An a posteriori error estimator for the Lamé equation based on equilibrated fluxes

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We derive a new a posteriori error estimator for the Lamé system based on H(div)-conforming elements and equilibrated fluxes. It is shown that the estimator gives rise to an upper bound where the constant is one up to higher-order terms. The lower bound is also established using Argyris elements. The reliability and efficiency of the proposed estimator are confirmed by some numerical tests.

Keywords: equilibrated fluxes; mixed finite elements; a posteriori error estimates; linear elasticity.

1. Introduction

The finite-element methods are commonly used in the numerical realization of many problems occurring in engineering applications, like the Laplace equation, the Lamé system, the Stokes system, etc. (see Brenner & Scott, 1994; Ciarlet, 1978). Adaptive techniques based on a posteriori error estimators have become indispensable tools for such methods. There exists nowadays a large number of publications devoted to the task of analysing finite-element approximations for problems in solid mechanics and obtaining locally defined a posteriori error estimates. We refer to the monographs of Ainsworth & Oden (2000), Babuška & Strouboulis (2001) and Verfürth (1996) for a good overview on this topic.

Usually upper and lower bounds are proved in order to guarantee the reliability and the efficiency of the proposed estimator. Most of the existing approaches involve constants depending on the shape regularity of the elements and/or of the jumps in the coefficients; but these dependences are often not given.

For the elasticity system, several different approaches leading to various estimators have been developed (see the review paper Verfürth, 1999). Let us quote the following methods: Residual-type error estimators measure the jump of the discrete flux in Babuška et al. (1992), Babuška & Miller (1987) and Verfürth (1999). Another approach is to solve local subproblems by using higher-order elements (see Babuška et al., 1992; Babuška & Rheinboldt, 1978; Bank & Weiser, 1985, 1990). Very simple and cheap error estimators are the so-called Zienkiewicz–Zhu estimators based on averaging techniques (cf. Ainsworth & Babuška, 1998; Ainsworth & Oden, 2000; Zhu & Zienkiewicz, 1988; Zienkiewicz & Zhu, 1987). Let us further cite estimates based on the adjoint problem in Becker & Rannacher (2001) and on dual problems in Neittaanmaäki & Repin (2004). Finally, we can mention estimators based on

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equilibrated fluxes and on the solution of local Neumann boundary-value problems (see Ainsworth & Oden, 1993; Brink & Stein, 1998; Kelly, 1984; Kelly & Isles, 1989; Ladevèze et al., 1986; Ladevèze & Leguillon, 1983; Ladevèze et al., 1991; Ohnimus et al., 2001; Stein & Ohnimus, 1997, 1999; Stein et al., 1999).

Here, we introduce a locally defined error estimator based on $H(\text{div})$-conforming approximations for the stress and on equilibrated fluxes. The solution of the local Neumann boundary-value problems is replaced by the construction of an explicit $H(\text{div})$-conforming approximation. That renders our estimator quite attractive since no supplementary problems have to be solved, and an upper bound with constant one, up to higher-order terms, for the energy norm of the discretization error can be guaranteed.

The outline of the paper is as follows: We recall in Section 2 the boundary-value problem and its numerical approximation. Section 3 is devoted to the introduction of the estimator and the proofs of the upper and lower bounds. The upper bound directly follows from the construction of the estimator, while the lower bound requires a suitable choice of a divergence-free element. Here, we use the Airy operator applied to Argyris elements. In Section 4, we focus on implementation aspects. Finally, some numerical tests are presented in Section 5 that confirm the reliability and efficiency of our estimator.

2. The boundary-value problem of elasticity and notation

In the context of elasticity, vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\sigma$ and $\tau$ will be denoted by $\sigma : \tau$ and is given by $a : b = a_{ij}b_{ij}$, the summation convention on repeated indices being invoked.

Consider a homogeneous, isotropic, linear elastic material body which occupies a bounded domain $\Omega$ in $\mathbb{R}^2$ with Lipschitz boundary $\Gamma$. For a prescribed body force $f \in \{L^2(\Omega)\}^2$, the governing equilibrium equation in $\Omega$ reads

$$-	ext{div} \sigma = f,$$

where $\sigma$ is the symmetric linearized Cauchy stress tensor. The infinitesimal strain tensor is defined as a function of the displacement $u$ by $\varepsilon(u) := \frac{1}{2}(\nabla u + [\nabla u]^T)$. With the fourth-order elasticity tensor denoted by $\mathcal{C}$, the constitutive equation reads

$$\sigma = \mathcal{C} \varepsilon(u) := \lambda (\text{tr} \varepsilon(u)) \mathbf{1} + 2\mu \varepsilon(u).$$

(2.1)

Here, $\mathbf{1}$ is the identity tensor and $\lambda$ and $\mu$ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body and are assumed to be positive. The displacement is assumed to satisfy homogeneous Dirichlet boundary conditions on a part $\Gamma_D$ of the boundary $\partial \Omega$, i.e. $u = \mathbf{0}$ on $\Gamma_D$, and a traction-free boundary condition on the remaining part $\Gamma_N$.

We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on $\Omega$ with the inner product and norm being denoted by $(\cdot, \cdot)_0$ and $\|\cdot\|_0$, respectively. The space $H^1(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on $\Gamma_D$ in the sense of traces. For the weak or variational formulations, we will require the space $V := [H^1(\Omega)]^2$ of displacements; this is an Hilbert space with inner product $(\cdot, \cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way, i.e. $(u, v)_1 := \sum_{i=1}^2(u_i, v_i)_1$, with the norm being induced by this inner product. To define our error estimator, we compute by a local postprocessing step an approximation of the symmetric stress $\sigma$. The stress is in $H^1(\text{div}; \Omega) := \{\tau | \tau_{ji} = \tau_{ij}, \tau_{ij} \in L^2(\Omega), \text{div} \tau \in [L^2(\Omega)]^2\}$ with the norm $\|\cdot\|_0 + \|\text{div}\cdot\|_0$ generated in the standard way by the $L^2$-norm $\|\cdot\|_0$.

Define the bilinear form $a(\cdot, \cdot)$ by

$$a : V \times V \rightarrow \mathbb{R}, \quad a(u, v) := \int_{\Omega} \mathcal{C} \varepsilon(u) : \varepsilon(v) \, dx.$$
Associated with the bilinear form is the energy norm \( \|v\|^2 := a(v,v), \ v \in V \). Then, the standard form of the weak problem for elasticity takes the following form: given \( f \in [L^2(\Omega)]^2 \), find \( u \in V \) that satisfies
\[
a(u,v) = (f,v)_0, \quad v \in V. \tag{2.2}
\]

Let \( \mathcal{T}_h \) be a shape regular triangulation of the polygonal domain \( \Omega \). We assume that all elements are affine equivalent to the reference triangle \( \hat{T} \) with corners \((0,0), (1,0)\) and \((0,1)\) or the reference square \( \hat{T} := (-1,1)^2 \). Note that in that last case only parallelograms are allowed. The diameter of an element \( T \) in \( \mathcal{T}_h \) is denoted by \( h_T \). The finite-element space \( V_h \subset [H^1_0(\Omega)]^2 \) for the displacement is taken to be the space of conforming finite elements of lowest order
\[
V_h = \{ v_h \in [H^1_0(\Omega)]^2 \mid v_h|_T \in V_T, T \in \mathcal{T}_h \},
\]
where \( V_T := [P_1(T)]^2 \) for a triangle and \( V_T := [Q_1(\hat{T})]^2 \circ F_T^{-1} \) for a quadrilateral. Here, \( F_T \) is the affine mapping from the reference square \( \hat{T} \) on \( T \). By \( u_h \in V_h \) we denote the finite-element solution of the variational problem (2.2), and \( \sigma_h \) stands for the elementwise computed discrete stress approximation, i.e., \( \sigma_h|_T = C e(u_h|_T) \). We note that for both types of elements, \( \sigma_h|_T \) is in \([P_1(T)]^{2 \times 2}\), and by definition it is symmetric.

### 3. Definition of the error estimator

Our error estimator is defined in terms of equilibrated fluxes. Equilibrated fluxes are well established and often used for an adaptive error control; we refer to the monographs of Ainsworth & Oden (2000), Babuška & Strouboulis (2001), Verfürth (1996) and the references therein as well, to the early works Kelly (1984), Ladevèze & Leguillon (1983) and Stein & Ohnimus (1997) for a special emphasis on the Lamé equation. *A posteriori* error estimates for the displacements in terms of local surface tractions are also considered in Brink & Stein (1998), Kelly & Isles (1989), Ohnimus et al. (2001), Stein & Ohnimus (1999) and Stein et al. (1999).

For convenience of the reader, we recall the basic ideas. The set of edges of the triangulation is denoted by \( \mathcal{E}_h \) and the set of vertices by \( \mathcal{P}_h \). With each edge, we associate one unit normal \( n_e \), and \( n_T \) stands for the outer unit normal of \( T \in \mathcal{T}_h \). We note that \( n_T \) restricted to the edge \( e \) is equal to \( \pm n_e \). The orientation of \( n_e \) is arbitrary but should be fixed. Moreover, we assume that the Dirichlet and the Neumann part of the boundary is exactly resolved by the mesh. Then, an equilibrated flux is characterized by the following problem: Find for each \( e \in \mathcal{E}_h \) a linear function \( g_e \in [P_1(e)]^2 \) such that the following local variational problem is satisfied on each element \( T \in \mathcal{T}_h \):
\[
a_T(u_h,v) = (f,v)_0 + \int_{\partial T} g_T \cdot v \, ds, \quad v \in V_T, \tag{3.1}
\]
where \( a_T(\cdot, \cdot) \) is the local contribution of the bilinear form \( a(\cdot, \cdot) \) restricted to the element \( T \) and \( g_T|_e := n_e \cdot n_T g_e \). Writing \( g_e \) as a linear combination of the two nodal basis functions associated with the two end-points of the edge \( e \), (3.1) yields a global system. However, introducing the moments or equivalently biorthogonal basis functions, the global system can be decoupled, and for each vertex \( p \in \mathcal{P}_h \), a local system for the moments is obtained (see Ainsworth & Oden, 2000; Ladevèze & Leguillon, 1983). The local matrix associated with this linear system is singular for inner vertices and vertices on the Neumann boundary and has an extremely simple structure. Existence of the solution follows from the fact that \( u_h \) is the finite-element solution. There are different possibilities to fix the additional degrees of freedom

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for the fluxes $g_e$. Basically, all approaches are motivated by the observation that for the lower bound of the discretization error, it is crucial to find an equilibrated flux which is as close as possible to the average of the discrete flux, i.e. to $\{\sigma_h n_e\} = \frac{1}{2} (\sigma_h^+ n_e + \sigma_h^- n_e)$, where $\sigma_h^+ = \sigma_h |_{T^+}$ is the restriction of $\sigma_h$ on $T^+$, and $T^\pm$ are the two elements that have $e$ as edge. In order to preserve the locality, we do not minimize $\sum_{e \in E_h} \| g_e - \{\sigma_h n_e\} \|_{0,e}^2$, but minimize the difference between the moments of the equilibrated flux and the average of the discrete flux. We refer to Ainsworth & Oden (2000) for details regarding the structure of the linear systems, the minimization step and the different cases of vertices associated with the interior of the domain or the boundary conditions. Note finally that for edges $e \subset T_N$, we can take $g_e = 0$ (which is quite natural because of the boundary condition $\sigma n = 0$ on $T_N$). To define an equilibrated error estimator, these locally computed fluxes are then used to define local Neumann boundary-value problems on each element $T$. In general, these local Neumann problems are solved numerically by higher-order elements, e.g. quadratic elements, and the resulting displacements are used to define the error estimator for which, up to higher-order terms, upper and lower bounds in terms of the discretization error can be shown. The influence of the order of the approximation of the Neumann problem is numerically investigated in Babuška & Strouboulis (2001). We remark that the displacements obtained from the local Neumann problems are in general discontinuous across the elements. Here, we proceed differently. We use the equilibrated fluxes to define locally on each element a $H^S(\text{div}; \Omega)$-conforming approximation of the stress. This approximation is then used to define the error estimator yielding the constant one in the upper bound for the error and higher-order terms only depending on the data oscillation and being zero if the applied volume force is elementwise linear. Moreover, in many applications the accuracy of the stress is of high importance. Using the hypercycle technique, one can define a new approximation for the stress for which the error estimator is asymptotically exact. For the case of the Laplace operator, we refer to Luce & Wohlmuth (2004), where Raviart–Thomas elements of lowest order are used on a finite-volume patch to carry out this idea.

3.1 Upper bounds in terms of mixed finite elements

Unfortunately, there exists no simple low-order conforming element for $H^S(\text{div}; \Omega)$. Possible remedies are to use macroelements or to weaken the symmetry condition, see Brezzi & Fortin (1991). However, if we weaken the symmetry, an additional factor due to the non-conformity of the approach occurs. Therefore, we work with the element proposed in Arnold & Winther (2002) for simplicial triangulations (see also Arnold & Winther, 2003). We refer to Wang (2004) for the application of this element to linear elasticity and for the construction of a preconditioner. A $H(\text{div})$-conforming element for quadrilaterals is given in Arnold & Awanou (2005); however, this element yields a local dimension of 45 compared to 31 when using the triangular element on two neighbouring triangles, see Lemma 3.1 below. We therefore cut each quadrilateral into two triangles. If there is a unique shortest diagonal, we use this one as cutting edge $e_T$, otherwise the choice is arbitrary but should globally be fixed. The new simplicial triangulation is then called $T_h^S$, and the set of edges being not in $E_h$ is denoted by $E_h^n$. If $T_h$ is a simplicial triangulation, we set $T_h^S := T_h$ and $E_h^n = \emptyset$. As before, we denote by $P_h^S$ and $E_h^S$ the set of vertices and edges of $T_h^S$, respectively. By construction, $P_h^S = P_h$ and $E_h^S = E_h \cup E_h^n$, and each $T \in T_h \setminus T_h^S$ is the union of two elements $t_1, t_2 \in T_h^S$. We refer to Fig. 1 for the different sets of edges and the notation. The elements of $E_h^n$ are indicated by dashed lines in the right picture of Fig. 1.

The element given in Arnold & Winther (2002) has locally 24 degrees of freedom, and a reduced variant has 21 degrees of freedom. In the following, we use the 24-dimensional space

$$S_T := \{ \tau_h \in [P_3(T)]^{2 \times 2} | (\tau_h)_{12} = (\tau_h)_{21}, \text{div} \tau_h \in [P_1(T)]^2 \}$$
as local space on a triangle and define the global $H^S(\text{div}; \Omega)$-conforming finite-element space on $\mathcal{T}_h^s$ for the stress approximation by

$$S_h^s := \{ \tau_h \in H^S(\text{div}; \Omega) | \tau_h|_T \in S_T, T \in \mathcal{T}_h^s \}. \quad (3.2)$$

It can be shown (see Arnold & Winther, 2002) that the global degrees of freedom given by

- the nodal values at the vertices $p \in \mathcal{P}_h^s$,
- the zero-order moment of $\tau_h n_e$ in direction $x_i$, $i \in \{1, 2\}$, on the edges $e \in \partial_h^s$,
- the first-order moment of $\tau_h n_e$ in direction $x_i$, $i \in \{1, 2\}$, on the edges $e \in \partial_h^s$,
- the mean values on the elements $T \in \mathcal{T}_h^s$

yield a conforming approximation of $S_h^s$, and moreover the pairing $(S_h^s, W_h^s)$, with $W_h^s := \{ w_h \in [L^2(\Omega)]^2 | w_h|_T \in [P_1(T)]^2, T \in \mathcal{T}_h^s \}$, satisfies a uniform inf–sup condition, see Arnold & Winther (2002). Thus, we have four degrees of freedom per edge, three per vertex and three per element. Originally, the degrees of freedom associated with the interior of the elements are given in terms of point values at the centre of gravity and not by a mean value. However, it is easy to see that the unique solvability is still guaranteed if we replace the point value by a mean value.

Because our finite-element solution $u_h$ is defined with respect to the mesh $\mathcal{T}_h$, it is not convenient to use the space $S_h^s$ being defined with respect to the mesh $\mathcal{T}_h^s$ for the stress approximation. For $T \in \mathcal{T}_h \setminus \mathcal{T}_h^s$, we set

$$S_T^q := \left\{ \tau_h \in S_h^s|_T \big| \text{div} \tau_h \in [P_1(T)]^2, \int_{e_T} (s - s_m) (\tau_h n_{e_T}) : t_{e_T} \, ds = 0 \right\},$$

where $s_m$ denotes the midpoint of the edge $e_T$, and $t_{e_T}$ stands for the tangential vector on the cutting edge $e_T$. In terms of this local space, we define the global space $S_h \subset S_h^s$ by

$$S_h := \{ \tau_h \in S_h^s|_{\mathcal{T}_h} \big| \tau_h|_T \in S_T^q, T \in \mathcal{T}_h \setminus \mathcal{T}_h^s \}. \quad \text{LEMMA 3.1}$$

Each element in $S_h$ is uniquely defined by its

- nodal values at the vertices $p \in \mathcal{P}_h$,
- zero-order moment of $\tau_h n_e$ in direction $x_i$, $i \in \{1, 2\}$, on the edges $e \in \partial_h$. 

![Fig. 1. Triangulation $\mathcal{T}_h$ (left) and triangulation $\mathcal{T}_h^s$ (right).](image-url)
• first-order moment of $\mathbf{t}_h n_e$ in direction $x_i$, $i \in \{1, 2\}$, on the edges $e \in \mathcal{E}_h$.
• mean values on the elements $T \in \mathcal{T}_h$.

**Proof.** The proof follows the lines given in Arnold & Winther (2002) for simplicial elements. Here, we have to consider the situation of decomposed quadrilaterals in more detail. Counting the number of constraints, it is sufficient to show that if $\mathbf{t}_h \in S_h$ satisfies $\mathbf{t}_h(p) = 0$, $p \in \mathcal{P}_h$, $\int_e (\mathbf{t}_h n_e) \cdot q_1 \, ds = 0$, $q_1 \in [P_1(e)]^2$, $e \in \mathcal{E}_h$, and $\int_T \mathbf{t}_h \, dx = 0$, $T \in \mathcal{T}_h$, then $\mathbf{t}_h = 0$. By definition of $S_h$, $\text{div} \mathbf{t}_h | T \in [P_1(T)]^2$, and we find

$$\int_T \text{div} \mathbf{t}_h \cdot \text{div} \mathbf{t}_h \, dx = - \int_T \mathbf{t}_h : \nabla \text{div} \mathbf{t}_h \, dx + \int_{\partial T} (\mathbf{t}_h n_T) \cdot \text{div} \mathbf{t}_h \, ds = 0.$$

Let $t \subset T$ be an element in $\mathcal{T}_h^s$, then Green’s formula yields for $v \in [P_0(t)]^2$

$$0 = \int_{\partial t} (\mathbf{t}_h n_t) \cdot v \, ds = \pm \int_{e_T} (\mathbf{t}_h n_{e_T}) \cdot v \, ds.$$

Similarly if $w = (x_2 - x_2^m, x_1^m - x_1)^T$, where $(x_1^m, x_2^m)$ denotes the midpoint of the edge $e_T$ and observing that $\mathbf{e}(w) = 0$, we get

$$0 = \int_{e_T} (\mathbf{t}_h n_{e_T}) \cdot w \, ds = \pm \int_{e_T} (s - s_m) (\mathbf{t}_h n_{e_T}) \cdot n_{e_T} \, ds.$$

The definition of $S_h$ and the previous results guarantee that the zero- and first-order moments of $\mathbf{t}_h n_e$ are zero on all edges $e \in \mathcal{E}_h^s$. Moreover, using $\text{div} \mathbf{t}_h = 0$, we find

$$\int_T \mathbf{t}_h : \nabla v = 0, \quad v \in [P_1(t)]^2,$$

and thus all degrees of freedom of $\mathbf{t}_h$ as an element in $S_h^s$ are zero. \hfill \Box

Now we proceed in two steps. In a first step, we define an approximation $\mathbf{t}_h \in S_h$ satisfying $\text{div} \mathbf{t}_h = -\Pi_1 \mathbf{f}$, where $\Pi_1$ is the $L^2$-projection on $W_h := \{w_h \in [L^2(\Omega)]^2 | w_h | T \in [P_1(T)]^2, T \in \mathcal{T}_h \}$. For this approximation, we show that in terms of $\mathbf{t}_h + \mathbf{t}^0$, where $\mathbf{t}^0 \in H_N^{1:S}(\text{div}; \Omega) := \{ e \in H^1(\Omega) | e \cdot n = 0 \text{ on } \Gamma_N \}$, we obtain an upper bound of the discretization error. In a second step, in order to obtain a lower bound, we have to fix a suitable divergence-free stress $\mathbf{t}^0$. The choice of which is motivated by being locally computable and being close to the discrete stress approximation $\mathbf{t}_h$. This second step is postponed to Section 3.2. We now concentrate on the first step.

**Definition 3.2** Let $\mathbf{t}_h \in S_h$ satisfy

• $\mathbf{t}_h(p) = 0$, $p \in \mathcal{P}_h$,
• $\int_e (\mathbf{t}_h n_e) \cdot q_1 \, ds = \int_e \mathbf{g}_e \cdot q_1 \, ds$, $q_1 \in [P_1(e)]^2$, $e \in \mathcal{E}_h$,
• $\int_T \mathbf{t}_h : \nabla v \, dx = a_T(u_h, v)$, $v \in [P_1(T)]^2$, $T \in \mathcal{T}_h$.

We note that the last condition is automatically satisfied for all rigid body motions. Thus, it yields only three independent equations on each element although $\dim [P_1(T)]^2 = 6$. Moreover, we observe that $\mathbf{g}_e = 0$ for all edges $e \subset \Gamma_N$. Then, Definition 3.2 guarantees that $\mathbf{t}_h n_e = 0$ on $e \subset \Gamma_N$. 


Lemma 3.3 \( \text{div}\tau_h = -\Pi_1 f \)

Proof. By definition of \( S_h \), we find that \( \text{div}\tau_h \in W_h \), and Green’s formula yields for all \( v_1 \in [P_1(T)]^2 \)

\[
\int_T \text{div}\tau_h \cdot v_1 \, dx = -\int_T \tau_h : \nabla v_1 \, dx + \int_{\partial T} (\tau_h n_T) \cdot v_1 \, ds
\]

\[
= -a_T(u_h, v_1) + \int_{\partial T} g_T v_1 \, ds = -\langle f, v_1 \rangle_{0,T}.
\]

Now, we consider elementwise the difference between the discrete stress approximation \( \sigma_h \) and \( \tau_h + \tau^0 \), and we define

\[
\eta(\tau^0)^2 := \sum_{T \in T_h} \left\| \mathcal{C}^{-\frac{1}{2}}(\tau_h + \tau^0 - \sigma_h) \right\|^2_{0,T}.
\]

Starting from (2.1), a straightforward computation shows that \( \mathcal{C}^{-\frac{1}{2}} e \) is given by

\[
\mathcal{C}^{-\frac{1}{2}} e = \frac{1}{\sqrt{2\mu}} \left( e - \frac{1}{2} \left( 1 - \frac{\mu}{\sqrt{\mu + \lambda}} \right) \text{tr} e \right).
\]

In most a posteriori error estimate, higher-order terms depending on the data occur. Here, we have to take into account

\[
osc_1(f)^2 := \sum_{T \in T_h} |T| \| f - \Pi_1 f \|^2_{0,T},
\]

where \( |T| \) denotes the area of the element \( T \). We note that if \( f \) is smooth, then \( \text{osc}_1(f) \) is of order \( h^2 \), whereas the discretization error in the energy norm is at most of \( O(h) \).

Theorem 3.4 For all \( \tau^0 \in H^{0:S}(\text{div}; \Omega) \), we obtain an upper bound for the energy norm of the discretization error in terms of \( \eta(\tau^0) \) and a higher-order data oscillation term. Namely, there exists a constant \( C < \infty \) independent of the mesh size such that

\[
\||u - u_h|| \leq \eta(\tau^0) + \text{osc}_1(f).
\]

Proof. We start with the definition of the energy norm, take into account the symmetry of the stress approximations, use \( [\tau_h + \tau^0]n_e = 0 \) across the edges of the triangulation (since \( \tau_h, \tau^0 \in H(\text{div}; \Omega) \)) and the boundary conditions \( u = u_h = 0 \) on \( \Gamma_D \) and \( \sigma n = \tau_h n = \tau^0 n = 0 \) on \( \Gamma_N \).

\[
\||u - u_h||^2 = \int_{\Omega} (\sigma - \sigma_h) : \varepsilon(u - u_h) \, dx = \int_{\Omega} (\sigma - \sigma_h) : \nabla(u - u_h) \, dx
\]

\[
= \int_{\Omega} (\tau_h + \tau^0 - \sigma_h) : \nabla(u - u_h) \, dx - \int_{\Omega} \text{div}(\sigma - \tau_h)(u - u_h) \, dx
\]

\[
= \int_{\Omega} \mathcal{C}^{-\frac{1}{2}}(\tau_h + \tau^0 - \sigma_h) : \mathcal{C}^{-\frac{1}{2}}\varepsilon(u - u_h) \, dx + \int_{\Omega} (f - \Pi_1 f)(u - u_h) \, dx
\]

\[
\leq \eta(\tau^0)||u - u_h|| + \text{osc}_1(f)||u - u_h||
\]

\[
\leq (\eta(\tau^0) + \text{osc}_1(f)||u - u_h||.
\]
In the last step, we have used Korn’s inequality and the boundedness of $C^{-1/2}$. Thus, the constant $C$ in the theorem only depends on the shape regularity of the triangulation, the Lamé parameter $\mu$ and the geometry.

**Remark 3.5** We note that the constant $C$ in Theorem 3.4 does not degenerate in the incompressible limit. More precisely, the operator $C^{-1/2}$ can be bounded in terms of $\mu$, and this bound does not depend on $\lambda/\mu \gg 1$.

### 3.2 Lower bounds in terms of Argyris elements

Although $\eta(\tau^0)$ provides for all $\tau^0 \in H^{0;S}_N(\text{div}; \Omega)$ an upper bound for the discretization error, we cannot expect to obtain a lower bound for arbitrary $\tau^0$. In principle, one could solve a global minimization problem for $\tau^0$, i.e. find $\tau^0 \in H^{0;S}_N(\text{div}; \Omega)$ such that

$$
\int_\Omega C^{-1} \tau^0 : e^0 \, dx = \int_\Omega (e(u_h) - C^{-1} \tau_h) : e^0 \, dx, \quad e^0 \in H^{0;S}_N(\text{div}; \Omega).
$$

In general, this is not reasonable, and thus we carry out two simplifications. Firstly, we replace $H^{0;S}_N(\text{div}; \Omega)$ by a suitable discrete approximation, and secondly we use a discrete norm equivalence to decouple the global minimization problem in local ones. We define by $S^0_{h;S} := \{ e_h \in S^0_{h} \mid \text{div} e_h = 0 \}$ the divergence-free subspace of $S^0_{h}$. The following result can be found in Arnold & Winther (2002). For convenience of the reader, we recall the proof.

**Lemma 3.6** Let $A_h \subset H^2(\Omega)$ be the space of Argyris elements with respect to the triangulation $\mathcal{T}_h$, then

$$
S^0_{h;S} = J(A_h), \quad J(v) := \begin{pmatrix} v_{x_2, x_2} & -v_{x_1, x_2} \\ -v_{x_1, x_2} & v_{x_1, x_1} \end{pmatrix},
$$

when $v_{x_i, x_j}$ means the second-order derivatives of $v$ in the $x_i$, $x_j$-directions.

**Proof.** We recall that Argyris elements are $C^1$-functions and locally in $P_5(T)$, see, e.g. Ciarlet (1978). Thus, it is easy to see that $J(A_h) \subset S^0_{h;S}$. For Argyris elements, six degrees of freedom are associated with the vertices (one nodal value, two first derivatives and three second derivatives), and one degree of freedom is associated with the normal derivative at the midpoint of the edge. Moreover, there exists an orthogonal decomposition of $S^0_h = S^0_{h;S} \oplus S^1_{h;S}$, and because of the inf–sup condition we know that the dimension of $S^1_{h;S}$ is greater than or equal to $6N_T$, where $N_T$ is the number of elements of $\mathcal{T}_h$. As a result we find that the dimension of $S^0_{h;S}$ is less than or equal to $3N_p + 4N_e - 3N_T$, where $N_p$ and $N_e$ are the number of vertices and edges, respectively, of $\mathcal{T}_h$. The dimension of $A_h$ is given by $6N_p + N_e$, and the dimension of the kernel of the Airy operator $J$ is three. Using $N_T + N_p - N_e = 1$, we get $6N_p + N_e - 3 = 3N_p + 4N_e - 3N_T$ and thus $S^0_{h;S} = J(A_h)$. □

Although the finite elements $S_T$ in combination with the degrees of freedom used in Lemma 3.1 are not affine equivalent, we can replace the $L^2$-norm by a discrete mesh-dependent norm. To do so, we
introduce the quantities $m_T^P(\eta_h)$, $m_T^e(\eta_h)$ and $m_T^i(\eta_h)$ by
\[
    m_T^P(\eta_h) := \sum_{p \in \mathcal{P}_T} \|\eta_h(p)\|^2, \\
    m_T^e(\eta_h) := \sum_{e \in \mathcal{E}_T} \left( \left\| \int_e \eta_h n_e \, ds \right\|^2 + \frac{1}{h_e} \int_e \eta_h n_e (s - s_m) \, ds \right)^2, \\
    m_T^i(\eta_h) := \left\| \int_T \eta_h \, dx \right\|^2,
\]
where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^2$ and $\mathbb{R}^{2 \times 2}$, and $\mathcal{P}_T$ stands for the set of vertices of $T$ and $\mathcal{E}_T$ is the set of edges of $T$.

**Lemma 3.7** Let $\eta_h \in \mathcal{S}_T$, $T \in \mathcal{K}_h \cap \mathcal{K}_h^s$, or $\eta_h \in \mathcal{S}_h^d$, $T \in \mathcal{K}_h \setminus \mathcal{K}_h^s$, then there exist constants $0 < c < C < \infty$ independent on the mesh size such that
\[
    c \|\eta_h\|_{0,T}^2 \leq |T|m_T^P(\eta_h) + m_T^e(\eta_h) + \frac{1}{|T|} m_T^i(\eta_h) \leq C \|\eta_h\|_{0,T}^2, \quad T \in \mathcal{K}_h.
\]

**Proof.** If $T \in \mathcal{K}_h$ is a triangle, we use the matrix Piola transformation
\[
    \eta_h(x) = B \hat{\eta}(\hat{x}) B^T,
\]
where the affine transformation $x = B \hat{x} + b$ maps the reference element $\hat{T}$ to $T$. The outer normal on $\hat{T}$ is denoted by $\hat{n}$. Since
\[
    n_T = \frac{1}{\|B^{-1}\hat{n}\|} B^{-1} \hat{n},
\]
we may write
\[
    \eta_h n_T = \frac{1}{\|B^{-1}\hat{n}\|} B \hat{n} \hat{\eta}.
\]
Therefore, the degrees of freedom on the edges are not preserved, but are transformed via the matrix $B$.

Now using Lemma 3.1 of Arnold & Winther (2002) on the reference triangle $\hat{T}$ and the fact that all norms are equivalent in a finite-dimensional space, we get
\[
    \|\hat{\eta}\|_{0,\hat{T}}^2 \sim m_{\hat{T}}^P(\hat{\eta}) + m_{\hat{T}}^e(\hat{\eta}) + m_{\hat{T}}^i(\hat{\eta}),
\]
where the constants of equivalence depend only on $\hat{T}$. The relations (3.4) and (3.5) and the equivalences $\|B\| \sim h_T$, $\|B^{-1}\| \sim h_T^{-1}$ yield
\[
    m_{\hat{T}}^e(\hat{\eta}) \sim h_T^{-4} \sum_{\hat{e} \in \mathcal{E}_{\hat{T}}} \left( \left\| \int_{\hat{e}} \frac{B \hat{n} \hat{\eta}}{\|B^{-1}\hat{n}\|} \, d\hat{s} \right\|^2 + \left\| \int_{\hat{e}} \frac{B \hat{n} \hat{\eta}}{\|B^{-1}\hat{n}\|} (\hat{s} - \hat{s}_m) \, d\hat{s} \right\|^2 \right) \sim h_T^{-6} m_{\hat{T}}^e(\hat{\eta}_h).
\]
Similar arguments give
\[
    m_T^P(\eta_h) \sim h_T^{-4} m_{\hat{T}}^P(\eta_h), \quad m_T^e(\eta_h) \sim h_T^{-4} m_{\hat{T}}^e(\eta_h), \quad \|\eta_h\|_{0,T} \sim h_T^2 \|\hat{\eta}\|_{0,\hat{T}},
\]
where we also used the equivalence $\text{det} B \sim h_T^2$. 

These equivalences used in (3.6) lead to (3.3) for a triangular element $T$.

If $T$ is a quadrilateral, we still use the matrix Piola transformation (3.4), with the affine transformation $x = B\hat{x} + b$ which maps the reference quadrilateral $\hat{T}$ to $T$. Here, we assume that $e_{\hat{T}}$ is mapped onto $e_T$ and that $e_{\hat{T}}$ has the end-points $(-1, -1)$ and $(1, 1)$. We note that the midpoint of $e_{\hat{T}}$ is given by $(0, 0)$. On one hand, we remark that (see Arnold & Winther, 2002, p. 411)

$$\text{div} \eta_h = B\text{div} \hat{\eta}.$$ 

Therefore, for $\eta_h$ in $S_q^T$ the property $\text{div} \eta_h \in [P_1(T)]^2$ is preserved. On the other hand, the property

$$\int_{e_T} (s - s_m)(\eta_h n_{e_T}) \cdot t_{e_T} \, ds = 0$$

is not preserved since

$$(\eta_h n_T) \cdot t_T = \frac{1}{\|B^{-1} \hat{n}\| \|B\hat{t}\|} (\hat{\eta} \hat{n}) \cdot B^T \hat{b}.$$ 

Therefore, we introduce the new space

$$\tilde{S}_h := \{ \tau \in S_h^s | \text{div} \tau \in [P_1(T)]^2, T \in \mathcal{T}_h \}.$$ 

Note that this space $\tilde{S}_h$ differs from $S_h^s$ only if the triangulation has some quadrilaterals because for a quadrilateral element $T$, for $\tau \in \tilde{S}_h$, we require $\text{div} \tau$ to be in $P_1$ in the whole of $T$, while it is only piecewise $P_1$ for $\tau \in S_h^s$. By Lemma 3.1 and the finite dimensionality, for $\eta_h \in \tilde{S}_h$ and $\hat{\eta} := B^{-1} \eta_h B^{-T}$ we can conclude that

$$\|\hat{\eta}\|_{0, \hat{T}}^2 \sim m_T^p(\hat{\eta}) + m_T^c(\hat{\eta}) + m_T^l(\hat{\eta}) + \left| \int_{e_{\hat{T}}} \hat{s}(\hat{\eta}) \cdot \hat{t} d\hat{s} \right|^2,$$ 

where the constants in the equivalence depend only on $\hat{T}$ but not on $T$.

The main point is to replace the last term by

$$\left| \int_{e_{\hat{T}}} \hat{s}(\hat{\eta}) \cdot B^T \hat{b} d\hat{s} \right|^2.$$ 

To start with, we rewrite $B^T \hat{b}$ as a linear combination of $\hat{t}$ and $\hat{n}$

$$B^T \hat{b} = a_1 \hat{t} + a_2 \hat{n},$$

where

$$a_1 = B^T \hat{b} \cdot \hat{t} = \|B\hat{t}\|^2 \sim h_T^2, \quad a_2 = B^T \hat{b} \cdot \hat{n},$$

and therefore $|a_2| \lesssim h_T^2$. The above identity is equivalent to

$$\hat{t} = \frac{1}{a_1} B^T \hat{b} - \frac{a_2}{a_1} \hat{n},$$

$$\hat{b} = \frac{1}{a_1} B^T \hat{b} \hat{t} - \frac{a_2}{a_1} B^T \hat{b} \hat{n},$$

$$\hat{n} = \frac{1}{a_1} B^T \hat{b} \hat{n} + \frac{a_2}{a_1} B^T \hat{b} \hat{t}.$$
and thus, we have
\[
\left| \int_{e\hat{T}} \hat{s}(\hat{n}) \cdot \hat{t} \, d\hat{\mathbf{s}} \right| \lesssim h^{-2} \int_{e\hat{T}} \hat{s}(\hat{n}) \cdot B^T B \hat{n} \, d\hat{\mathbf{s}} + \int_{\hat{T}} \hat{s}(\hat{n}) \cdot \hat{n} \, d\hat{\mathbf{s}}.
\]  
(3.9)

To estimate this last term, we use an argument already used in Lemma 3.1, namely, we take \( \hat{\nu} := (\hat{x}_2, -\hat{x}_1)^T \) and use Green’s formula as well as the property \( \varepsilon(\hat{\nu}) = 0 \) to obtain
\[
\int_{e\hat{T}} \hat{s}(\hat{n}) \cdot \hat{n} \, d\hat{\mathbf{s}} = -\int_{\partial\hat{T} \setminus e\hat{T}} (\hat{n}) \cdot \hat{\nu} \, d\hat{\mathbf{s}} + \int_{\hat{T}} \text{div} \hat{\mathbf{u}} \cdot \hat{\nu} \, d\hat{\mathbf{x}},
\]  
(3.10)

where \( \hat{\nu} \) is the subtriangle of \( \hat{T} \) with vertices \((-1, -1), (-1, 1) \) and \((1, 1) \). Since \( \hat{\nu} \) belongs to \( [P_1(\hat{T})]^2 \) and \( \partial\hat{\nu} \setminus e\hat{T} \subset \partial\hat{T} \), we deduce that
\[
\left| \int_{\partial\hat{T} \setminus e\hat{T}} (\hat{n}) \cdot \hat{\nu} \, d\hat{\mathbf{s}} \right| \lesssim m_e(\hat{\mathbf{n}}).
\]  
(3.11)

It then remains to estimate the second term on the right-hand side of (3.10). For this purpose, writing
\[
\text{div} \hat{\mathbf{u}} = \left( \alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2 + \alpha_0 \right),
\]  
with some real numbers \( \alpha_i, \beta_i, i = 0, 1, 2 \), we see that
\[
\int_{\hat{T}} \text{div} \hat{\mathbf{u}} \cdot \hat{\nu} \, d\hat{\mathbf{x}} = \int_{\hat{T}} ((\alpha_1 - \beta_2)\hat{x}_1 \hat{x}_2 + \alpha_2 \hat{x}_2^2 - \beta_1 \hat{x}_1^2 + \alpha_0 \hat{x}_2 - \beta_0 \hat{x}_1) \, d\hat{\mathbf{x}}.
\]

Since one readily checks that
\[
0 = \int_{\hat{T}} \hat{x}_1 \hat{x}_2 \, d\hat{\mathbf{x}} = 2 \int_{\hat{T}} \hat{x}_1 \hat{x}_2 \, d\hat{\mathbf{x}}, \quad \int_{\hat{T}} \hat{x}_1^2 = 2 \int_{\hat{T}} \hat{x}_1^2 \, d\hat{\mathbf{x}}, \quad \int_{\hat{T}} \hat{x}_2^2 = 2 \int_{\hat{T}} \hat{x}_2^2 \, d\hat{\mathbf{x}},
\]

the above identity becomes
\[
\int_{\hat{T}} \text{div} \hat{\mathbf{u}} \cdot \hat{\nu} \, d\hat{\mathbf{x}} = \frac{1}{2} \int_{\hat{T}} \text{div} \hat{\mathbf{u}} \cdot \hat{\nu} \, d\hat{\mathbf{x}} + \int_{\hat{T}} (\alpha_0 \hat{x}_2 - \beta_0 \hat{x}_1) \, d\hat{\mathbf{x}}.
\]

Applying Green’s formula on \( \hat{T} \), we obtain
\[
\int_{\hat{T}} \text{div} \hat{\mathbf{u}} \cdot \hat{\nu} \, d\hat{\mathbf{x}} = \frac{1}{2} \int_{\partial\hat{T}} (\hat{n}) \cdot \hat{\nu} \, d\hat{\mathbf{s}} + \int_{\hat{T}} (\alpha_0 \hat{x}_2 - \beta_0 \hat{x}_1) \, d\hat{\mathbf{x}}.
\]

On the other hand, one may remark that
\[
4(\alpha_0, \beta_0)^T = \int_{\hat{T}} \text{div} \hat{\mathbf{u}} \, d\hat{\mathbf{x}} = \int_{\partial\hat{T}} \hat{\mathbf{n}} \cdot \hat{\nu} \, d\hat{\mathbf{s}}.
\]

These two identities imply that
\[
\left| \int_{\hat{T}} \text{div} \hat{\mathbf{u}} \cdot \hat{\nu} \, d\hat{\mathbf{x}} \right|^2 \lesssim m_e(\hat{\mathbf{n}}).
\]  
(3.12)
Therefore, the integral term on $e_{\tilde{T}}$ of the proof yields
\[
\left| \int_{e_{\tilde{T}}} \hat{s}(\hat{\eta}) \cdot \hat{n} \, d\hat{s} \right|^2 \lesssim m_{T}^{c}(\hat{\eta}).
\]
This estimate used in (3.9) leads to
\[
\left| \int_{e_{\tilde{T}}} \hat{s}(\hat{\eta}) \cdot \hat{i} \, d\hat{s} \right|^2 \lesssim h_{T}^{-2} \left( \int_{e_{\tilde{T}}} \hat{s}(\hat{\eta}) \cdot B_{T} \hat{B} \hat{i} \, d\hat{s} \right)^2 + m_{T}^{c}(\hat{\eta}).
\]
Inserting this estimate in (3.8) allows us to conclude with
\[
\|\hat{\eta}\|_{0;\tilde{T}}^2 \lesssim m_{p}(\hat{\eta}) + m_{e}(\hat{\eta}) + m_{\tilde{T}}(\hat{\eta}) + \frac{1}{h_{T}^4} \left( \int_{e_{\tilde{T}}} \hat{s}(\hat{\eta}) \cdot B_{T} \hat{B} \hat{i} \, d\hat{s} \right)^2.
\] (3.13)

Similarly, we can prove that for $\hat{\eta} := B^{-1} \eta_{h} B^{-T}$
\[
m_{p}(\hat{\eta}) + m_{e}(\hat{\eta}) + m_{\tilde{T}}(\hat{\eta}) + \frac{1}{h_{T}^4} \left( \int_{e_{\tilde{T}}} \hat{s}(\hat{\eta}) \cdot B_{T} \hat{B} \hat{i} \, d\hat{s} \right)^2 \lesssim \|\hat{\eta}\|_{0;\tilde{T}}^2.
\] (3.14)

Let $\eta_{h} \in S_{h} \subset \tilde{S}_{h}$ and $T \in \mathcal{P}_{h} \setminus \mathcal{P}_{h}^{s}$, then the definition of $S_{h}$, (3.4) and (3.7) yield
\[
\int_{e_{\tilde{T}}} \hat{s}(\hat{\eta}) \cdot B_{T} \hat{B} \hat{i} \, d\hat{s} = 0.
\]
Therefore, the integral term on $e_{\tilde{T}}$ in (3.13) and (3.14) disappears and, as for triangular elements, going back to $T$, we get the norm equivalence. \(\square\)

This lemma motivates our choice for the definition of $\mathbf{r}^{0}$. Let us denote by $\phi_{p}^{ij}, p \in \mathcal{P}_{h}, 1 \leq i \leq j \leq 2$, the nodal Argyris basis function associated with the vertex $p$ and the second derivative with respect to the coordinates $x_{i}$ and $x_{j}$; the nodal Argyris basis function associated with the edge is denoted by $\phi_{e}, e \in \mathcal{E}_{h}^{s}$. This one is rescaled such that $\int_{e}(\partial \phi_{e})/(\partial n_{e}) \, ds = h_{e}$. We denote the subset of $A_{h}$ spanned by these basis functions by $A_{h}^{0}$, i.e. $A_{h}^{0} := \text{span}(\phi_{p}^{ij}, p \in \mathcal{P}_{h}, 1 \leq i \leq j \leq 2, \phi_{e}, e \in \mathcal{E}_{h}^{s})$.

**Lemma 3.8** Let $\phi \in A_{h}^{0}$, then
\[
\int_{e} J(\phi) n_{e} \, ds = 0, \quad \int_{e} (s - s_{m})(J(\phi) n_{e}) \cdot n_{e} \, ds = 0, \quad e \in \mathcal{E}_{h}^{s}.
\]

Let $\phi = \sum_{p \in \mathcal{P}_{h}} \sum_{1 \leq i \leq j \leq 2} a_{ij}^{p} \phi_{p}^{ij} + \sum_{e \in \mathcal{E}_{h}^{s}} \beta_{e} \phi_{e}$, then
\[
\int_{e} (s - s_{m})(J(\phi) n_{e}) \cdot t_{e} \, ds = \beta_{e} h_{e}, \quad e \in \mathcal{E}_{h}^{s}.
\]

**Proof.** A straightforward computation shows
\[
J(\phi) n_{e} = \begin{pmatrix} \phi_{2i}^{1e} \\ \phi_{1i}^{1e} \end{pmatrix}, \quad (J(\phi) n_{e}) \cdot n_{e} = \phi_{0e}^{1e},
\]
where \( \phi_{i,T_e} \) has to be understood as the second-order derivative of \( \phi \) in \( x_i \) and \( t_e \) direction (i.e. \( \phi_{i,T_e} = \sum_{i,j=1,2} t_{e,i} \phi_{x_i,x_j} \) as \( t_e = (t_{e,1}, t_{e,2})^T \)) and \( \phi_{t_e,T_e} \) as second-order derivative in \( t_e \) direction (i.e. \( \phi_{t_e,T_e} = \sum_{i,j=1,2} t_{e,i} t_{e,j} \phi_{x_i,x_j} \)). From this and the fact that \( \phi \) and all its first-order derivatives are zero at all vertices, the first part of the assertion follows by integration. Now, we consider the tangential component of \( J(\phi)n_e \) and find \( (J(\phi)n_e) \cdot t_e = -\phi_{n_e,T_e} \). Then, integration by parts yields
\[
\int_{e} (s - s_m)(J(\phi)n_e) \cdot t_e \, ds = \int_{e} \phi_{n_e} \, ds = \beta_e h_e.
\]

Now, we define our divergence-free contribution \( r_0 := J(\psi) \), where \( \psi \) is a linear combination of these Argyris basis functions. We note that we do not use all Argyris functions and that other choices are possible. Let \( Q: V_h \rightarrow [V_h^s]^{2 \times 2:S} \), where \([V_h^s]^{2 \times 2:S}\) is the space of symmetric tensors of which the components are in \( V_h^s \), \( V_h := \{ v \in H^1(\Omega), v|_T \in P_1(T), T \in \mathcal{T}_h \} \), be a locally defined operator such that for all \( T \in \mathcal{T}_h \)
\[
\| Qv_h - \mathcal{C} e(v_h) \|^2_{0,T} \leq C \sum_{e \in \mathcal{E}_T} h_e \| [\mathcal{C} e(v_h)]n_e \|^2_{0,e},
\]
where \( \mathcal{E}_T^{\text{loc}} \) is a suitable set of edges being in the neighbourhood of \( T \). The jump of \( \mathcal{C} e(v_h) \) through an interior \( e \) is given by \( [\mathcal{C} e(v_h)] = \mathcal{C} e(v_h)|_{T^+} - \mathcal{C} e(v_h)|_{T^-} \), where \( T^\pm \) are the two elements sharing the edge \( e \) and such that \( n_e \) is directed in the direction of \( T^+ \). For a Dirichlet boundary edge \( e \in \Gamma_D \), we set \( [\mathcal{C} e(v_h)] = 0 \) and we define \( [\mathcal{C} e(v_h)] = \mathcal{C} e(v_h)|_{T_e} \) for an edge \( e \in \Gamma_N \), where \( e \in \partial T_e \). Such an operator is uniquely defined by its values at the vertices and can be quite easily constructed. For a homogeneous material with constant Lamé parameters, it is sufficient to consider the mean value
\[
\alpha_p := \frac{1}{N_T} \sum_{T \in \mathcal{T}_p} [\mathcal{C} e(v_h)|_T](p),
\]
where \( \mathcal{T}_p \) is the set of all elements \( T \in \mathcal{T}_h \) such that \( p \) is a vertex of \( T \), and \( N_T^p \) is the number of elements of \( \mathcal{T}_p \). Then, we can define the value of \( Qv_h \) at a vertex \( p \) by
\[
Qv_h(p) := \alpha_p, \quad p \notin \Gamma_N, \quad Qv_h(p) := t_p^T \alpha_p t_p t_p^T, \quad p \in \Gamma_N,
\]
where \( t_p \) is the unit tangential vector at \( p \) if it is well-defined and the zero vector if \( p \) is a corner point of \( \Gamma_N \). We note that this definition guarantees that \( Qv_hn_e = 0 \) for all Neumann boundary edges \( e \subset \Gamma_N \). For this choice, standard scaling arguments lead to
\[
\| Qv_h - \mathcal{C} e(v_h) \|^2_{0,T} \leq C \sum_{e \in \mathcal{E}^{\text{loc}}} h_e \| [\mathcal{C} e(v_h)] \|^2_{0,e}.
\]
Then, the tangential component of \( \mathcal{C} e(v_h) \) can be removed in the upper bound. To see this, we start with the observation that each symmetric \( 2 \times 2 \) tensor \( \xi \) can be written in terms of two linear independent vectors \( r_1 \) and \( r_2 \), where we assume that both vectors have length one. More precisely,
\[
\xi = ar_1r_1^T + br_2r_2^T + c(r_1r_2^T + r_2r_1^T),
\]
where \( a := s_1^T \xi s_2/(r_1^T s_2)^2 \), \( b := s_1^T \xi s_1/(r_2^T s_1)^2 \) and \( c := s_2^T \xi s_1/(r_2^T s_1 r_1^T s_2) \), where \( s_1 \) and \( s_2 \) are orthogonal to \( r_1 \) and \( r_2 \), respectively, and have length one. Using this for the choice \( r_1 := n_e \) and \( r_2 := t_e \), we find that

\[
\| \mathcal{E}(v_h) \|_{0;e}^2 \leq C ( \| n^T \mathcal{E}(v_h)n_e \|_{0;e}^2 + \| [T^T \mathcal{E}(v_h) n_e]_e \|_{0;e}^2 + \| [T^T \mathcal{E}(v_h) t_e]_e \|_{0;e}^2 )
\]

\[
\leq C ( \| \mathcal{E}(v_h) n_e \|_{0;e}^2 + \| [T^T \mathcal{E}(v_h) t_e]_e \|_{0;e}^2 ).
\]

Now, we consider the second term on the right side in more detail. Observing that \( \text{grad}(v_h) t_e = 0 \), the definition of \( \mathcal{E} \) yields

\[
[T^T \mathcal{E}(v_h) t_e] = 2 \mu [T^T \mathcal{E}(v_h) t_e] + \lambda [\text{tr} \mathcal{E}(v_h)]
\]

\[
= 2 \mu [T^T \text{grad}(v_h) t_e] + \lambda [\text{tr} \mathcal{E}(v_h)] = \lambda [\text{tr} \mathcal{E}(v_h)]
\]

\[
= \lambda [n^T \text{grad}(v_h) n_e] = \frac{\lambda}{2 \mu + \lambda} [n^T \mathcal{E}(v_h) n_e]
\]

and thus \( \| \mathcal{E}(v_h) \|_{0;e} \leq C \| \mathcal{E}(v_h) n_e \|_{0;e} \). We note that more sophisticated choices of \( Q \) are possible. Then weights depending on the area of the elements and on the Lamé parameters in case of discontinuities enter.

Now, we define

\[
\psi := \sum_{p \in \mathcal{T}_h} \sum_{1 \leq i \leq j \leq 2} \alpha_{ij} \phi_p^{ij} + \sum_{e \in \mathcal{E}_h^n} \beta_e \phi_e,
\]

(3.16)

where the coefficients are given by \( \beta_e := 1/h_e \int_e (s - s_m) \sigma_h n_e \cdot t_e \, ds \) and \( \alpha_p^{11} := (Qu_h(p))_{22}, \alpha_\mu^{12} := -(Qu_h(p))_{12}, \alpha_p^{22} := (Qu_h(p))_{11} \). The discrete surface traction \( \sigma_h n_e \) is well-defined on \( e \in \mathcal{E}_h^n \). We note that if the two elements \( T_1, T_2 \in \mathcal{T}_h \) sharing the edge \( e \in \mathcal{E}_h \) are triangles, then \( \{ \sigma_h n_e \} \in P_0(e) \) and thus \( \int_e (s - s_m) (\sigma_h n_e) \cdot t_e \, ds = 0 \). This observation motivates the use of the edges \( e \in \mathcal{E}_h^n \) in the definition (3.16), but to skip the ones in \( \mathcal{E}_h \). By construction of \( Q \), it is also easy to see that \( J(\psi)n_e = 0 \) for all \( e \subset T_N \) and therefore \( J(\psi) \in H_0^0(\text{div}; \Omega) \).

Now, we are in the setting to define our error estimator

\[
\eta^2 := \sum_{T \in \mathcal{T}_h} \eta_T^2 := \sum_{T \in \mathcal{T}_h} \| \mathcal{E}^{-1/2} (\mathcal{T} + J(\psi) - \mathcal{A}) \|_{0;T}^2.
\]

(3.17)

Because \( J(\psi) \in H_0^0(\text{div}; \Omega) \), we are in the setting of Theorem 3.4 and \( \eta \) provides an upper bound of the discretization error. The following theorem shows that \( \eta_T \) provides a local lower bound.

**Theorem 3.9** There exists a constant independent of the mesh size such that

\[
c \eta_T^2 \leq \| \mathcal{E}^{1/2} (u - u_h) \|_{0;\omega_T}^2 + \text{osc}_{1;\omega_T}(f), \quad T \in \mathcal{T}_h,
\]

where \( \omega_T \) is the union of all elements \( \tilde{T} \in \mathcal{T}_h \) such that \( \partial T \cap \partial \tilde{T} \neq \emptyset \).

**Proof.** We start with the observation that the components of \( \mathcal{A}_h = \mathcal{E}(u_h) \) restricted to \( T \in \mathcal{T}_h \) are in \( P_1(T) \). Moreover, the definition of \( \beta_e \) in (3.16) and Lemma 3.8 yield that

\[
\int_{e_T} (s - s_m) ((J(\psi) - \mathcal{A}_h)n_e) \cdot t_e \, ds = 0, \quad e_T \in \mathcal{E}_h^n.
\]
and thus $J(\psi) - \sigma_h$ is an element in $S_T$ if $T \in \mathcal{T}_h \cap \mathcal{T}_h^s$ and in $S_T^d$ for $T \in \mathcal{T}_h \setminus \mathcal{T}_h^s$, and we can apply Lemma 3.7 to $\tau_h + J(\psi) - \sigma_h$. Using Lemma 3.7, we have to estimate the quantities $m_T^e(\cdot), m_T^p(\cdot)$ and $m_T^l(\cdot)$. In a first step, we consider $m_T^l(\cdot)$

$$m_T^l(\tau_h + J(\psi) - \sigma_h) = \left\| \int_T (\tau_h + J(\psi) - \sigma_h) \, dx \right\|^2 = \left\| \int_T J(\psi) \, dx \right\|^2,$$

since the definition of $\tau_h$ yields $\int_T \tau_h \, dx = \int_T \sigma_h \, dx$. Let $\nu \in \text{RM}^e(T)$, where $\text{RM}^e(T) := \text{span}\{(x_2, x_1)^T, (x_1, 0)^T, (0, x_2)^T\}$, then we find

$$\int_T J(\psi) : \nabla \nu \, dx = - \int_T \text{div}(J(\psi)) \cdot \nu \, dx + \int_{\partial T} (J(\psi) \mathbf{n}_T) \cdot \nu \, ds = \int_{\partial T} (J(\psi) \mathbf{n}_T) \cdot \nu \, ds.$$

Using Lemma 3.8 and the definition (3.16) of $\psi$, this right-hand side is zero. As the set $\{\nabla \nu | \nu \in \text{RM}^e(T)\}$ spans the set of symmetric constant matrices, we find that

$$\int_T J(\psi) \, dx = 0$$

and therefore

$$m_T^l(\tau_h + J(\psi) - \sigma_h) = 0.$$

Secondly, we consider $m_T^p(\cdot)$. The definition of $\tau_h$ gives $\tau_h(p) = 0$, and the assumption (3.15) on the operator $Q$ gives

$$m_T^p(\tau_h + J(\psi) - \sigma_h) = m_T^p(J(\psi) - \sigma_h) = \sum_{p \in \mathcal{P}_T} \| Qu_h(p) - \mathcal{C} \mathcal{E}(u_h)|_{T}(p) \|^2$$

$$\leq \frac{C}{|T|} \| Qu_h - \mathcal{C} \mathcal{E}(u_h) \|_{0,T}^2 \leq \frac{C}{|T|} \sum_{e \in \mathcal{E}^e_T} h_e \| \sigma_h \mathbf{n}_e \|_{0,e}^2.$$

In a third step, we focus on $m_T^e(\cdot)$. The definition of $\tau_h$ allows us to replace $\tau_h \mathbf{n}_e$ by $\mathbf{g}_e$. Moreover, using Lemma 3.8 and the definition (3.16) of $\psi$, we find that $J(\psi) \mathbf{n}_e$ does not contribute to the integrals in the definition of $m_T^e(\cdot)$. Then, the Cauchy–Schwarz inequality yields

$$m_T^e(\tau_h + J(\psi) - \sigma_h) \leq C \sum_{\mathcal{E}^e_T} h_e \| \mathbf{g}_e - \sigma_h \mathbf{n}_e \|_{0,e}^2.$$

By Lemma 3.7, we find the upper bound

$$\eta_T^2 \leq C \sum_{\mathcal{E}^e_T} h_e \| \mathbf{g}_e - \sigma_h \mathbf{n}_e \|_{0,e}^2.$$

This term is a part of the equilibrated error estimator, and we refer to Ainsworth & Oden (2000) for an upper bound in terms of the discretization error.

We note that the higher-order terms are, in general, expressed in terms of $\| f - \Pi_0 f \|_{0,T}$ instead of $\| f - \Pi_1 f \|_{0,T}$. Following the lines in Verfürth (1996) and using element bubble functions as weights, we can replace $\| f - \Pi_0 f \|_{0,T}$ by $\| f - \Pi_1 f \|_{0,T}$.
Remark 3.10 By defining in a local postprocess a new approximation for the stress by $\sigma_{h}^{\text{post}} := \frac{1}{2}(\sigma_{h} + \tau_{h} + J(\psi))$, we find that the error estimator $0.5\eta$ is up to higher-order terms equal to the error $\|\varepsilon^{-1/2}(\sigma - \sigma_{h}^{\text{post}})\|_{0}$, see, e.g. Luce & Wohlmuth (2004) for the Laplace equation, and Ladevèze & Rougeot (1997) for more general concepts and basic techniques.

Remark 3.11 We note that the constant in the higher-order term for the upper bound does not degenerate in the nearly incompressible limit. In that case, the standard displacement-based formulation shows volume locking and has to be replaced by a stable approach. This can be realized, e.g. by replacing the bilinear form $a(\cdot, \cdot)$ by a mesh-dependent one. Then the upper bound also holds. This observation makes the approach quite interesting in the nearly incompressible range.

Remark 3.12 In the case of inhomogeneous boundary conditions which cannot be resolved by the finite-element approximations, additional terms appear in the bounds. These terms reflect the boundary oscillations and are of higher order if the boundary conditions are smooth enough.

4. Local realization of the error estimator

In this section, we provide a possibility to compute the error estimator from $\mathbf{g}_e$ by solving scalar equations and using basis functions on the reference element. Without loss of generality, we restrict ourselves to the case of a simplicial triangulation $T_h = T_h^s$. Using the Piola transformation for tensor quantities, the nodal basis functions on the reference element can be transformed. However, this does not guarantee an affine equivalent family. Here, we proceed differently and do not use a nodal basis with respect to the specified degrees of freedom, but a basis which forms a lower triangular stiffness matrix with many zero entries.

Let us denote by $\lambda_e$ the nodal quadratic Lagrange finite-element functions associated with the midpoint $m_e$ of the edge $e$, i.e. $\lambda_e(m_{\tilde{e}}) = \delta_{e\tilde{e}}$. Concerning the basis functions for the Argyris element, there exists locally a second set of degrees of freedom such that the arising nodal basis functions are affine equivalent, see, e.g. Ciarlet (1978). We use the symbol $\phi$ for the nodal Argyris basis functions associated with the original set of degrees of freedom and the symbol $\tilde{\phi}$ for the ones associated with the affine set. The main difference compared to the standard set is that the derivatives at the vertices $p_j$, 1 $\leq j \leq 3$, are not prescribed with respect to the coordinates $x_i$, 1 $\leq i \leq 2$, but with respect to $p_{j+1} - p_j$ and $p_{j+2} - p_j$, see Fig. 2.

![Fig. 2. Local notation on an element.](image-url)
Here and in the following, the notations $j + 1$ and $j + 2$ have to be understood in the sense of $\text{mod}_3(j + 1)$ and $\text{mod}_3(j + 2)$, respectively, and 0 is identified with 3. The normal derivatives at the midpoints are also replaced. The degrees of freedom associated with the normal derivative at the midpoint $m_{e_i}$ of the edges $e_i$ is substituted by the mean value of the derivative in direction $p_i - m_{e_i}$. We specify elementwise the following set of 24 symmetric matrix-valued functions:

\[
\begin{align*}
\varphi_T &:= \lambda_{e_i} t_{e_i} t_{e_i}^T, \quad 1 \leq i \leq 3, \\
\varphi_{e_i}^{n:0} &:= \lambda_{e_i} n_{e_i} n_{e_i}^T, \quad 1 \leq i \leq 3, \\
\varphi_{e_i}^{T:0} &:= \lambda_{e_i} (t_{e_i} n_{e_i}^T + n_{e_i} t_{e_i}^T), \quad 1 \leq i \leq 3, \\
\varphi_{e_i}^{n:1} &:= J \left( \tilde{\varphi}_{p_i+2}^{j+1} - \tilde{\varphi}_{p_i+1}^{j+2} \right), \quad 1 \leq i \leq 3, \\
\varphi_{e_i}^{T:1} &:= J (\varphi_{e_i}), \quad 1 \leq i \leq 3, \\
\varphi_{p_i}^{k;j} &:= J (\varphi_{p_i}^{k;j}), \quad 1 \leq i \leq 3, \quad 1 \leq k \leq j \leq 2,
\end{align*}
\]

where $e_i$ and $p_i$ are the edges and vertices of the triangles, and $\tilde{\varphi}_{p_i}^{j}$ is the nodal basis function satisfying

\[\nabla \tilde{\varphi}_{p_i}^{j} (p_i) \cdot (p_j - p_i) = 1, \quad i \neq j,\]

whereas all other degrees of freedom are zero. Upper indices 0 and 1 stand for the zero- and first-order moments and upper indices $n$ and $t$ stand for the normal and tangential direction, respectively. It is obvious that all given functions are in $S_T$ (note that the nodal function $\tilde{\varphi}_{p_i}^{j}$ is a linear combination of the standard nodal Argyris basis function). We show that these 24 functions generate any element of $S_T$. Moreover, using this set, we shall see that the matrix defining $\tau_h + J (\psi)$ is a lower triangular matrix that is easily computable. Indeed, in the rest of this section, we provide the construction which will be carried out in six steps. We set locally

\[\tau_h^g = \sum_{i=1}^{3} \left( a_i \varphi_T^i + b_i \varphi_{e_i}^{n:0} + c_i \varphi_{e_i}^{T:0} + d_i \varphi_{e_i}^{n:1} + e_i \varphi_{e_i}^{T:1} + \sum_{1 \leq k \leq j \leq 2} f_i^{k;j} \varphi_{p_i}^{k;j} \right).\]

To start, we consider the degrees of freedom associated with the lowest and first-order moment in normal direction. In the first step, we define the coefficients $b_i$ by

\[b_i := \frac{\int_{e_i} g_T \cdot n_T \, ds}{\int_{e_i} (\varphi_{e_i}^{n:0} n_T) \cdot n_T \, ds} = \frac{\int_{e_i} g_T \cdot n_T \, ds}{\int_{e_i} \lambda_{e_i} \, ds} = \frac{3}{2 h_{e_i}} \int_{e_i} g_T \cdot n_T \, ds,\]

and observe that $(\varphi_{e_i}^{n:0} n_T) \cdot n_T = (\lambda_{e_i} n_T n_{e_i}^T n_T) n_T = \lambda_{e_i} n_T n_T = \lambda_{e_i}$. In the second step, we set the coefficients $d_i$ to

\[d_i := \frac{\int_{e_i} (s - s_m) g_T \cdot n_T \, ds}{\int_{e_i} (s - s_m) (\varphi_{e_i}^{n:1} n_T) \cdot n_T \, ds} = \frac{1}{h_{e_i}} \int_{e_i} (s - s_m) g_T \cdot n_T \, ds,\]

and use the definition of $\varphi_{e_i}^{n:1}$. We note that the steps 1 and 2 are independent. Now, we define in steps 3 and 4 the coefficients associated with the moments in tangential direction. Let $q_1 := \sum_{i=1}^{3} d_i \varphi_{e_i}^{n:1}$, then
we define in step 3
\[
\epsilon_i := \int_{e_i} \left( g_T - q_1 n_T \right) \cdot t_T \, ds = \frac{3}{2h_{e_i}} \int_{e_i} \left( g_T - q_1 n_T \right) \cdot t_T \, ds.
\]
Here, we have used \((t_T n_T^0 + n_T t_T^0)n_T \cdot t_T = t_T \cdot t_T = 1\) and the definition of \(\varphi_{e_i}^{t_0}\). Step 4 is given by
\[
\epsilon_i := \int_{e_i} \left( s - s_m \right) \left( g_T - q_1 n_T \right) \cdot t_T \, ds = \frac{1}{h_{e_i}} \int_{e_i} \left( s - s_m \right) \left( g_T - q_1 n_T \right) \cdot t_T \, ds.
\]
We note that the ordering of the steps 3 and 4 is arbitrary. Step 5 is independent of all the previous four steps. Following the definition (3.16), the coefficients \(f_i^{k_j}\) are defined by
\[

t_i^{11} := (Q u_h (p_i))_{22},
\]
\[

t_i^{12} := -(Q u_h (p_i))_{12},
\]
\[

t_i^{22} := (Q u_h (p_i))_{11}.
\]
Finally in the last step, we fix the degrees of freedom associated with the interior of the element. Let \(q_2 := \tau_h - \sum_{i=1}^{3} a_i \varphi_{e_i}^T\), then we define
\[
a_i := \frac{\int_T (\sigma_h - q_2) : (n_{e_{i+1}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T) \, dx}{\int_T \varphi_{e_i}^T : (n_{e_{i+1}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T) \, dx} = \frac{-3h_i^2 h_{e_{i+1}} h_{e_{i+2}}}{8 |T|^3} \int_T (\sigma_h - q_2) : (n_{e_{i+1}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T) \, dx.
\]
Here, we have used that
\[
(n_{e_{i+1}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T) : t_{e_i} t_{e_i}^T = 2n_{e_{i+1}} n_{e_{i+2}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T t_{e_i} t_{e_i} = -4 |T|^2 / (h_i^2 h_{e_{i+1}} h_{e_{i+2}}).
\]
lemma 4.1 We find that
\[
\tau_h = (\tau_h + J (\psi)) |T|.
\]
\textbf{Proof.} A straightforward computation taking into account the definition (4.1) and the properties of the Argyris functions and the Airy operator shows the equality. We note that the elements \(n_{e_{i+1}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T\), \(1 \leq i \leq 3\), form a basis of the symmetric matrices with constant coefficients. Moreover, we observe that
\[
(n_{e_{i+1}} n_{e_{i+2}}^T + n_{e_{i+2}} n_{e_{i+1}}^T) t_{e_j} t_{e_j}^T = 0 \quad \text{for} \quad j \in \{i+1, i+2\}.\]
Due to this biorthogonality, the coefficients \(a_i\) decouple and can be computed by scalar equations. \(\square\)

\textbf{Remark 4.2} The construction of the error estimator and the proof of Theorem 3.9 show that we can replace the definition of \(\epsilon_i\) by
\[
\frac{\int_{e_i} (s - s_m)(-q_1 n_T) \cdot t_T \, ds}{\int_{e_i} (s - s_m) \varphi_{e_i}^T n_T \cdot t_T \, ds} = \frac{1}{h_{e_i}} \int_{e_i} (s - s_m)(q_1 n_T) \cdot t_T \, ds
\]
and still get upper and lower bounds for the discretization error.

\textbf{Remark 4.3} The construction of the error estimator can also be applied for quite general meshes. The proof of the upper bound shows that it also holds true for general quadrilateral meshes. In the lower bound, we exploit the property that all elements are affine equivalent to the unit square or a reference triangle, hence for shortness the affine equivalence has been assumed in the whole paper.
5. Numerical results

5.1 L-shaped domain

In the first example, we consider the problem

\[- \text{div } \sigma = 0 \quad \text{in } \Omega\]

with Dirichlet boundary conditions computed from the exact solution, where \(\Omega\) is the L-shaped domain described by the polygon \((0, 0), (1, 0), (1, 1), (1, -1), (-1, -1), (0, -1)\). We set the Lamé parameters \(\lambda = 4.6\) and \(\mu = 0.63\), the material coefficients of lead. The exact solution for this problem is

\[u(r, \theta) = \left( \begin{array}{c} r^\alpha (-1 + A_\alpha) \sin(\alpha \theta) + \sin((\alpha - 2) \theta) \\ r^\alpha (-\cos(\alpha \theta) + \cos((\alpha - 2) \theta)) \end{array} \right)\]

with \(A_\alpha = \frac{2(\lambda + 3\mu)}{(\lambda + \mu)\alpha}\) and \(\alpha = 0.5788772315\).

Figure 3 shows the initial and refined meshes. Here, we use a quadrilateral mesh and the refinement is based on a standard maximum strategy. The solution has a singularity at the re-entrant corner, so the adapted meshes are much more refined there.

One computes the convergence rates \(s\), as the square root of the quotient of the number \(N_i\) of nodes in step \(i\) and the number of nodes in step \(i + 1\) to the power of \(s\) is the quotient of the error \(\text{err}_{i+1}\) in step \(i + 1\) and the error in step \(i\), i.e.

\[
\left( \frac{N_i}{N_{i+1}} \right)^s = \frac{\text{err}_{i+1}}{\text{err}_i}.
\]

The averaged value is 1.0816 for the estimated and 1.0898 for the exact error for adaptive refinement. For uniform refinement, the square root of the quotient of the number of nodes in step \(i\) and the number of nodes in step \(i + 1\) is approximately 1/2, so we compute \(s\) from \((1/2)^s = \text{err}_{i+1}/\text{err}_i\). In that case, the convergence rates tend to \(\alpha\) as theoretically expected.

Figure 4 shows on the left the ratio between estimated and exact error. Asymptotically, this ratio tends to a constant value which is a measure for the quality of the error estimator. The closer it is to one, the better is the error estimator because that value is the inverse of the constant in the upper bound for the discretization error. Here, our ratio is around 3, so our estimator can be considered quite satisfactory.
On the right, the estimated and exact error in the energy norm for adaptive and uniform refinement is displayed. The slopes of the lines are the convergence rates computed before. Observe the much better error reduction for the adaptive algorithm.

5.2 Plate with hole under traction

In the last example, we consider a plate with a circular hole, subject to a shearing load on the right side, see Carstensen et al. (2005).

Because of the symmetry, it is sufficient to consider only a quarter of the plate, if one enforces symmetry boundary conditions on the axis of symmetry of the entire plate, e.g. on the left homogeneous Dirichlet boundary conditions for the first component and homogeneous Neumann boundary conditions for the second component and below the other way around.

On the right, we have Neumann boundary conditions with $\sigma \cdot n = (\sigma_0, 0)^T$ and homogeneous Neumann boundary conditions elsewhere. For this example, we use $\sigma_0 = 1$, $E = 100000$ and $\nu = 0.3$. In Fig. 5, the problem setting and initial mesh as well as adaptively refined meshes are depicted.
As expected, the elements around the hole are much more refined because the stress gradients are higher there.

In the case of an infinitely large, thin plate, a closed-form solution of this problem exists. Then, the stress normal to the vertical plane of symmetry at point $P$ (see Fig. 5) is $\sigma_{xx} = 3\sigma_0$.

Figure 6 shows $\sigma_{xx}$ as a function of the number of degrees of freedom for the adaptive and the uniform algorithm. In both cases, the value tends to the exact solution, but for the adaptive algorithm, we have a much faster convergence.

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