \} \]

into itself. Therefore \( S \) has at least a fixed point \( \sigma_h \) in the ball \( B_h \) and such a fixed point is a solution of

\[
H_h(\sigma_h) = 0.
\]

Since \( \sigma_h \in B_h \), \( \rho_- \leq Ch^{\frac{2}{s} - \alpha} \) and using (4.23), we have

\[
\| \sigma - \sigma_h \|_{X_+} \leq Ch^{\frac{2}{s} - \alpha},
\]

which is the desired result. \( \blacksquare \)

**Remark.** Smooth solutions corresponds to the case \( \alpha = 0 \) and in that case the estimate (4.25) yield the quasi-optimal order of convergence \( h^{\frac{2}{s}} \) with \( s > 2 \) but tending to 1 as \( s \) goes to 2.

**V. A POSTERIORI ERROR ANALYSIS**

Our a posteriori error analysis is based on the next abstract nonlinear theory, a modified version of the theory developed by Verfürth [20] and then on the a posteriori error analysis performed for the Stokes system in [26].

**A. An Abstract Nonlinear Setting**

Consider five Banach spaces \( X \hookrightarrow X_+ \hookrightarrow X_- \) and \( Y_+ \hookrightarrow Y \) with continuous and dense embeddings. We further assume that \( X_- \) and \( Y \) are reflexive. Let \( H \) be a continuously differentiable mapping from \( X \) to \( Y^* \) (where \( Y^* \) denotes the dual space of \( Y \)).

**Theorem 5.1.** Let \( u_0 \in X \) be a solution of

\[
H(u_0) = 0. \tag{5.1}
\]

Assume that \( DH(u_0)^* \) is an isomorphism from \( Y_+ \) into \( X_+^* \) and that there exist two positive real numbers \( R_0 \) and \( \beta \) such that

\[
\| [DH(u_0) - DH(u_0 + tw)]w \|_{Y_+^*} \leq \beta t \| w \|_{X_+} \| w \|_{X_-}, \tag{5.2}
\]

for all \( w \in X \) such that \( \| w \|_{X_+} \leq R_0 \) and all \( t \in [0, 1] \). Set

\[
R := \min \left\{ R_0, \beta^{-1} \| DH(u_0) \|_{L(Y_+, X_+)}^{-1}, 2\beta^{-1} \| DH(u_0) \|_{L(Y_+, X_+)} \right\}.
\]

Then the following error estimate holds

\[
C_1 \| H(u) \|_{Y_+^*} \leq \| u_0 - u \|_{X_-} \leq C_2 \| H(u) \|_{Y_+^*}, \tag{5.3}
\]

for all \( u \in X \) such that \( \| u_0 - u \|_{X_+} \leq R \), where the two positive constants \( C_1 \) and \( C_2 \) depend on \( DH(u_0) \) and are given by

\[
C_1 = \frac{1}{2} \| DH(u_0) \|_{L(X_-, Y_+)}^{-1}, \quad C_2 = 2 \| DH(u_0) \|_{L(Y_+, X_-)}^{-1}.
\]

**Proof.** The proof is similar to the one of Proposition 2.1 of [20], we therefore omit it. \( \blacksquare \)
B. A Posteriori Error Estimates for our Navier-Stokes System

Our Navier-Stokes system (3.7) enters in the abstract setting of the previous subsection by taking:

\[ X := \Sigma \times M, \]
\[ X_+ := (L^2(\Omega))^2 \times L^2_0(\Omega) \times (L^2(\Omega))^2, \]
\[ X_- := \mathbb{H}^2, \]
\[ Y := \tilde{X}^*, \]
\[ \tilde{X} := (L^{r_1}(\Omega))^2 \times L^2_0(\Omega) \times (L^2(\Omega))^2, \]
\[ Y_+ := \mathbb{H}^2, \]

where for shortness we write

\[ \mathbb{H}^2 := (L^2(\Omega))^2 \times L^2_0(\Omega) \times (L^2(\Omega))^2, \]

all being equipped with their natural norms except \( X_- = Y_+ = \mathbb{H}^2 \) which is equipped with the norm depending on \( \nu \):

\[ \|((\sigma, p); u)\|_{\mathbb{H}^2} = \|\sigma\|_{0, \Omega} + \|p\|_{0, \Omega} + \nu \|u\|_{0, \Omega}, \]

and \( r > 2 \) is fixed as in the previous section.

Clearly we have \( X \hookrightarrow X_+ \hookrightarrow X_-, Y_+ \hookrightarrow Y \) and \( H \in C^1(X, X^*) \). Now we want to check the assumptions on \( DH \):

**Lemma 5.2.** Assume that the parameter \( r_1 \) further satisfies \( r_1 \leq 3 \). Let \( f \in (L^2(\Omega))^2 \) and let \( ((\sigma, p); u) \in X \) be a solution of (3.7). Then \( DH((\sigma, p); u) \) maps \( \mathbb{H}^2 \) into itself.

**Proof.** We recall that

\[ DH((\sigma, p); u)((\tau, q); v) = ((\tau, q); v) + S \left( \frac{1}{\nu} (\sigma \cdot v + \tau \cdot u) \right). \]

Therefore \( DH \) maps \( \mathbb{H}^2 \) into itself if and only if \( S \) maps \( \mathbb{H}^2 \) into itself. In other words, for any \( ((\tau, q); v) \in \mathbb{H}^2 \), we must show that

\[ ((\tilde{\sigma}, \tilde{p}); \tilde{u}) := S \left( \frac{1}{\nu} (\sigma \cdot v + \tau \cdot u) \right) \]

also belongs to \( \mathbb{H}^2 \). Setting \( \tilde{f} := \frac{1}{\nu} (\sigma \cdot v + \tau \cdot u) \), we know that \((\tilde{u}, \tilde{p})\) is the (weak) solution of the Stokes system with datum \( \tilde{f} \) and that \( \tilde{\sigma} = \nu \nabla \tilde{u} \). By Hölder’s inequality, the term \( \sigma \cdot v \) belongs to \((L'^s(\Omega))^2\), where \( s > 1 \) is defined by the relation \( \frac{1}{s} = \frac{1}{2} + \frac{1}{r_1} \) while the term \( \tau \cdot u \) belongs to \((L'^q(\Omega))^2\), where \( q > 1 \) is defined by the relation \( \frac{1}{q} = \frac{1}{2} + \frac{1}{r_2} \). Since the assumption \( r_1 \leq 3 \) guarantees that \( q \geq s \), we deduce that

\[ \tilde{f} \in (L'^s(\Omega))^2. \]
By regularity results for the Stokes system (cf. Theorem 4.1 of [2]), \((\tilde{u}, \tilde{p})\) belongs to \((W^{2,s}(\Omega))^2 \times W^{1,s}(\Omega)\), since \(s\) satisfies
\[
2 - \frac{2}{s} < \min_j \xi_s(\omega_j),
\]
consequence of the inequality \(2 - \frac{2}{s} < 1 - \alpha\). Since by the Sobolev embedding theorem \(W^{1,s}(\Omega) \hookrightarrow L^2(\Omega)\), we can conclude that \(((\tilde{\sigma} = \nu \nabla \tilde{u}, \tilde{p}); \tilde{u})\) belongs to \(IL^2\).

Lemma 5.3. Assume that the parameter \(r_1\) further satisfies \(r_1 \leq 3\). Let \(f \in (L^2(\Omega))^2\) and let \(((\sigma, p); u) \in X\) be a nonsingular solution of (3.7). Then \(DH((\sigma, p); u)\) is an isomorphism from \(IL^2\) into itself.

Proof. The result directly follows from the two following properties:
\[
\text{ind}_{IL^2} DH((\sigma, p); u) = 0, \quad (5.4)
\]
\[
\ker_{IL^2} DH((\sigma, p); u) \subset \ker_X DH((\sigma, p); u), \quad (5.5)
\]
where \(\text{ind}_{IL^2}\) means the index of the operator from \(IL^2\) into itself and \(\ker_{IL^2}\) (resp. \(\ker_X\)) means the kernel of the operator from \(IL^2\) into itself (resp. \(X\) into itself).

Let us start with the first property: Denote by \(K\) the linear operator from \(IL^2\) into itself defined by
\[
K((\tau, q); v) := S\left(\frac{1}{\nu}(\sigma \cdot v + \tau \cdot u)\right).
\]
The well-posedness of \(K\) was checked in the previous Lemma.

Let us now show that \(K\) is a compact operator. Indeed, let be given a sequence \(((\tau_n, q_n); v_n)\) such that
\[
\|\tau_n\|_{0,\Omega} + \|q_n\|_{0,\Omega} + \|v_n\|_{0,\Omega} \leq C, \forall n \in \mathbb{N}, \quad (5.6)
\]
for some \(C > 0\). Then denote by
\[
((\tilde{\sigma}_n, \tilde{p}_n); \tilde{u}_n) := K((\tau_n, q_n); v_n).
\]
Then by Lemma 5.2, there exists \(C_1 > 0\) such that
\[
\|\tilde{\sigma}_n\|_{1,\Omega} + \|\tilde{p}_n\|_{1,\Omega} + \|\tilde{u}_n\|_{1,\Omega} \leq C_1 \frac{1}{\nu} \|\sigma \cdot v_n + \tau_n \cdot u\|_{0,\Omega, \Omega}, \forall n \in \mathbb{N},
\]
where \(\frac{1}{3} = \frac{1}{2} + \frac{1}{r_1}\).

By Hölder’s inequality we obtain
\[
\|\tilde{\sigma}_n\|_{1,\Omega} + \|\tilde{p}_n\|_{1,\Omega} + \|\tilde{u}_n\|_{1,\Omega} \leq C_2 \frac{1}{\nu} (\|\sigma\|_{\Omega, \Omega} \|v_n\|_{0,\Omega} + \|u\|_{0,\Omega} \|\tau_n\|_{0,\Omega}), \forall n \in \mathbb{N},
\]
for some \(C_2 > 0\). Consequently the assumption (5.6) implies the existence of a positive constant \(C_3\) (depending on \(u\) and \(\sigma\)) such that
\[
\|\tilde{\sigma}_n\|_{1,\Omega} + \|\tilde{p}_n\|_{1,\Omega} + \|\tilde{u}_n\|_{1,\Omega} \leq C_3, \forall n \in \mathbb{N}.
\]
By the compact embedding of \( W^{1,s}(\Omega) \) into \( L^2(\Omega) \), we can conclude that \( K \) is compact. Since \( DH((\sigma, p); u) = I + K \), the property (5.4) follows.

Let us now pass to the second property: Fix \((\tau, q); v) \in \text{ker}_{L^2} DH((\sigma, p); u)\), which means that

\[
((\tau, q); v) + S\left( \frac{1}{\nu}(\sigma \cdot v + \tau \cdot u) \right) = 0.
\]

This is also equivalent to

\[
((\tau, q); v) = -S\left( \frac{1}{\nu}(\sigma \cdot v + \tau \cdot u) \right).
\]

Again thanks to Lemma 5.2 we can say that \((\tau, q); v) \) belongs to \( (W^{1,s}(\Omega))^2 \times \times (W^{2, r_1}(\Omega))^2 \) (since \( 2 - \frac{2}{r_1} \geq -\frac{2}{r_2} \)), we deduce that \((\tau, q); v) \) belongs to \( X \).

It remains to check the local Lipschitz property of \( DH \):

**Lemma 5.4.** Under the assumptions of Lemma 5.2, for all \((\tau, q); v) \in X\) and all \( t \in [0, 1]\), we have

\[
\| [DH((\sigma, p); u) - DH((\sigma + t\tau, p + tq); u + tv)]((\tau, q); v) \|_{L^2} \leq \frac{\beta}{\nu^2} t\|((\tau, q); v)\|_{L^2} \times \times ((\tau, q); v)\|_{X}, \tag{5.7}
\]

for some \( \beta > 0 \) independent of \( v \).

**Proof.** By the definition of \( DH \), we notice that

\[
[DH((\sigma, p); u) - DH((\sigma + t\tau, p + tq); u + tv)]((\tau, q); v) = -\frac{2t}{\nu} S(\tau \cdot v).
\]

By the properties of \( S \) proved in Lemma 5.2 for \( p' > 1 \) such that \( \frac{1}{r} + \frac{1}{r'} = \frac{1}{r} \), there exists \( C > 0 \) such that

\[
\| [DH((\sigma, p); u) - DH((\sigma + t\tau, p + tq); u + tv)]((\tau, q); v) \|_{L^2} \leq C \frac{2t}{\nu} S(\tau \cdot v)_{L^2}
\]

\[
\leq C \frac{2t}{\nu} \|\tau \cdot v\|_{L^2_{0,p';\Omega}}
\]

\[
\leq C \frac{2t}{\nu} \|\tau\|_{L^2_{0,r;\Omega}} \|v\|_{L^2_{0,\Omega}},
\]

this last inequality following from Hölder’s inequality. This proves the Lemma.

Because we checked all the assumptions of Theorem 5.1, we arrive at the next result:

**Theorem 5.5.** Assume that the parameter \( r_1 \) further satisfies \( r_1 \leq 3 \). Let \( f \in (L^2(\Omega))^2 \) and let \((\sigma, p); u) \in X\) be a nonsingular solution of (3.7). Then there exist three positive constants

$C_1$, $C_2$, and $h_0$ depending on $\nu$ and $f$ such that for all $h \leq h_0$, any solution $((\sigma_h, p_h); u_h) \in X_h$ of (4.5) verifying the estimate (4.25) satisfies

$$C_1 \|((\sigma, p); u) - ((\sigma_h, p_h); u_h)\|_{L^2} \leq \left\| S\left( f - \frac{1}{\nu} \sigma_h \cdot u_h \right) - S_h\left( f - \frac{1}{\nu} \sigma_h \cdot u_h \right) \right\|_{L^2}$$

$$\leq C_2 \|((\sigma, p); u) - ((\sigma_h, p_h); u_h)\|_{L^2}. \quad (5.8)$$

**Proof.** By Theorem 5.1, we may write

$$C_1 \|((\sigma, p); u) - ((\sigma_h, p_h); u_h)\|_{L^2} \leq \|H((\sigma_h, p_h); u_h)\|_{L^2} \leq C_2 \|((\sigma, p); u) - ((\sigma_h, p_h); u_h)\|_{L^2},$$

since by Theorem 4.7

$$\|((\sigma, p); u) - ((\sigma_h, p_h); u_h)\|_{X^r_s} \leq C h^{\frac{1}{2} - \alpha} s^{-\alpha} r^{s-\frac{2}{\alpha}}.$$

As $H_h((\sigma_h, p_h); u_h) = 0$, we directly see that

$$H((\sigma_h, p_h); u_h) = H((\sigma_h, p_h); u_h) - H_h((\sigma_h, p_h); u_h) = S_h\left( f - \frac{1}{\nu} \sigma_h \cdot u_h \right) - S\left( f - \frac{1}{\nu} \sigma_h \cdot u_h \right),$$

by the definition of $H$ and $H_h$.

In our setting, by Theorem 5.1, the constants $C_1$ and $C_2$ appearing in the estimates (5.8) are given by

$$C_1 = \frac{1}{2} \|DH((\sigma, p); u)^{-1}\|_{L^2}^{-1}, \quad C_2 = 2\|DH((\sigma, p); u)\|_{L^2}.$$

Hence a careful analysis of $DH((\sigma, p); u)$ allows to obtain appropriate bounds for $C_1$ and $C_2$:

**Lemma 5.6.** Assume that the parameter $r_1$ further satisfies $r_1 \leq 3$. Then there exists a positive constant $C$ that does not depend on $\nu$ and $f$ such that

$$\|DH((\sigma, p); u)\|_{L^2} \leq 1 + C \frac{C_1(\nu, f)}{\nu^2}, \quad (5.9)$$

where we recall that $C_1(\nu, f)$ was introduced in Theorem 2.1.

On the other hand there exists $c > 0$ that does not depend on $\nu$ and $f$ such that if

$$\frac{C_1(\nu, f)}{\nu^2} < c, \quad (5.10)$$

then any $((\sigma, p); u) \in X$ solution of (3.7) is non-singular and

$$\|DH((\sigma, p); u)^{-1}\|_{L^2} \leq \frac{c}{c - \frac{C_1(\nu, f)}{\nu^2}}. \quad (5.11)$$

**Proof.** Since

$$DH((\sigma, p); u)((\tau, q); v) = ((\tau, q); v) + \frac{1}{\nu} S(\sigma \cdot v + \tau \cdot u),$$

we directly deduce that
\[
\|DH((\sigma, p); u)((\tau, q); v)\|_{L^2} \leq \|((\tau, q); v)\|_{L^2} + \frac{1}{\nu}\|S(\sigma \cdot v + \tau \cdot u)\|_{L^2}.
\]
Hence it remains to estimate the second term of this estimate. But Hölder’s inequality (see the arguments of the proof of Lemma 5.2) implies that
\[
\|\sigma \cdot v + \tau \cdot u\|_{0, s, \Omega} \leq \|\sigma\|_{0, r_1, \Omega} \|v\|_{0, \Omega} + \|u\|_{0, r_2, \Omega} \|\tau\|_{0, \Omega},
\]
where \(\frac{1}{s} = \frac{1}{2} + \frac{1}{r_1} + \frac{1}{r_2} \).

Therefore by the estimate (3.4) of Theorem 3.1, we obtain
\[
\|\sigma \cdot v + \tau \cdot u\|_{0, s, \Omega} \leq C C_1(\nu, f) \nu \|((\tau, q); v)\|_{L^2}.
\]
By the arguments of the proof of Lemma 5.2 we get
\[
\|S(\sigma \cdot v + \tau \cdot u)\|_{L^2} \leq C \|\sigma \cdot v + \tau \cdot u\|_{0, s, \Omega}
\]
and therefore
\[
\|S(\sigma \cdot v + \tau \cdot u)\|_{L^2} \leq C \frac{C_1(\nu, f)}{\nu} \|((\tau, q); v)\|_{L^2}. \tag{5.12}
\]
This estimate shows that
\[
\|DH((\sigma, p); u)((\tau, q); v)\|_{L^2} \leq \|((\tau, q); v)\|_{L^2} \left(1 + \frac{C_1(\nu, f)}{\nu^2}\right)
\]
and hence the estimate (5.9).

For the second estimate by setting \(DH((\sigma, p); u)((\tau, q); v) = \zeta\), we may write
\[
((\tau, q); v) = \zeta - \frac{1}{\nu}S(\sigma \cdot v + \tau \cdot u),
\]
and therefore by the estimate (5.12) we get
\[
\|((\tau, q); v)\|_{L^2} \leq \|\zeta\|_{L^2} + \frac{C_1(\nu, f)}{\nu^2} \|((\tau, q); v)\|_{L^2}.
\]
By sending the second term of this right-hand side in the left-hand side we get
\[
\left(c - \frac{C_1(\nu, f)}{\nu^2}\right) \|((\tau, q); v)\|_{L^2} \leq c \|\zeta\|_{L^2},
\]
with \(c = \frac{1}{c'}\). This shows that under the constraint (5.10) \(DH((\sigma, p); u)\) is injective and that (5.11) holds.

From the previous Theorem 5.5, we see that the a posteriori error estimate of the Navier-Stokes system is reduced to the a posteriori error estimate for the Stokes system with datum \(f - \frac{1}{\nu}\sigma_h \cdot u_h\). Therefore applying our former results [26], we arrive at

Theorem 5.7. Under the assumptions of Theorem 5.5, there exists a positive constant \( c_1 \) independent of \( \nu \) and \( f \) and there exist two positive constants \( c_2, c_3 \) and a meshsize \( h_0 \) small enough depending on \( \nu \) such that for all \( h \leq h_0 \), the following error estimates hold

\[
\| \sigma - \sigma_h \|_{0,\Omega} + \| p - p_h \|_{0,\Omega} + \nu \| u - u_h \|_{0,\Omega} \leq \frac{c_1}{C_1} (\eta + \xi),
\]

and

\[
\eta \leq c_2 (\| \sigma - \sigma_h \|_{0,\Omega} + \| p - p_h \|_{0,\Omega} + \nu \| u - u_h \|_{0,\Omega}) + c_3 \xi,
\]

where the constant \( C_1 \) is the one from Theorem 5.5 and the global estimator \( \eta \) and approximation term \( \xi \) are defined by

\[
\eta^2 := \sum_{K \in \mathcal{T}_h} \eta^2_K, \quad \xi^2 := \sum_{K \in \mathcal{T}_h} \xi^2_K,
\]

where the local estimators and approximation terms are given by

\[
\eta^2_K := h^2_K \| r \|_{0,K}^2 + \| \text{tr} \sigma_h \|_{0,K}^2
\]

\[
+ h^2_K \| \sigma_h \|_{0,K}^2 + \sum_{E \subset \partial K} h_E \| \left[ \sigma_h \cdot t_E \right]_E \|_{0,E}^2,
\]

\[
\xi^2_K := h^2_K \| f - P^0_h f \|_{0,K}^2,
\]

where \( t_E \) is a fixed tangent vector along an edge \( E \), \( \left[ \sigma_h \cdot t_E \right]_E \) is the jump of \( \sigma_h \cdot t_E \) across \( E \), and \( P^0_h \) is the \((L^2(\Omega))^2\)-orthogonal projection on \( M_h \) or equivalently

\[
(P^0_h f)_K = \frac{1}{|K|} \int_K f(x) dx, \quad \forall f \in (L^2(\Omega))^2, \quad K \in \mathcal{T}_h.
\]

As usual the exact residual is

\[
R := \nabla \cdot (\sigma - \sigma_h - (p - p_h)\delta) - \frac{1}{\nu} (\sigma \cdot u - \sigma_h \cdot u_h) = -f - \nabla \cdot (\sigma_h \cdot p_h \delta) + \frac{1}{\nu} (\sigma_h \cdot u_h),
\]

while the approximated residual is

\[
r := -P^0_h f - \nabla \cdot (\sigma_h \cdot p_h \delta) + \frac{1}{\nu} (\sigma_h \cdot u_h).
\]

Proof. The main point is to express explicitly the element residual \( R \) in terms of \( f, \sigma_h \) and \( u_h \). Indeed the identity (4.2) (recall that (4.1)–(4.2) is equivalent to (4.5)) is equivalent to

\[
\nabla \cdot (\sigma_h - p_h \delta) = \frac{1}{\nu} P^0_h (\sigma_h \cdot u_h) - P^0_h f.
\]

In a same manner the identity (3.2) is equivalent to

\[
\nabla \cdot (\sigma - p \delta) = \frac{1}{\nu} \sigma \cdot u - f.
\]

Subtracting these two identities, we arrive at

\[ R = -(f - P_h^0 f) + \frac{1}{v} (\sigma_h \cdot u_h - P_h^0 (\sigma_h \cdot u_h)), \]

(5.15)

These identities further yield

\[ R - r = -(f - P_h^0 f). \]

The upper bound (5.13) is now a direct consequence of this identity, the estimate (5.8), and Theorem 4.7 of [26], which showed that

\[ \| S (f - \frac{1}{v} \sigma_h \cdot u_h) - S_h (f - \frac{1}{v} \sigma_h \cdot u_h) \|^2_{L^2} \leq C \sum_{K \in T_h} \left( \| \text{tr} \sigma_h \|^2_{0,K} + h_K^2 \| \sigma_h \|^2_{0,K} \right. \\
\left. + \sum_{E \subset \partial K} h_E \left\| \left[ (\sigma - \delta) - (\sigma_h - p_h \delta) \right] : \nabla w_K \right\|_{0,E}^2 + h_K^2 \| R \|^2_{0,K} \right), \]

for some positive constant \( C \) independent of \( v \).

For the lower bound (5.14), using Theorem 4.8 of [26] and the estimate (5.8), we have

\[ \sum_{K \in T_h} \left( \| \text{tr} \sigma_h \|^2_{0,K} + h_K^2 \| \sigma_h \|^2_{0,K} + \sum_{E \subset \partial K} h_E \left\| \left[ (\sigma - \delta) - (\sigma_h - p_h \delta) \right] : \nabla w_K \right\|_{0,E}^2 \right) \]

\[ \leq C \left\| S (f - \frac{1}{v} \sigma_h \cdot u_h) - S_h (f - \frac{1}{v} \sigma_h \cdot u_h) \right\|^2_{L^2} \]

\[ \leq CC \| ((\sigma, p); u) - ((\sigma_h, p_h); u_h) \|^2_{L^2}, \]

(5.16)

for some positive constant \( C \) independent of \( v \). So it remains to estimate the element residual. As usual taking \( w_K = r b_K \), where \( b_K \) is the element bubble function associated with \( K \), we have

\[ \| r \|^2_{0,K} \leq C \int_K \mathbf{r} \cdot w_K = C \left( \int_K R \cdot w_K + \int_K (\mathbf{r} - R) \cdot w_K \right). \]

Using the definition of \( R \) and Green’s formula we obtain

\[ \| r \|^2_{0,K} \leq C \int_K (f - P_h^0 f) \cdot w_K \]

\[ - C \int_K [(\sigma - p \delta) - (\sigma_h - p_h \delta)] : \nabla w_K \]

\[ - \frac{C}{\nu} \int_K [\sigma \cdot u - \sigma_h \cdot u_h] \cdot w_K. \]

(5.17)

For this last term we write

\[ \int_K [\sigma \cdot u - \sigma_h \cdot u_h] \cdot w_K = \int_K [(\sigma - \sigma_h) \cdot u - \sigma_h \cdot (u_h - u)] \cdot w_K \]
and applying Hölder’s inequality we obtain
\[
\left| \int_K \left[ \sigma \cdot u - \sigma_h \cdot u_h \right] \cdot w_K \right| \leq C \left( \| \sigma - \sigma_h \|_{0,K} \| u \|_{0,4,K} + \| \sigma_h \|_{0,4,K} \| u - u_h \|_{0,K} \right) \| w_K \|_{0,4,K}.
\]

Now using standard inverse inequality we get
\[
\left| \int_K \left[ \sigma \cdot u - \sigma_h \cdot u_h \right] \cdot w_K \right| \leq C \left( \| \sigma - \sigma_h \|_{0,K} \| u \|_{0,4,K} + \| \sigma_h \|_{0,4,K} \| u - u_h \|_{0,K} \right) h^{-1}_K \| r \|_{0,K}.
\]

Using Theorem 4.7, we get
\[
h^{1/2}_K \| \sigma - \sigma_h \|_{0,K} \| u \|_{0,4,K} + \| \sigma_h \|_{0,K} \| u - u_h \|_{0,K} \leq C \left( \| \sigma - \sigma_h \|_{0,K} + \| u - u_h \|_{0,K} \right),
\]
where \( C > 0 \) may depend on \( \nu \) and therefore
\[
\left| \int_K \left[ \sigma \cdot u - \sigma_h \cdot u_h \right] \cdot w_K \right| \leq C h^{-1}_K \left( \| \sigma - \sigma_h \|_{0,K} + \| u - u_h \|_{0,K} \right) \| r \|_{0,K}.
\]

This estimate in (5.17) and again Cauchy-Schwarz’s inequality and inverse estimates yield
\[
h_K \| r \|_{0,K} \leq C \max \left\{ 1, \frac{1}{\nu^2} \right\} \left( \| \sigma - \sigma_h \|_{0,K} + \| p - p_h \|_{0,K} + \nu \| u - u_h \|_{0,K} + \xi_K \right).
\]

Summing the square of this estimate and using (5.16), we obtain the requested lower bound. \( \blacksquare \)

**Remark.** By Lemma 5.6, if (5.10) holds, the constant \( C_1 \) appearing in the upper bound (5.13) can be estimated explicitly in terms of \( \nu \) and the \( L^2 \)-norm of \( f \). Hence this estimate can be successfully used for an adaptive algorithm. Note further that the proof of (5.14) gives a more precise bound on \( c_2 \) and \( c_3 \) at least for the main part of the estimator.

**References**


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