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On deformations of transversely homogeneous foliations

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Abstract

A transversely homogeneous foliation is a foliation whose transverse model is a homogeneous space G/H. In this paper we consider the class of transversely homogeneous foliations $\mathscr{F}$ on a manifold M which can be defined by a family of 1-forms on M fulfilling the Maurer–Cartan equation of the Lie group G. This class includes as particular cases Lie foliations and certain homogeneous spaces foliated by points. We develop, for the foliations belonging to this class, a deformation theory for which both the foliation $\mathscr{F}$ and the model homogeneous space G/H are allowed to change. As the main result we show that, under some cohomological assumptions, there exist a versal space of deformations of finite dimension for the foliations of the class and when the manifold M is compact. Some concrete examples are discussed. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Deformations; Foliations; Homogeneous spaces

0. Introduction

A foliation $\mathscr{F}$ on a manifold M is called a transversely homogeneous foliation modeled on the homogeneous space G/H if it can be defined by a family of local submersions $f_i: U_i \to G/H$ such that $\{U_i\}$ is an open covering of M and $f_j = g_{ji} \cdot f_i$ with $g_{ji} \in G$ and where the dot denotes the action on G/H of the elements of G. If the subgroup H of G is trivial one says that $\mathscr{F}$ is a Lie foliation modeled on the Lie group G. In this paper we discuss the deformations of Lie foliations, and more generally

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of a class of transversely homogeneous foliations, for which both the foliation $\mathscr{F}$ and the model Lie group G, or the model homogeneous space G/H are allowed to change. The main purpose is to show the existence of a versal space of deformations of finite dimension.

The theory of deformations of geometric structures was initiated by the work of Kodaira and Spencer [10] when they tried to extend to compact complex manifolds of arbitrary dimension the classical construction of moduli spaces of Riemann surfaces. They proved that, under some cohomological assumptions, there exists a family of deformations of the given complex structure parametrized by a smooth finite dimensional space which is versal in the sense that each other complex structure close enough to the original one is isomorphic to a complex structure contained in the family. In [11] Kuranishi proved a completely general theorem of existence of a versal space for deformations of compact complex manifolds but obtaining as a parameter space an analytic space that can be singular. Subsequently, similar theories of deformations were constructed for some other geometric structures. For instance Girbau et al. [7] proved the existence of a versal space of deformations for each transversely holomorphic foliation on a compact manifold and Griffiths [8] developed a deformation theory for a large class of G-structures.

In all the above constructions, the geometric structure under consideration has a unique local model which remains fixed under the deformations. This fact permits to identify the space of infinitesimal deformations with the first cohomology group of the sheaf Θ of the infinitesimal transformations of the structure. Moreover, the sheaf Θ admits a resolution associated to an elliptic complex. The versal space is then obtained using a Hodge decomposition theorem for that complex.

Our interest in this subject comes mainly from a question raised by Ghys [6], where he conjectures the existence of a versal space of deformations for every Riemannian foliation on a compact manifold. This problem is more intricate because the deformations one has to consider are no longer locally trivial and the potential theoretic methods do not apply anymore. In this article, we have restricted ourselves to the particular case of Lie foliations which are in particular Riemannian foliations. Within this category, even if the local model is allowed to vary, we are able to prove the existence of a finite-dimensional versal space. If the model group is kept fixed and one just deforms the foliations the existence of a versal space is well known (cf. [16,6]).

One of the differences with other deformation theories is that we cannot work directly with the cocycle $\{g_{ji}\}$ and define deformations of the Lie foliation in terms of deformations $\{g^s_{ji}\}$ of this cocycle with values in a family G_s of Lie groups. Such a definition would be too restrictive since there is not a version with parameters of Ado's theorem. Instead of using this point of view we consider the Lie foliation $\mathscr{F}$ as defined by a differential system $\omega^1 = \cdots = \omega^q = 0$, where ω^i are 1-forms on M which are linearly independent at each point and fulfill the Maurer–Cartan equation of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Then the deformations of $\mathscr{F}$ are given by families ω^i_s fulfilling the Maurer–Cartan equation of a suitable family of Lie algebras $\mathfrak{g}_s$. The existence of a versal space for Lie foliations relies deeply in the fact that the deformations of a Lie algebra (of a fixed dimension) depend only on a finite number of parameters.

The construction made for describing the deformations of Lie foliations can be extended to study the deformations of a certain class of transversely homogeneous foliations. Let $e_1, \ldots, e_m$ be a fixed basis of $\mathfrak{g}$ such that $e_{q+1}, \ldots, e_m$ generates the Lie subalgebra $\mathfrak{h} = \mathrm{Lie}(H)$ and let K^k_{ij} be the structure constants of G corresponding to this basis. With this choice (cf. [1]) a transversely homogeneous foliation $\mathscr{F}$ on M modeled on G/H is locally determined by a collection of 1-forms

$\omega^1, \dots, \omega^q, \omega^{q+1}, \dots, \omega^m$ fulfilling the Maurer–Cartan equations with coefficients K^k_{ij} and such that $\omega^1, \dots, \omega^q$ are linearly independent at each point. Then $\mathscr{F}$ is defined by the differential system $\omega^1 = \cdots = \omega^q = 0$. The transversely homogeneous foliations considered here are those for which the forms ω^i can be defined globally on M. This condition is equivalent to the triviality of a certain principal H-bundle over M naturally associated to $\mathscr{F}$. This class of foliations includes the homogeneous spaces (foliated by points) for which the fiber bundle $G \to G/H$ is trivial. For instance we analyze in the last section the deformations of different homogeneous structures over the circle $\mathbb{S}^1$.

This paper is organized as follows: in the first two sections we define the class of transversely homogeneous foliations that we consider and their deformations. In Section 3 we construct a differential complex $\mathscr{A}$ whose first cohomology group $H^1(\mathscr{A})$ describes the infinitesimal deformations of the given foliation. The complex $\mathscr{A}$ is not elliptic but we prove in Section 4 that it behaves like an elliptic one and that it admits a Hodge decomposition. This implies in particular that the cohomology $H^*(\mathscr{A})$ has finite dimension when the manifold M is compact. Using the Hodge decomposition for the complex $\mathscr{A}$ and following the arguments of Kuranishi as they are explained in [2,13] we prove in Section 5 the existence of a versal space of deformations under some cohomological conditions and when the manifold M is compact. More precisely, we prove that (i) if $H^1(\mathscr{A}) = 0$ then each deformation of $\mathscr{F}$ is trivial and (ii) if $H^2(\mathscr{A}) = 0$ then the germ $(H^1(\mathscr{A}),0)$ parametrizes a family of deformations $\mathscr{F}_s$ of $\mathscr{F} = \mathscr{F}_0$ which is versal in the sense that for any other family $\mathscr{F}_{t'}$ of deformations of $\mathscr{F}$ parametrized by $(T',0)$ there is a smooth map $\varphi : (T',0) \to (H^1(\mathscr{A}),0)$ such that $\mathscr{F}_{t'}$ and $\mathscr{F}_{\varphi(t')}$ are isomorphic. In the case of a Lie foliation and without any cohomological assumption we prove the existence of a family of deformations $\mathscr{F}_s$ of $\mathscr{F}$ parametrized by an analytic space S of finite dimension (eventually singular) which is weakly versal: the elements of the family represent all the Lie foliations close enough to $\mathscr{F}$ up to isomorphism. If S is smooth then the family is versal. Finally, in Section 6, we discuss several examples to illustrate the theory: the three homogeneous structures on $\mathbb{S}^1$ (Abelian, affine and projective), the Lie foliations modeled on an Abelian Lie group and the homogeneous foliations, that is foliations on a manifold $M = \Gamma \backslash G$, where Γ is a discrete and cocompact subgroup of the Lie group G, whose leaves are the left cosets defined by a subgroup H of G.

All the objects considered along the paper will be assumed to be of class C^∞.

1. Preliminaries

Let $\mathfrak{g}$ be a Lie algebra of dimension m and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. We fix a basis $e_1, \dots, e_m$ of $\mathfrak{g}$ such that $e_{q+1}, \dots, e_m$ span $\mathfrak{h}$ and denote by $\theta^1, \dots, \theta^m$ the corresponding dual basis. One has $[e_i, e_j] = \sum_k K^k_{ij} e_k$, where the *structure constants* K^k_{ij} fulfill the relations

$$K^k_{ij} = -K^k_{ji}, \tag{1}$$

$$\sum_i (K^k_{ij} K^i_{rs} + K^k_{ir} K^i_{sj} + K^k_{is} K^i_{jr}) = 0 \quad \text{(Jacobi identity)}, \tag{2}$$

$$K^k_{ij} = 0 \text{ if } k \leqslant q \quad \text{and} \quad q+1 \leqslant i,j. \tag{3}$$

The set of constants K_{ij}^k satisfying (1) and (2) determines the Lie algebra structure of $\mathfrak{g}$ while (3) states that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. We denote by G the simply connected Lie group with Lie algebra $\mathfrak{g}$ and by H the connected Lie subgroup of G corresponding to the Lie subalgebra $\mathfrak{h}$.

We shall denote by θ the $\mathfrak{g}$-valued 1-form on G which is the identity over the left-invariant vector fields on G, i.e. $\theta = \sum_k \theta^k \otimes e_k$. Let $\omega = \sum_k \omega^k \otimes e_k$ be a $\mathfrak{g}$-valued 1-form on a manifold M. An element $g \in G$ transforms ω into the $\mathfrak{g}$-valued form $\mathrm{Ad}_g\, \omega$ where $\mathrm{Ad}_g\, \omega(X) = \mathrm{Ad}_g\, (\omega(X))$ for any vector field X on M. Once the basis $e_1, \ldots, e_m$ of $\mathfrak{g}$ has been fixed, we shall identify ω with the n-tuple of scalar 1-forms $(\omega^1, \ldots, \omega^m)$. In particular $\theta = (\theta^1, \ldots, \theta^m)$.

Definition 1. Let a $\mathfrak{g}$-valued 1-form $\omega = (\omega^1, \ldots, \omega^m)$ on a connected manifold M be given. Assume that ω fulfills the Maurer–Cartan equation $\mathrm{d}\omega + 1/2[\omega,\omega] = 0$, i.e.

$$\mathrm{d}\omega^k = -\frac{1}{2} \sum_{i,j=1}^{m} K_{ij}^k\, \omega^i \wedge \omega^j \tag{4}$$

and that $\omega^1, \ldots, \omega^q$ are linearly independent. Then the differential system $\omega^1 = \cdots = \omega^q = 0$ is integrable and defines a codimension q foliation $\mathscr{F}$. We shall say that $\mathscr{F}$ is a $\mathfrak{g}/\mathfrak{h}$-foliation defined by the $\mathfrak{g}$-valued form ω.

Main example. Let $M = G$. Then $\theta = (\theta^1, \ldots, \theta^m)$ defines a $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}_{G,H}$ whose leaves are the left cosets of H.

Remark 1. The notion of $\mathfrak{g}/\mathfrak{h}$-foliation includes several classes of geometric structures

(a) If $q = \dim M$ and H is closed then a $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$ defines a structure of locally homogeneous space on M; that is, the manifold M is locally modeled on the homogeneous space G/H with coordinate changes given by left translations by elements of G and $\mathscr{F}$ is the foliation by points. The homogeneous space G/H is endowed with a $\mathfrak{g}/\mathfrak{h}$-foliation when the projection $G \to G/H$ admits a global section.
(b) When $\mathfrak{h} = 0$, $\mathfrak{g}/\mathfrak{h}$-foliations are just Lie foliations modeled over G. For instance a nonsingular closed 1-form ω on M defines a Lie foliation modeled over $\mathbb{R}$.
(c) If H is closed then a $\mathfrak{g}/\mathfrak{h}$-foliation is a transversely homogeneous foliation modeled over the homogeneous space G/H. Every transversely homogeneous foliation is given locally by a collection of 1-forms $\omega^1, \ldots, \omega^m$ fulfilling (4) (cf. [1]). If these forms are global then they define a $\mathfrak{g}/\mathfrak{h}$-foliation. This is the case if $H^1(M, H) = 0$ (cf. [1]).
(d) In general, when H is not necessarily closed, a $\mathfrak{g}/\mathfrak{h}$-foliation is a locally transversely homogeneous foliation as it is defined in [12] or [9].

Let $\mathscr{F}$ be a $\mathfrak{g}/\mathfrak{h}$-foliation on M defined by ω. A map $\varphi: N \to M$ transverse to $\mathscr{F}$ induces a $\mathfrak{g}/\mathfrak{h}$-foliation $\varphi^*\mathscr{F}$ on N which is defined by $\varphi^*\omega$. We say that $\varphi^*\mathscr{F}$ is the *pull-back* of $\mathscr{F}$ by φ. In particular, the universal covering space $\tilde{M}$ of M is endowed with the $\mathfrak{g}/\mathfrak{h}$-foliation $\tilde{\mathscr{F}}$ defined by $\pi^*\omega$ where $\pi: \tilde{M} \to M$ is the canonical projection. The following proposition states that the $\mathfrak{g}/\mathfrak{h}$-foliation $\tilde{\mathscr{F}}$ on $\tilde{M}$ is a pullback of the $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}_{G,H}$ on G which was considered as the main example.

Proposition 1. *Let $\mathscr{F}$ be a $\mathfrak{g}/\mathfrak{h}$-foliation on M defined by ω and let $\tilde{\mathscr{F}} = \pi^*\mathscr{F}$ be its pull-back to the universal covering space $\tilde{M}$ of M. There are a map $\mathscr{D}: \tilde{M} \to G$ and a group representation $\rho: \pi_1(M) \to G$ such that*

(i) *$\mathscr{D}$ is $\pi_1(M)$-equivariant, i.e. $\mathscr{D}(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot \mathscr{D}(\tilde{x})$ for any $\gamma \in \pi_1(M)$, and*
(ii) *$\tilde{\omega} := \pi^*\omega = \mathscr{D}^*\theta$, i.e. $\tilde{\mathscr{F}} = \mathscr{D}^*\mathscr{F}_{G,H}$.*

The map $\mathscr{D}$ is called the developing map of $\mathscr{F}$ and it is uniquely determined up to left translations by elements of G.

This statement is proved by the same arguments used in the construction of the *developing map* of a Lie foliation or a transversely homogeneous foliation [4,1]. It follows from Frobenius theorem for differential ideals that for any point in $\tilde{M}$ there are a neighbourhood U of it and a map $f: U \to G$ such that $f^*\theta = \tilde{\omega}$. The maps f are unique up to left translations by elements of G. Since $\tilde{M}$ is simply connected the maps f can be chosen in such a way that they glue together giving rise to the developing map $\mathscr{D}$.

Let us consider the space

$$\tilde{Q} = \{(\tilde{x}, g) \in \tilde{M} \times G \mid \mathscr{D}(\tilde{x}) \in g \cdot H\}.$$

The projection $\tilde{Q} \to \tilde{M}$ onto the first factor defines $\tilde{Q}$ as the total space of a principal H-bundle over $\tilde{M}$. The map $\tilde{s}: \tilde{M} \to \tilde{Q}$ given by $\tilde{s}(\tilde{x}) = (\tilde{x}, \mathscr{D}(\tilde{x}))$ is a section. Therefore, the map

$$\tilde{M} \times H \to \tilde{Q},$$
$$(\tilde{x}, h) \mapsto (\tilde{x}, \mathscr{D}(\tilde{x}) \cdot h)$$

is an isomorphism and the bundle is trivial. Under this identification the projection onto the second factor $\varpi: \tilde{Q} \to G$ is given by $\varpi(\tilde{x}, h) = \mathscr{D}(\tilde{x}) \cdot h$ and one has the commutative diagram

$$\begin{array}{ccc} \tilde{M} \times H \cong \tilde{Q} & \xrightarrow{\ \varpi\ } & G \\ \uparrow{\scriptstyle \tilde{s}} & \nearrow{\scriptstyle \mathscr{D}} & \\ \tilde{M} & & \end{array}$$

The map ϖ is a submersion. Therefore, the $\mathfrak{g}$-valued 1-form $\tilde{\zeta} = \varpi^*\theta$ defines a Lie foliation $\mathscr{F}_{\tilde{Q}}$ on $\tilde{Q}$. Clearly $\tilde{s}^*\tilde{\zeta} = \tilde{\omega}$. Since $\mathscr{D}$ is equivariant by the $\pi_1(M)$-action the principal H-bundle $\tilde{Q}$, the section $\tilde{s}$ and the $\mathfrak{g}$-valued 1-form $\tilde{\zeta}$ project onto an H-bundle Q over M, a section $s: M \to Q$ and a $\mathfrak{g}$-valued form ζ defining a Lie foliation $\mathscr{F}_Q$ on Q such that $\omega = s^*\zeta$. So we obtain the following proposition.

Proposition 2. *Let $\mathscr{F}$ be a $\mathfrak{g}/\mathfrak{h}$-foliation on M defined by ω. There is a triple (Q, ζ, s), where $Q \to M$ is a principal H-bundle, ζ is a $\mathfrak{g}$-valued 1-form on Q and $s: M \to Q$ is a section (in particular the bundle Q is trivial) such that*

(i) *The form* $\zeta = (\zeta^1, \ldots, \zeta^m)$ *fulfills the Maurer–Cartan equation* (4) *and defines a Lie foliation* $\mathscr{F}_Q$ *on* Q.

(ii) *The form* ζ *is* Ad(H)-*invariant, i.e. the right translation* R_h *by an element* $h \in H$ *transforms* ζ *into* $\mathrm{Ad}_{h^{-1}}(\zeta)$, *and* $\zeta^i(e^*_{q+k}) = \delta^i_{q+k}$ *for* $i = 1, \ldots, m$ *and* $k = 1, \ldots, m-q$, *where* e^*_{q+k} *denotes the fundamental vector field associated to* e_{q+k}.

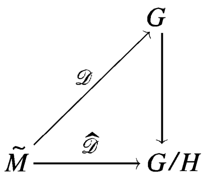
(iii) s *is transverse to* $\mathscr{F}_Q$ *and* $\omega = s^*\zeta$, *i.e.* $\mathscr{F} = s^*\mathscr{F}_Q$.

The triple (Q, ζ, s) *determines the* $\mathfrak{g}/\mathfrak{h}$-*foliation* $\mathscr{F}$ *on* M.

Remark 2. (a) It follows from conditions (i) and (ii) that the form ζ is determined by its values over the restriction of the tangent bundle TQ to the section $s(M)$.

(b) In the case of a Lie foliation $\mathfrak{h} = 0$, Q coincides with M and $\zeta = \omega$. Therefore, the triple (Q, ζ, s) determining the foliation can be replaced by ω.

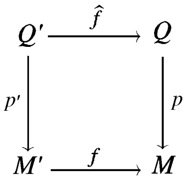
Suppose now that H is closed in G. In this case, ω defines a transversely homogeneous foliation on M and the composition of the developing map $\mathscr{D}$ with the projection $G \to G/H$ is a submersion $\hat{\mathscr{D}}$. This map has the property that the leaves of the foliation $\tilde{\mathscr{F}}$ on $\tilde{M}$ defined by $\tilde{\omega}$ are the connected components of the fibers of $\hat{\mathscr{D}}$. It follows from the commutative diagram



that any $\pi_1(M)$-invariant map $\mathscr{D}' : \tilde{M} \to G$ projecting onto $\hat{\mathscr{D}}$ defines the same transversely homogeneous foliation on $\tilde{M}$, and thus on M, although $(\mathscr{D}')^*\theta$ can be different from $\tilde{\omega}$. The different choices of $\pi_1(M)$-invariant maps $\mathscr{D}$ projecting onto $\hat{\mathscr{D}}$ correspond exactly to the different choices of the section $s: M \to Q$. This fact motivates the following definition.

Definition 2. Let ω and ω' define a $\mathfrak{g}/\mathfrak{h}$-foliation and a $\mathfrak{g}'/\mathfrak{h}'$-foliation on M and M', respectively. Let (Q, ζ, s) and (Q', ζ', s') be the corresponding triples given by Proposition 2. We say that the two foliations are isomorphic if there is a triple $(f, \hat{f}, v)$, where $f: M' \to M$ and $\hat{f}: Q' \to Q$ are diffeomorphisms and $v: \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra isomorphism with $v(\mathfrak{h}') = \mathfrak{h}$, such that

(i) the diagram



where p and p' denote the natural projections, is commutative,

(ii) $\hat{f}^*\zeta = v \circ \zeta'$.

Remark 3. (a) It follows from the above definition that, if we still denote by $v: H' \to H$ the isomorphism of Lie groups induced by the Lie algebra isomorphism, then $\hat{f}$ is v-equivariant in the sense that $\hat{f} \circ R_{h'} = R_{v(h')} \circ \hat{f}$ for any element h' of H'.

(b) Notice that we do not require $\hat{f} \circ s'$ to be equal to $s \circ f$.

(c) It follows from (ii) that $v \circ \omega' = v \circ s'^*\zeta = s'^*\hat{f}^*\zeta$.

(d) Let a triple $(f, \hat{f}, v)$, where $f: M \to M$ is a diffeomorphism, $\hat{f}: Q \to Q$ is a H-bundle isomorphism over f and $v: \mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism with $v(\mathfrak{h}) = \mathfrak{h}$, be given. Then there is a unique $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}'$ on M for which $(f, \hat{f}, v)$ is an isomorphism identifying $\mathscr{F}'$ with $\mathscr{F}$. The foliation $\mathscr{F}'$, called the *foliation induced by* $(f, \hat{f}, v)$, will be denoted by $\mathscr{F} \circ (f, \hat{f}, v)$. It is determined by the triple (Q, ζ', s) where ζ' is defined by $\hat{f}^*\zeta = v \circ \zeta'$.

(e) In the case of Lie foliations the description of isomorphisms is simpler. An isomorphism between a $\mathfrak{g}'$-foliation $\mathscr{F}'$ on M' defined by ω' and a $\mathfrak{g}$-foliation $\mathscr{F}$ on M defined by ω is given by a couple (f, v) where $f: M' \to M$ is a diffeomorphism and $v: \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra isomorphism such that $f^*\omega = v \circ \omega'$.

2. Deformations of $\mathfrak{g}/\mathfrak{h}$-foliations

From now on we suppose that a $\mathfrak{g}$-valued 1-form ω defining a $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$ on a connected manifold M has been fixed. We shall denote by $(T,0)$ the germ at 0 of a real analytic set T defined in a neighbourhood of the origin of an Euclidean space $\mathbb{R}^{\ell}$.

Definition 3. A *family of deformations* $\mathscr{F}_t$ of the $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$ parametrized by $(T,0)$ is given by a collection of 1-forms $\omega_t^1, \ldots, \omega_t^m$ on M, depending smoothly on $t \in T$, and a set of smooth functions $K_{ij}^k(t)$ such that conditions (1)–(4) are fulfilled for each $t \in T$. So for every $t \in T$ the set of constants $K_{ij}^k(t)$ defines a Lie algebra $\mathfrak{g}_t$ and a Lie subalgebra $\mathfrak{h}_t$ and the forms $\omega_t = (\omega_t^1, \ldots, \omega_t^m)$ define a $\mathfrak{g}_t/\mathfrak{h}_t$-foliation $\mathscr{F}_t$ on M. Moreover we require $\omega_0 = \omega$.

One cannot expect an analogous of Proposition 2 for families of deformations because there is not a version of Ado's theorem with parameters. Nevertheless, if the manifold M is compact then a weaker statement, which will be sufficient for our purposes, is still true.

Let $\omega_t = (\omega_t^1, \ldots, \omega_t^m)$ and $K_{ij}^k(t)$ define a family of deformations of the given $\mathfrak{g}/\mathfrak{h}$-foliation on M. The third Lie theorem, stating that there is a local Lie group realizing a given Lie algebra, is based on the theorem of existence of solutions for ordinary differential equations and on Frobenius theorem for involutive distributions (cf. [15]). Therefore, it admits a version with parameters and we deduce that there are a family $\mathscr{G}_t$ of local Lie groups and a family $\mathscr{H}_t$ of local Lie subgroups such that the corresponding Lie algebras $\mathfrak{g}_t$ and $\mathfrak{h}_t$ are those defined by the family of structure constants $K_{ij}^k(t)$. In fact, one can find vector fields $e_{1,t}, \ldots, e_{m,t}$, on a neighbourhood of the origin in $\mathbb{R}^m$, depending smoothly on $t \in T$ and such that $[e_{i,t}, e_{j,t}] = \sum_k K_{ij}^k(t) e_{k,t}$.

Proposition 3. *Let ω define a $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$ on a compact manifold M and let $\omega_t = (\omega_t^1, \ldots, \omega_t^m)$ and $K_{ij}^k(t)$ define a family of deformations of $\mathscr{F}$ parametrized by $t \in T$. Let $\mathfrak{g}_t$ and $\mathfrak{h}_t$ be the families of Lie algebras and Lie subalgebras defined by $K_{ij}^k(t)$ and denote, respectively, by $\mathscr{G}_t$ and $\mathscr{H}_t$ the corresponding families of local Lie groups and Lie subgroups.*

There is a family ζ_t of $\mathfrak{g}_t$-valued 1-forms on the trivial $\mathscr{H}_t$-bundle $M \times \mathscr{H}_t$, defined for small t, such that:

(i) *For each t, the form ζ_t fulfills the Maurer–Cartan equation and defines a Lie foliation on $M \times \mathscr{H}_t$ modeled over $\mathscr{G}_t$.*
(ii) *The form ζ_t is $\mathrm{Ad}(\mathscr{H}_t)$-invariant and $\zeta_t^i(e_{q+k,t}^*) = \delta_{q+k}^i$ for $i = 1, \ldots, m$ and $k = 1, \ldots, m-q$, where $e_{q+k,t}^*$ denotes the fundamental vector field associated to $e_{q+k,t}$.*
(iii) *The zero section $s(x) = (x, e)$ is transverse to the Lie foliation on $M \times \mathscr{H}_t$ and $\omega_t = s^*\zeta_t$.*

The family of vector 1-forms ζ_t can be constructed on the product $U \times \mathscr{H}_t$, where U is a simply connected open subset of M and t is small enough, as in the proof of Proposition 2. The forms ζ_t are determined by their restriction to the zero section (cf. Remark 2) and therefore all these constructions must agree. Since M is compact ζ_t are defined on $M \times \mathscr{H}_t$ with t in a suitable neighbourhood of 0 in T.

Let a family of deformations $\mathscr{F}_t$ of $\mathscr{F}$ defined by ω_t and $K_{ij}^k(t)$ and parametrized by $(T,0)$ be given. Each smooth map of germs of analytic sets $\varphi : (T',0) \to (T,0)$ induces a family $\mathscr{F}_{\varphi(t')}$ parametrized by $(T',0)$ which is given by $\omega_{\varphi(t')}$ and $K_{ij}^k(\varphi(t'))$. We say that $\mathscr{F}_{\varphi(t')}$ is the pull-back of $\mathscr{F}_t$ by φ.

Two families of deformations $\mathscr{F}_t$ and $\mathscr{F}'_t$ of $\mathscr{F}$ parametrized by the same space of parameters $(T,0)$ and defined, respectively, by ω_t, $K_{ij}^k(t)$ and by ω'_t and $K'^k_{ij}(t)$ will be said to be *equivalent* if there is a triple $(f_t, \hat{f}_t, v_t)$, where $f_t : M \to M$ and $\hat{f}_t : M \times \mathscr{H}_t \to M \times \mathscr{H}'_t$ are families of diffeomorphisms depending smoothly on $t \in T$ and $v_t : \mathfrak{g}_t \to \mathfrak{g}'_t$ is a Lie algebra isomorphism with $v_t(\mathfrak{h}_t) = \mathfrak{h}'_t$, such that (i) $p' \circ \hat{f}_t = f_t \circ p$, where p and p' denote the natural projections, (ii) $\hat{f}_t^* \zeta'_t = v_t \circ \zeta_t$ and (iii) $f_0, \hat{f}_0$ and v_0 are the identity maps.

A family of deformations of $\mathscr{F}$ parametrized by $(T,0)$ is called *trivial* if it is equivalent to the constant family.

3. Infinitesimal deformations

The fixed $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$ defined by ω can be perturbed by taking a new family of 1-forms $\omega'^k = \omega^k + \sigma^k$ and a new set of constants $K'^k_{ij} = K_{ij}^k + C_{ij}^k$ in such a way that conditions (1)–(4) are still fulfilled. Therefore, a new $\mathfrak{g}/\mathfrak{h}$-foliation near $\mathscr{F}$ is specified by a couple (σ, ψ), where

(a) $\sigma = (\sigma^1, \ldots, \sigma^m)$ is a m-tuple of differential 1-forms on M close to zero and
(b) $\psi \in \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$ is a $\mathfrak{g}$-valued 2-form over $\mathfrak{g}$ close to zero,

$$\psi = -\frac{1}{2} \sum_{i,j,k} C_{ij}^k \theta^i \wedge \theta^j \otimes e_k$$

such that the integrability conditions

$$d(\omega^k + \sigma^k) + \frac{1}{2} \sum_{i,j} (K_{ij}^k + C_{ij}^k)(\omega^i + \sigma^i) \wedge (\omega^j + \sigma^j) = 0 \tag{5}$$

and

$$\sum_i (K'^k_{ij}K'^i_{rs} + K'^k_{ir}K'^i_{sj} + K'^k_{is}K'^i_{jr}) = 0, \tag{6}$$

where $K'^k_{ij} = K^k_{ij} + C^k_{ij}$, are satisfied. Moreover, since we want the set of constants K'^k_{ij} to define, not only a new Lie algebra $\mathfrak{g}'$ but also a Lie subalgebra $\mathfrak{h}'$, we require the first q components $\psi^1,\dots,\psi^q$ of ψ to belong to the ideal in $\bigwedge^*\mathfrak{g}^*$ generated by $\theta^1,\dots,\theta^q$.

Let Ω^r be the space of differential forms on M of degree r. We denote by $\mathscr{R} = (\mathscr{R}^1,\dots,\mathscr{R}^m)$ the linear map from $\bigwedge^r\mathfrak{g}^*\otimes\mathfrak{g}$ into $(\Omega^r)^m$ given by

$$\mathscr{R}^k(\theta^J\otimes e_i) = \delta^k_i\omega^J, \tag{7}$$

where $J = (j_1,\dots,j_r)$ and $\theta^J = \theta^{j_1}\wedge\cdots\wedge\theta^{j_r}$.

Given an element $\sigma = (\sigma^1,\dots,\sigma^m)\in(\Omega^r)^m$ we denote by $\hat{d}_M\sigma$ the element of $(\Omega^{r+1})^m$ whose components are given by

$$(\hat{d}_M\sigma)^k = \mathrm{d}\sigma^k + \sum_{i,j} K^k_{ij}\omega^i\wedge\sigma^j. \tag{8}$$

In a similar way, we introduce an operator $\hat{d}_{\mathfrak{g}}:\bigwedge^r\mathfrak{g}^*\otimes\mathfrak{g}\to\bigwedge^{r+1}\mathfrak{g}^*\otimes\mathfrak{g}$ acting on an element $\psi\in\bigwedge^r\mathfrak{g}^*\otimes\mathfrak{g}$ by

$$\hat{d}_{\mathfrak{g}}\psi = \sum_k (\mathrm{d}\psi^k + \sum_{i,j} K^k_{ij}\theta^i\wedge\psi^j)\otimes e_k, \tag{9}$$

where here d denotes the exterior derivative on the Lie group G. Notice that the operators $\hat{d}_M$ and $\hat{d}_{\mathfrak{g}}$ are formally the same.

The linearization of Eqs. (5) and (6) with respect to the variables σ^k and C^k_{ij} are, respectively,

$$\begin{aligned} 0 &= \mathrm{d}\sigma^k + \sum_{i,j} K^k_{ij}\omega^i\wedge\sigma^j + \frac{1}{2}\sum_{i,j} C^k_{ij}\omega^i\wedge\omega^j \\ &= (\hat{d}_M\sigma)^k - \mathscr{R}^k(\psi) \end{aligned} \tag{10}$$

and

$$0 = \sum_i (K^k_{ij}C^i_{rs} + C^k_{ij}K^i_{rs} + K^k_{ir}C^i_{sj} + C^k_{ir}K^i_{sj} + K^k_{is}C^i_{jr} + C^k_{is}K^i_{jr}).$$

This second equation can be written in terms of the vector form ψ as

$$0 = \sum_k (\mathrm{d}\psi^k + \sum_{i,j} K^k_{ij}\theta^i\wedge\psi^j)\otimes e_k = \hat{d}_{\mathfrak{g}}\psi. \tag{11}$$

Let V^r denote the space of elements $\xi\in\bigwedge^r\mathfrak{g}^*\otimes\mathfrak{g}$ whose first q components $\xi^1,\dots,\xi^q$ belong to the ideal in $\bigwedge^*\mathfrak{g}^*$ generated by $\theta^1,\dots,\theta^q$. When $r = 0$ this means that $V^0 = \mathfrak{h}$. For $r\in\mathbb{N}$, we set

$$\mathscr{A}^r = (\Omega^r)^m\oplus V^{r+1}.$$

Notice that the couple (σ,ψ) defining a perturbation of $\mathscr{F}$ is an element of $\mathscr{A}^1$ and that the integrability equations (5) and (6) as well as their linearization (10) and (11) are identities in $\mathscr{A}^2$. In fact, if we define

$$D:\mathscr{A}^r\to\mathscr{A}^{r+1}$$

by

$$D(\sigma,\psi) = (\hat{d}_M \sigma - \mathscr{R}\psi, -\hat{d}_{\mathfrak{g}}\psi)$$

then the linearized equations (10) and (11) can be written as

$$D(\sigma,\psi) = 0 \tag{12}$$

and the complete integrability equations (5) and (6) are of the form

$$D(\sigma,\psi) + P(\sigma,\psi) = 0, \tag{13}$$

where $P(\sigma,\psi)$ is a sum of quadratic and cubic terms.

A straightforward computation using the Jacobi identity shows that

Proposition 4. $D^2 = 0$.

Therefore we obtain the differential complex $\mathscr{A}$:

$$0 \to \mathscr{A}^0 \overset{D}{\to} \mathscr{A}^1 \overset{D}{\to} \mathscr{A}^2 \to \cdots. \tag{14}$$

We denote by $H^*(\mathscr{A})$ the cohomology of this complex, i.e.

$$H^k(\mathscr{A}) = \mathrm{Ker}\{D\colon \mathscr{A}^k \to \mathscr{A}^{k+1}\}/\mathrm{Im}\{D\colon \mathscr{A}^{k-1} \to \mathscr{A}^k\}.$$

It will be proved in the next section that this cohomology has finite dimension when the manifold M is compact.

The assertion of Proposition 4 is equivalent to say that $\hat{d}_M^2 = 0$ and $\hat{d}_{\mathfrak{g}}^2 = 0$. Therefore if we denote $\mathscr{A}_0^k = (\Omega^k)^m$ we can also consider the complexes $\mathscr{A}_0$ and $\mathscr{V}$ given by

$$0 \to \mathscr{A}_0^0 \overset{\hat{d}_M}{\to} \mathscr{A}_0^1 \overset{\hat{d}_M}{\to} \mathscr{A}_0^2 \to \cdots \tag{15}$$

and

$$0 \to V^0 \overset{\hat{d}_{\mathfrak{g}}}{\to} V^1 \overset{\hat{d}_{\mathfrak{g}}}{\to} V^2 \to \cdots. \tag{16}$$

Notice that the projections $\mathscr{A}^r \to V^{r+1}$ defined by $(\sigma,\psi) \mapsto \psi$ induce a complex morphism from $\mathscr{A}$ onto the complex

$$0 \to V^1 \to V^2 \to V^3 \to \cdots$$

whose kernel is the complex $\mathscr{A}_0$.

Now we denote by Ξ the space of integrable elements of $\mathscr{A}^1$, that is the space of couples (σ,ψ) in $\mathscr{A}^1$ fulfilling the integrability equations (5) and (6). A family $\mathscr{F}_t$ of deformations of $\mathscr{F}$ parametrized by $(T,0)$ is given by a family of couples (ω_t, η_t) where

$$\eta_t = -\frac{1}{2}\sum_{i,j,k} K_{ij}^k(t)\theta^i \wedge \theta^j \otimes e_k.$$

If we set $K_{ij}^k(t) = K_{ij}^k + C_{ij}^k(t)$, $\eta = -\frac{1}{2}\sum_{i,j,k} K_{ij}^k \theta^i \wedge \theta^j \otimes e_k$ and $\psi_t = -\frac{1}{2}\sum_{i,j,k} C_{ij}^k(t)\theta^i \wedge \theta^j \otimes e_k$ then

$$(\omega_t, \eta_t) = (\omega, \eta) + (\sigma_t, \psi_t)$$

and the family $\mathscr{F}_t$ can be viewed as a smooth map from T into $\mathscr{A}^1$, sending t into $(\sigma_t, \psi_t) \in \mathscr{A}^1$, whose image is contained in Ξ.

In particular, if the family of deformations is parametrized by an interval of the real line then $t \mapsto (\sigma_t, \psi_t)$ is a smooth curve in Ξ passing trough the origin. The tangent vector to this curve

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma_t, \psi_t) \tag{17}$$

fulfills Eq. (12). Therefore, it is a cocycle and defines a cohomology class in $H^1(\mathscr{A})$ which is called the *infinitesimal deformation* of the family $\mathscr{F}_t$.

More generally, in case T is smooth there is a well-defined map

$$\varrho: T_0 T \to H^1(\mathscr{A})$$

sending a vector $v \in T_0 T$ into the cohomology class defined by the derivative of (σ_t, ψ_t) at the origin and in the direction v. We say that ϱ is the *Kodaira–Spencer* map associated to the family. Corollary 1 below states that if the family of deformations is trivial then the vector (17) is a coboundary. Therefore, we shall say that the elements of $H^1(\mathscr{A})$ are the infinitesimal deformations of $\mathscr{F}$.

In the proofs of the proposition below and the versality theorem we shall need to parametrize fibered automorphisms of the principal H-bundle $Q \cong M \times H$ over diffeomorphisms of M by vector fields. This can be done in the following way. Let N denote the restriction $T(M \times H)|_{M \times \{e\}}$ of the tangent bundle of $M \times H$ to $M \times \{e\}$. Fix a complete Riemannian metric on M and denote by $\exp^M: TM \to M$ the associated geodesic exponential map. Let $\exp^H: T_e H \to H$ be the exponential map of the Lie group H. Then we define $\Phi: N \to M \times H$ by

$$\Phi(x, v) = (\exp_x^M(v_M), \exp^H(v_H)),$$

where $v = (v_M, v_H) \in N_{(x,e)} = T_x M \times T_e H$.

Let χ be a smooth section of N. We associate to χ the pair $(f_\chi, \hat{f}_\chi)$, where $f_\chi : M \to M$ is given by $f_\chi(x) = \exp_x^M(\chi_M(x))$ and $\hat{f}_\chi : M \times H \to M \times H$ is given by

$$\hat{f}_\chi(x, h) = \Phi(x, \chi(x)) \cdot h.$$

Here the dot denotes the action of $h \in H$ on the principal bundle $M \times H$. If χ is small enough then f_χ and $\hat{f}_\chi$ are diffeomorphisms and $\hat{f}_\chi$ is a H-bundle isomorphism over f_χ. Notice that any pair $(f, \hat{f})$, where $\hat{f}$ is a H-bundle morphism over f, close enough to the identity can be obtained in this way.

Assume now that T is a neighbourhood of zero in the real line and let χ_t be a smooth family of sections of N parametrized by T with $\chi_0 = 0$ and v_t a smooth family of linear isomorphisms of the vector space $\mathfrak{g}$ preserving the subspace $\mathfrak{h}$ and such that $v_0 = \mathrm{id}$. We can write

$$\chi_t = t(X, Y) + O(t^2), \tag{18}$$

where X is a vector field on M and $Y_x \in T_e H \cong \mathfrak{h}$, and

$$v_t = \mathrm{id} + tv + O(t^2). \tag{19}$$

The triple $(f_t, \hat{f}_t, \nu_t)$, where $f_t = f_{\chi_t}$ and $\hat{f}_t = \hat{f}_{\chi_t}$, induces a trivial family of deformations $\mathscr{F}_t$ of $\mathscr{F}$ parametrized by T. More precisely, for each $t \in T$, $\mathscr{F}_t$ is the $\mathfrak{g}/\mathfrak{h}$-foliation on M which is identified to $\mathscr{F} = \mathscr{F}_0$ by means of the isomorphism $(f_t, \hat{f}_t, \nu_t)$. We recall that this means the following. The $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$ is determined by a vector 1-form ζ on $M \times H$ defining a $\mathfrak{g}$-Lie foliation on $M \times H$ in such a way that $\omega = s^*\zeta$, where $s : M \to M \times H$ is the zero section $s(x) = (x, e)$. Now we denote by ζ_t the vector 1-form on $M \times H$ defined by $\nu_t(\zeta_t) = \hat{f}_t^*\zeta$. For each $t \in T$, $\zeta_t = (\zeta_t^1, \ldots, \zeta_t^m)$ defines a $\mathfrak{g}$-Lie foliation on $M \times H$ with set of constants $K_{ij}^k(t)$ given by the Maurer–Cartan equation

$$\mathrm{d}\zeta_t^k = -\frac{1}{2}\sum K_{ij}^k(t)\, \zeta_t^i \wedge \zeta_t^j.$$

Then $\mathscr{F}_t$ is the $\mathfrak{g}/\mathfrak{h}$-foliation defined by the vector 1-form $\omega_t = s^*\zeta_t$ and the set of constants $K_{ij}^k(t) = K_{ij}^k + C_{ij}^k(t)$. If we set $\omega_t = \omega + \sigma_t$ and

$$\psi_t = -\frac{1}{2}\sum C_{ij}^k(t)\, \theta^i \wedge \theta^j \otimes e_k,$$

then the curve (σ_t, ψ_t) in the space Ξ of integrable elements of $\mathscr{A}^1$ represents the trivial family of deformations $\mathscr{F}_t$.

Proposition 5. *In the above situation one has*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\sigma_t, \psi_t) = D(\omega(X) + Y, \nu). \tag{20}$$

Proof. Recall that if F_t is a family of local diffeomorphisms of $\mathbb{R}^m$ of the form $F_t^k(x) = x^k + t\, Z^k(x) + O(t^2)$ and α is a 1-form then

$$F_t^*\alpha = \alpha + t\, L_Z\alpha + O(t^2),$$

where L_Z denotes the Lie derivative with respect to the vector field $Z = (Z^1, \ldots, Z^m)$. Applying this fact to the family $\hat{f}_{\chi_t}$ of diffeomorphisms of $M \times H$ associated to the family χ_t of sections of N and using (18) one obtains

$$\hat{f}_{\chi_t}^*\zeta = \zeta + t\, L_{(X,Y)}\zeta + O(t^2).$$

It follows from (19) that

$$\zeta_t = \nu_t^{-1}(\hat{f}_{\chi_t}^*\zeta) = \zeta + t\, L_{(X,Y)}\zeta - t\, \nu(\zeta) + O(t^2).$$

Computing $\omega_t = s^*\zeta_t$ reduces to evaluate ζ_t on the horizontal vectors $(v_M, 0) \in N$. A direct calculation shows that

$$\begin{aligned} L_{(X,Y)}\zeta(v_M, 0) &= \left[\mathrm{d}(\omega^k(X) + Y^k) + \sum_{ij} K_{ij}^k(\omega^j(X) + Y^j)\,\omega^i \right](v_M) \\ &= (\hat{d}_M(\omega(X) + Y))(v_M) \end{aligned}$$

and that $v^k(\zeta)(v_M,0) = \sum v_i^k \omega^i(v_M)$. One can also calculate $\mathrm{d}\zeta_t^k$ obtaining

$$\mathrm{d}\zeta_t^k = -\frac{1}{2}\sum_{ij}\left(K_{ij}^k + t\left(\sum_l K_{il}^k v_j^l + K_{lj}^k v_i^l - K_{ij}^l v_l^k\right)\right)\zeta_t^i \wedge \zeta_t^j + O(t^2).$$

This implies that

$$\psi_t = t\sum_k\left(-dv^k - \sum_{ij} K_{ij}^k \theta^i \wedge v^j\right)\otimes e_k + O(t^2) = -t\,\hat{d}_{\mathfrak{g}} v + O(t^2).$$

Collecting all these computations and using the expression in components of the operator D one deduces (20).

From the above result the corollary follows.

Corollary 1. *Assume that the parameter space* $(T,0)$ *is smooth. If the family of deformations* $\mathscr{F}_t$ *is trivial then the Kodaira–Spencer map* $\varrho : T_0 T \to H^1(\mathscr{A})$ *vanishes identically.*

4. A twisted Hodge theory

In this section, we begin by recalling the basic facts of the Hodge theory for elliptic differential operators and elliptic complexes, especially the Hodge decomposition theorem. For the details we refer to Warner [17]. Then we show how this theory can be extended to a larger class of complexes which includes the complex $\mathscr{A}$ constructed in the precedent section. In particular, we obtain the finiteness of its cohomology when the manifold M is compact.

Through this section the manifold M will be assumed to be connected, compact, of dimension n and orientable with an orientation given by a volume element μ. In fact all the results remain valid for M nonorientable as can be seen by passing to the double-covering space.

Let $E \to M$ be a complex vector bundle of rank d equipped with an Hermitian metric h. The space $\Gamma(E)$ of C^∞-sections of E is a Fréchet space with the C^∞-topology. The Hermitian metric h induces a Hermitian product on $\Gamma(E)$ given by

$$\langle \alpha,\beta \rangle = \int_M h_x(\alpha_x,\beta_x)\,\mathrm{d}\mu(x),$$

were α,β are elements of $\Gamma(E)$ and α_x, β_x denote their values at a given point $x \in M$. The corresponding norm is $\| \; \| = \sqrt{\langle \, , \rangle}$. We denote by $L^2(E)$ the associated Hilbert space. The natural inclusion $\Gamma(E) \hookrightarrow L^2(E)$ is continuous. In the next section, we will also consider the Sobolev s-completion $W^s(E)$ of $\Gamma(E)$. The following proposition states the basic properties of Sobolev spaces.

Proposition 6. (i) *Let* $m \in \mathbb{N}$. *If* $s \geqslant [n/2] + m$ *then any section* $\alpha \in W^s(E)$ *is of class* C^s.

(ii) *For any* $s \in \mathbb{N}$ *there is an inclusion* $W^{s+1}(E) \subset W^s(E)$ *and the natural injection* $W^{s+1}(E) \hookrightarrow W^s(E)$ *is compact.*

(iii) $\Gamma(E) = \bigcap_{s \in \mathbb{N}} W^s(E)$.

A continuous linear functional on $\Gamma(E)$, equipped with the C^∞-topology, is called a *current* on E. The space of currents on E will be denoted by $\mathscr{C}(E)$. An element $\beta \in L^2(E)$ induces a current $T_\beta : \Gamma(E) \to \mathbb{C}$ given by

$$T_\beta(\alpha) = \langle \alpha, \beta \rangle.$$

The injection $L^2(E) \to \mathscr{C}(E)$ defined by $\beta \mapsto T_\beta$ is continuous.

Let E, F be the total spaces of complex vector bundles over M of respective ranks k and l. A *differential operator* of order m from E into F is a linear map $L : \Gamma(E) \to \Gamma(F)$ such that in local trivializations of E and F and with respect to local coordinates of M can be written as

$$L = \sum_{|r| \leqslant m} a_r D^r,$$

where $D^r = \partial^{|r|}/\partial x_1^{r_1} \ldots \partial x_n^{r_n}$ and a_r is a $k \times l$-matrix (a_{ij}) whose coefficients a_{ij} are C^∞-functions. Given a covector $\xi = (\xi_1, \ldots, \xi_n) \in T_x^*M$ the *symbol* of L at (x,ξ) is the linear map $\sigma(L)(x,\xi) : E_x \to F_x$ given by

$$\sigma(L)(x, \xi) = \sum_{|r| = m} \xi_1^{r_1} \ldots \xi_n^{r_n} a_r(x).$$

The differential operator L is called *elliptic* if the symbol $\sigma(L)(x, \xi)$ is an isomorphism for every $x \in M$ and every nonzero covector $\xi \in T_x^*M$. In this case, the ranks k and l are equal. Suppose now that $E = F$ and the order m of L is even, then L is said *strongly elliptic* if the quadratic form Q defined at (x, ξ) by

$$Q(x, \xi)(v) = (-1)^{m/2} h_x(\sigma(L)(x, \xi)(v), v)$$

is positive definite for each couple (x,ξ). Here h denotes the Hermitian metric on E and v is an element of E_x. Every strongly elliptic operator is elliptic. The properties of elliptic operators stated in the following proposition are the key for proving the Hodge decomposition theorem. For the proof one can see [17, p. 248].

Proposition 7. *Let $L : \Gamma(E) \to \Gamma(F)$ be a differential operator. Then it extends to $L : \mathscr{C}(E) \to \mathscr{C}(F)$.*

(i) *Suppose that L is elliptic. Given $\gamma \in \Gamma(F)$ every solution $T \in \mathscr{C}(E)$ of the equation $L(T) = \gamma$ is in fact an element of $\Gamma(E)$, that is a C^∞ section of E.*
(ii) *Suppose that $E = F$ and that L is strongly elliptic and selfadjoint and let α_j be a sequence in $\Gamma(E)$ such that $\|\alpha_j\| \leqslant C$ and $\|L(\alpha_j)\| \leqslant C$ for a given constant C. Then α_j admits a Cauchy subsequence for the L^2-norm $\| \; \|$.*

Let us consider now a finite family (E^i, d_i), $i = 0,1, \ldots, q$, of Hermitian vector bundles $E^i \to M$ and differential operators $d_i : \Gamma(E^i) \to \Gamma(E^{i+1})$ of order 1 such that $d_{i+1} \circ d_i = 0$. This family gives rise to a differential complex of global sections $\mathscr{E}$:

$$0 \to \Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \cdots \to \Gamma(E^q) \to 0.$$

Its cohomology is denoted by $H^*(\mathscr{E})$. We say that $\mathscr{E}$ is an *elliptic complex* if the corresponding symbol sequence

$$0 \to E_x^0 \xrightarrow{\sigma_0(x,\xi)} E_x^1 \xrightarrow{\sigma_1(x,\xi)} \cdots \to E_x^q \to 0$$

is exact for each $x \in M$ and each nonzero covector $\xi \in T_x^* M$. If $q = 1$ the elliptic complex $\mathscr{E}$ reduces to just one elliptic operator $\Gamma(E^0) \to \Gamma(E^1)$.

For each i, let $d_i^* : \Gamma(E^{i+1}) \to \Gamma(E^i)$ be the adjoint operator of d_i. Then

$$\nabla_i = d_i^* d_i + d_{i-1} d_{i-1}^*$$

is a self-adjoint differential operator on $\Gamma(E^i)$. It is easy to prove that the complex $\mathscr{E}$ is elliptic if and only if for every i the operator ∇_i is strongly elliptic. Let $\mathbb{H}(E^i)$ be the kernel of ∇_i. Then

$$\mathbb{H}(E^i) = \operatorname{Ker} d_i \cap \operatorname{Ker} d_{i-1}^*.$$

The elements of $\mathbb{H}(E^i)$ are called *harmonic sections* of E. The Hodge decomposition theorem can now be stated as

Hodge Theorem. *Let M be a compact manifold and let (E_i, d_i) be an elliptic complex on M. Then for each i*

(i) *the space $\mathbb{H}(E^i)$ is finite dimensional and*
(ii) *there is an orthogonal decomposition*

$$\begin{aligned}\Gamma(E^i) &= \mathbb{H}(E^i) \oplus \nabla_i(\Gamma(E^i)) \\ &= \mathbb{H}(E^i) \oplus d_{i-1}(\Gamma(E^{i-1})) \oplus d_i^*(\Gamma(E^{i+1})).\end{aligned}$$

As a consequence, the cohomology $H^i(\mathscr{E})$ of the complex $\mathscr{E}$ of global sections is finite dimensional and isomorphic to $\mathbb{H}(E^i)$.

Now we want to extend the preceding theorem to a class of differential complexes which are not elliptic but which are closely related. Suppose that we are given

1. An elliptic complex $\mathscr{E}$ of Hermitian vector bundles over M,

$$0 \to \Gamma(E^0) \xrightarrow{d_0} \Gamma(E^1) \xrightarrow{d_1} \cdots \xrightarrow{d_{q-1}} \Gamma(E^q) \to 0.$$

2. A complex of Hermitian finite-dimensional vector spaces,

$$0 \to V^0 \xrightarrow{\delta_0} V^1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{q-1}} V^q \to 0.$$

3. An injective linear map $R_i : V^i \to \Gamma(E^{i+1})$, for each i, such that the following diagram is commutative

$$\begin{array}{ccc} V^{i-1} & \xrightarrow{\delta_{i-1}} & V^i \\ \downarrow{\scriptstyle R_{i-1}} & & \downarrow{\scriptstyle R_i} \\ \Gamma(E^i) & \xrightarrow{d_i} & \Gamma(E^{i+1}) \end{array}$$

On $\Gamma(E^i) \oplus V^i$ we define the Hermitian product $\langle\, , \rangle$ by

$$\langle(\alpha, u),(\beta, v)\rangle = \langle\alpha, \beta\rangle + \langle u, v\rangle.$$

The corresponding norm is $\|(\alpha, u)\| = (\|\alpha\|^2 + \|u\|^2)^{1/2}$.

Let $d_i^* : \Gamma(E^{i+1}) \to \Gamma(E^i)$, $\delta_i^* : V^{i+1} \to V^i$ and $R_i^* : \Gamma(E^{i+1}) \to V^i$ be the respective adjoints operators of d_i, δ_i and R_i. Notice that R_i^* is well defined and bounded since V^i is finite dimensional; so it admits a bounded extension $R_i^* : L^2(E) \to V^i$. Let $L_i : \Gamma(E^i) \oplus V^i \to \Gamma(E^{i+1}) \oplus V^{i+1}$ be defined by

$$L_i(\sigma_i, v) = (d_i\sigma - R_i v, -\delta_i v).$$

A straightforward computation shows that $L_{i+1} \circ L_i = 0$ for each i. So we get a differential complex $\mathscr{V}$:

$$0 \to \Gamma(E^0) \oplus V^0 \xrightarrow{L_0} \Gamma(E^1) \oplus V^1 \xrightarrow{L_1} \cdots \xrightarrow{L_{q-1}} \Gamma(E^q) \oplus V^q \to 0.$$

We denote by $H^i(\mathscr{V})$ its cohomology. Notice that the complex $\mathscr{V}$ reduces to $\mathscr{E}$ when $V^i = \{0\}$ for each i.

The complex $\mathscr{V}$ is not elliptic since the involved spaces are not spaces of sections of vector bundles and the involved operators are not differential operators. We will show however that it behaves like an elliptic one and in particular that the Hodge decomposition theorem remains valid. This is a key ingredient in the proof of the existence of a versal space of deformations for $\mathfrak{g}/\mathfrak{h}$-foliations.

Let $L_i^* : \Gamma(E^{i+1}) \oplus V^{i+1} \to \Gamma(E^i) \oplus V^i$ be the adjoint operator of L_i. It is defined by the formula

$$L_i^*(\sigma, v) = (d_i^*\sigma, -R_i^*\sigma - \delta_i^* v).$$

Set $\Delta_i = L_i^* L_i + L_{i-1} L_{i-1}^*$. Then Δ_i is an operator on $\Gamma(E^i) \oplus V^i$ which can be written as

$$\Delta_i(\sigma, v) = ((\nabla_i + \mathscr{R}_i)(\sigma) + A_i(v), B_i(\sigma) + C_i(v)),$$

where

$$\nabla_i = d_i^* d_i + d_{i-1} d_{i-1}^*,$$
$$\mathscr{R}_i = R_{i-1} R_i^*,$$
$$A_i = R_{i-1}\delta_i^* - d_i^* R_i,$$
$$B_i = \delta_{i-1} R_i^* - R_i^* d_i,$$
$$C_i = \delta_{i-1}\delta_{i-1}^* + \delta_i^*\delta_i + R_i^* R_i.$$

Notice that the assumption made on the complex $\mathscr{E}$ implies that $\nabla_i : \Gamma(E^i) \to \Gamma(E^i)$ are strongly elliptic selfadjoint operators.

From Proposition (7) one can deduce the following

Lemma 1 (Regularity lemma). *Denote by the same symbol ∇_i the extension of ∇_i to the space of currents $\mathscr{C}(E^i)$. Let $(\sigma, u) \in \mathscr{C}(E^i) \oplus V^i$ such that $\Delta_i(\sigma, u) = 0$. Then $(\sigma, u) \in \Gamma(E^i) \oplus V^i$.*

Proposition 8. *Let (α_j, v_j) be a sequence in $\Gamma(E^i) \oplus V^i$ and assume that $\|(\alpha_j, v_j)\| \leqslant C$ and $\|\Delta_i(\alpha_j, v_j)\| \leqslant C$ for a given positive constant C. Then (α_j, v_j) admits a Cauchy subsequence for the L^2-norm $\| \ \|$.*

Proof. Since $\|(\alpha_j, v_j)\| = (\|\alpha_j\|^2 + \|v_j\|^2)^{1/2} \leqslant C$ the sequences α_j and v_j are bounded. So v_j admits a Cauchy subsequence in the finite-dimensional vector space V^i which, for convenience, we can assume to be v_j itself. Now, since the operators $\mathscr{R}_i$ and A_i are bounded and $\|\Delta_i(\alpha_j, v_j)\| = (\|\nabla_i(\alpha_j) + \mathscr{R}_i(\alpha_j) + A_i(v_j)\|^2 + \|B_i(\alpha_j) + C_i(v_j)\|^2)^{1/2} \leqslant C$, the sequence $\|\nabla_i(\alpha_j)\|$ is also bounded. Then it follows from part (ii) of Proposition 7, applied to the selfadjoint strongly elliptic operator ∇_i, that the sequence α_j admits a Cauchy subsequence. This completes the proof. □

Corollary 2. *The kernel $\mathbb{H}^i$ of $\Delta_i : \Gamma(E^i) \oplus V^i \to \Gamma(E^i) \oplus V^i$ has finite dimension.*

Now we are in position to give the statement and the proof of the Hodge decomposition theorem for the complex $\mathscr{V}$.

Theorem 1 (Twisted Hodge theorem). (i) *The space $\mathbb{H}^i = \operatorname{Ker} \Delta_i$ is finite dimensional.*

(ii) *There is an orthogonal decomposition*

$$\begin{aligned} \Gamma(E^i) \oplus V^i &= \mathbb{H}^i \oplus \operatorname{Im} \Delta_i \\ &= \mathbb{H}^i \oplus \operatorname{Im} L_{i-1} \oplus \operatorname{Im} L_i^*. \end{aligned}$$

As a consequence, the cohomology space $H^i(\mathscr{V})$ of the complex $\mathscr{V}$ is finite dimensional and isomorphic to $\mathbb{H}^i$.

Proof. Part (i) is just Corollary 2. Let us prove part (ii). Since $\mathbb{H}^i$ has finite dimension we have an orthogonal decomposition

$$\Gamma(E^i) \oplus V^i = \mathbb{H}^i \oplus (\mathbb{H}^i)^{\perp}.$$

So it suffices to prove the equality $\operatorname{Im} \Delta_i = (\mathbb{H}^i)^{\perp}$. The inclusion $\operatorname{Im} \Delta_i \subset (\mathbb{H}^i)^{\perp}$ is clear. To prove that $(\mathbb{H}^i)^{\perp}$ is contained in $\operatorname{Im} \Delta_i$ is equivalent to prove that for any given (α, u) in $(\mathbb{H}^i)^{\perp}$ the equation

$$\Delta_i(\sigma, v) = (\alpha, u) \tag{21}$$

has a solution (σ, v) in $\Gamma(E^i) \oplus V^i$. Following the proof of the analogous statement given in [17, p. 224] and using the regularity lemma (1), one can construct a *weak solution* of (21), that is a continuous linear functional T on $(\Gamma(E^i) \oplus V^i, \| \, \|)$ such that for every $(\beta, w) \in \Gamma(E^i) \oplus V^i$ we have

$$T(\Delta_i(\beta, w)) = \langle (\alpha, u), (\beta, w) \rangle.$$

But this linear functional extends, by Hahn–Banach theorem, to a continuous linear functional $\tilde{T}$ on the Hilbert space $L^2(E^i) \oplus V^i$ satisfying the same property as T, i.e. for every $(\beta, w) \in L^2(E^i) \oplus V^i$ we have

$$\tilde{T}(\Delta_i(\beta, w)) = \langle (\alpha, u), (\beta, w) \rangle.$$

Since $L^2(E^i) \oplus V^i$ is a Hilbert space, there exists $(\sigma, v) \in L^2(E^i) \oplus V^i$ such that for every $(\gamma, a) \in L^2(E^i) \oplus V^i$ we have

$$\tilde{T}(\gamma, a) = \langle (\sigma, v), (\gamma, a) \rangle$$

and

$$\begin{aligned} \tilde{T}(\Delta_i(\beta, w)) &= \langle (\sigma, v), \Delta_i(\beta, w) \rangle \\ &= \langle \Delta_i(\sigma, v), (\beta, w) \rangle \\ &= \langle (\alpha, u), (\beta, w) \rangle \end{aligned}$$

for every $(\beta, w) \in L^2(E^i) \oplus V^i$. Therefore, we have

$$\Delta_i(\sigma, v) = (\alpha, u)$$

in the sense of currents. In particular,

$$\nabla_i(\sigma) = \alpha - \mathscr{R}_i(\sigma) - A_i(v).$$

Since α, $\mathscr{R}_i(\sigma)$ and $A_i(v)$ are C^∞-sections of E^i it follows from part (i) of Proposition 7 that σ is also a C^∞-section of E^i. This shows that $\operatorname{Im} \Delta_i = (\mathbb{H}^i)^\perp$ ending the proof. □

Remark 4. All the results in this sections apply to the case of differential operators acting on sections of real vector bundles. One has just to complexify the vector bundles and to extend the differential operators in the obvious way.

5. The versal space of deformations

Let $\mathscr{F}$ be a $\mathfrak{g}/\mathfrak{h}$-foliation on a compact manifold M defined by a $\mathfrak{g}$-valued form ω.

Definition 4. A family of deformations $\mathscr{F}_s$ of $\mathscr{F}$ parametrized by a smooth space of parameters $(S, 0)$ will be called *versal* if for any other family $\mathscr{F}_{t'}$ of deformations of $\mathscr{F}$ parametrized by $(T', 0)$ there is a smooth map $\varphi : (T', 0) \to (S, 0)$ such that $\mathscr{F}_{t'}$ and $\mathscr{F}_{\varphi(t')}$ are equivalent. Moreover, the differential $d_0 \varphi$ of φ at 0 is unique. Such a map φ, which need not to be unique, will be called versal.

If a family of deformations $\mathscr{F}_t$ of $\mathscr{F}$ parametrized by a smooth space $(T,0)$ verifies the same versal property as $\mathscr{F}_s$, then the versal map $\varphi:(T,0)\to(S,0)$ is an isomorphism. This means that the space $(S,0)$, which is called the versal space of deformations of $\mathscr{F}$, is unique up to isomorphisms.

The aim of this section is to sketch the proof of the following

Theorem 2. *Let $\mathscr{F}$ be a $\mathfrak{g}/\mathfrak{h}$-foliation on a compact manifold M defined by a $\mathfrak{g}$-valued form ω. Let $\mathscr{A}$ be the complex associated to ω as defined in Section* 3. *Then*

(i) *There is a Hodge decomposition for the complex $\mathscr{A}$. In particular its cohomology $H^*(\mathscr{A})$ has finite dimension.*
(ii) *If $H^1(\mathscr{A})=0$ then any family of deformations of $\mathscr{F}$ is trivial.*
(iii) *Assume that $H^2(\mathscr{A})=0$. There is a versal family of deformations $\mathscr{F}_s$ of $\mathscr{F}$ parametrized by the germ $(H^1(\mathscr{A}),0)$ of $H^1(\mathscr{A})$ at the origin. Let $\mathscr{F}'_{t'}$ be a family of deformations of $\mathscr{F}$ parametrized by a smooth space $(T',0)$. The differential $\mathrm{d}_0\varphi$ of φ at 0 of a versal map $\varphi:(T',0)\to(H^1(\mathscr{A}),0)$ coincides with the Kodaira–Spencer map $\varrho: T_0T'\to H^1(\mathscr{A})$ of the family $\mathscr{F}_{t'}$.*
(iv) *Let $\mathscr{F}_t$ be a family of deformations of $\mathscr{F}$ parametrized by a smooth space $(T,0)$. If the Kodaira–Spencer map $\varrho: T_0T\to H^1(\mathscr{A})$ of $\mathscr{F}_t$ is an isomorphism then the family $\mathscr{F}_t$ is versal.*

The proof of this theorem follows the ideas used to prove the Kodaira–Spencer–Kuranishi theorem for deformations of complex structures, as they are given by Morrow and Kodaira [13] and Douady [2], and its generalization to transversely holomorphic foliations given by Girbau et al. [7]. Here we just indicate the main steps emphasizing the differences with the above results.

Fix a Riemannian metric on the manifold M and scalar products on the vector subspaces V^k of $\bigwedge^k\mathfrak{g}^*$ defined is Section 3. They induce scalar products on the spaces $\mathscr{A}^i=(\Omega^i)^m\oplus V^{i+1}$ that are used to define the adjoint D^* of D. We set $\Delta=DD^*+D^*D$. The complex $\mathscr{A}$ is in the hypothesis of the twisted Hodge theorem and therefore

$$\mathscr{A}^i=\mathbb{H}^i\oplus\Delta(\mathscr{A}^i)=\mathbb{H}^i\oplus\operatorname{Im}D\oplus\operatorname{Im}D^*,$$

where $\mathbb{H}^i$ is the kernel of $\Delta:\mathscr{A}^i\to\mathscr{A}^i$, and $H^i(\mathscr{A})\cong\mathbb{H}^i$ has finite dimension.

Let r be an integer big enough with respect to the dimension of m and let ${}^r\mathscr{A}^i$ denote the Sobolev r-completion of $\mathscr{A}^i$. The set

$$\Sigma=\{\tau\in {}^r\mathscr{A}^1\,|\,D^*\tau=0 \text{ and } D^*(D\tau+P\tau)=0\}$$
$$=\{\tau\in {}^r\mathscr{A}^1\,|\,D^*(D\tau+P\tau)+DD^*\tau=0\}$$

is, in a neighbourhood of the origin, an analytic submanifold of ${}^r\mathscr{A}^1$ of finite dimension whose tangent space at 0 is $\mathbb{H}^1\cong H^1(\mathscr{A})$. One can see that finding solutions of $D^*(D\tau+P\tau)+DD^*\tau=0$ reduces, as in the Regularity Lemma (1), to solve an elliptic equation and therefore the elements of Σ are smooth.

We denote by $(S,0)$ the germ at the origin of the analytic subset S of the integrable elements of Σ, that is

$$S=\Sigma\cap{}^r\Xi=\{\tau\in\Sigma\,|\,D\tau+P\tau=0\},$$

where ${}^{r}\Xi$ denotes the space of elements in ${}^{r}\mathscr{A}^1$ fulfilling the integrability condition $D\tau + P\tau = 0$. For a given element $\tau \in {}^{r}\mathscr{A}^1$ let $\psi = \psi(\tau)$ be the element of ${}^{r-1}\mathscr{A}^2$ defined as

$$\vartheta(\tau) = D\tau + P\tau.$$

A long but straightforward computation shows that the components of $D\vartheta = DP\tau$ can be written in terms of products of the coefficients of ϑ and τ. One can deduce that $D\vartheta$ fulfills the following estimate with respect to the Sobolev norms

$$\|D\vartheta\|_{r-2} \leqslant \text{const.}\|\vartheta\|_{r-2}\|\tau\|_{r-2}. \tag{22}$$

Now, using (22), it can be proved as in [11] (cf. also [13, p. 163]) that S can also be described as

$$S = \{\tau \in \Sigma \,|\, \text{harmonic part of } P\tau = 0\},$$

in particular S is smooth and coincides with Σ when $H^2(\mathscr{A}) = 0$. The elements of S define a family $\mathscr{F}_s$ of deformations of $\mathscr{F}$ parametrized by $(S,0)$.

From now on we will assume that the space S is smooth, possibly reduced to a point (this will happen if $H^1(\mathscr{A}) = 0$). We shall show that in this case the family $\mathscr{F}_s$ parametrized by $(S,0)$ is versal. This will prove parts (ii) and (iii) of the theorem.

The family of deformations $\mathscr{F}_s$ of $\mathscr{F}$ defines a family of Lie algebras $\mathfrak{g}_s$ and Lie subalgebras $\mathfrak{h}_s$ with $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{h}_0 = \mathfrak{h}$. Each $\mathfrak{g}_s$ and $\mathfrak{h}_s$ are identified as vector spaces to $\mathfrak{g}$ and $\mathfrak{h}$ through the linear isomorphism determined by $e_{k,s} \mapsto e_k$. We denote by $\mathscr{G}_s$ and $\mathscr{H}_s$ the local Lie groups corresponding to $\mathfrak{g}_s$ and $\mathfrak{h}_s$, respectively. Furthermore, using Proposition 3 we can construct a family ζ_s of $\mathfrak{g}_s$-valued 1-forms on the trivial $\mathscr{H}_s$-bundle $M \times \mathscr{H}_s$ which defines a family of Lie foliations on $M \times \mathscr{H}_s$ modeled over $\mathscr{G}_s$ and such that the family of deformations $\mathscr{F}_s$ is given by the family of 1-forms $\omega_s = s^*\zeta_s$, where $s(x) = (x, e)$ is the zero section.

Recall that in Section 3 we used the fixed Riemannian metric on M to construct a correspondence

$$\chi \mapsto (f_\chi, \hat{f}_\chi),$$

where χ is a smooth section of $N = T(M \times H)|_{M \times \{e\}}$ close to zero, f is a diffeomorphism of M and $\hat{f}$ is a H-bundle isomorphism of $M \times H$ over f. This correspondence defines a parametrization of pairs $(f, \hat{f})$ close to the identity by sections of N. We denote now by N_s the family of vector bundles $T(M \times \mathscr{H}_s)|_{M \times \{e\}}$ over M and by $\Phi_s : N_s \to M \times \mathscr{H}_s$ the map defined as

$$\Phi_s(x, v) = (\exp_x^M(v_M), \exp^{\mathscr{H}_s}(v_{\mathscr{H}_s})),$$

where $v = (v_M, v_{\mathscr{H}_s}) \in N_{s,(x,e)} = T_x M \times T_e \mathscr{H}_s$ and $\exp^{\mathscr{H}_s}$ denotes the exponential map associated to the lie group $\mathscr{H}_s$. A smooth section χ_s of N_s defines by the above construction a couple $(f_{\chi_s}, \hat{f}_{\chi_s})$, where f_{χ_s} is the diffeomorphism of M given by $f_{\chi_s}(x) = \exp_x^M(\chi_{sM}(x))$ and $\hat{f}_{\chi_s}$ is the principal $\mathscr{H}_s$-bundle isomorphism over f_{χ_s} given by

$$\hat{f}_{\chi_s}(x, h) = \Phi_s(x, \chi_s(x)) \cdot h.$$

Here the dot denotes the action of $h \in \mathscr{H}_s$ on the principal bundle $M \times \mathscr{H}_s$.

Let $X_{s,1}, \ldots, X_{s,q}$ be the vector fields on M which are orthogonal to the foliation $\mathscr{F}_s$ and are determined by the condition $\omega_s^j(X_{s,i}) = \delta_i^j$, for $i,j = 1, \ldots, q$. This vector fields can be thought as global sections of N_s. We denote by $X_{s,q+1}, \ldots, X_{s,m}$ the restriction to $M \times \{e\}$ of the fundamental vector fields $e^*_{s,q+1}, \ldots, e^*_{s,m}$, associated to $e_{s,q+1}, \ldots, e_{s,m}$, of the principal $\mathscr{H}_s$-bundle $M \times \mathscr{H}_s$. Then $X_{s,1}, \ldots, X_{s,m}$ are sections of N_s such that $\zeta_s(X_{s,k}) = e_{s,k}$ for $k = 1, \ldots, m$.

An element of $\mathscr{A}^0$ is a couple (κ,v) where $\kappa = (\kappa^1, \ldots, \kappa^m) \in (\Omega^0)^m$ and $v \in V^1 \subset \mathfrak{g}^* \otimes \mathfrak{g}$. Notice that v is a linear isomorphism of the vector space $\mathfrak{g}$ such that $v(\mathfrak{h}) = \mathfrak{h}$. Given a couple $(s,(\kappa,v))$ where $(\kappa,v) \in \mathscr{A}^0$ is close to zero and $s \in S$ we denote by κ_s the section of N_s defined as $\kappa_s = \sum_{k=1}^m \kappa^k X_{s,k}$ and by v_s the linear isomorphism of $\mathfrak{g}_s$ sending $\mathfrak{h}_s$ into $\mathfrak{h}_s$ obtained from v through the linear identification $\mathfrak{g}_s \cong \mathfrak{g}$ stated above. Therefore we can associate to $(s,(\kappa,v))$ the triple $(f_{\kappa_s}, \hat{f}_{\kappa_s}, v_s)$. The $\mathfrak{g}_s/\mathfrak{h}_s$-foliation $\mathscr{F}_s \circ (f_{\kappa_s}, \hat{f}_{\kappa_s}, v_s)$ induced as a pull-back of $\mathscr{F}_s$ by this triple is defined by an element $(\omega,\eta) + (\sigma, \psi) \in \mathscr{A}^1$ with $(\sigma, \psi) \in \Xi \subset \mathscr{A}^1$ close to zero. We denote the element (σ, ψ) by $\rho(s,(\kappa,v))$. The map $\rho : S \times \mathscr{A}^0 \to \Xi \subset \mathscr{A}^1$ defined in this way extends to a smooth map

$$\rho : S \times {}^{r+1}\mathscr{A}^0 \to {}^r\Xi \subset {}^r\mathscr{A}^1$$

which is properly defined in a neighbourhood of (0,0). Notice that the restriction of ρ to $S \equiv S \times \{0\}$ is the natural inclusion and that the restriction of ρ to $\{0\} \times {}^{r+1}\mathscr{A}^0$ has as differential map at the origin the linear map $D: {}^{r+1}\mathscr{A}^0 \to {}^r\mathscr{A}^1$. This follows from Proposition 5.

Let us consider the subspace of ${}^r\mathscr{A}^1$ containing ${}^r\Xi$ which is defined by

$${}^r\Psi = \{\tau \in {}^r\mathscr{A}^1 \mid D^*(D\tau + P\tau) = 0\}.$$

It follows from the implicit function theorem for Banach spaces and from the decomposition Hodge theorem that ${}^r\Psi$ is a Banach submanifold in a neighbourhood of the origin having as tangent space at the origin the kernel of the linear map $D: {}^r\mathscr{A}^1 \to {}^{r-1}\mathscr{A}^2$.

Let A a topological supplementary to the subspace $\mathbb{H}^0$ of ${}^{r+1}\mathscr{A}^0$. The tangent linear map at the origin of the restriction

$$\rho : S \times A \to {}^r\Xi \subset {}^r\Psi$$

is a linear isomorphism from $\mathbb{H}^1 \times A$ into the kernel of $D : {}^r\mathscr{A}^1 \to {}^{r-1}\mathscr{A}^2$. Therefore ${}^r\Xi = {}^r\Psi$ and ρ identifies $S \times A$ with ${}^r\Xi$ in a neighbourhood of the origin. From this it follows that the family $\mathscr{F}_s$ is versal as it is explained in [2].

Finally, assume that there is a family $\mathscr{F}_t$ parametrized by a smooth space T in such a way that the corresponding Kodaira–Spencer map $\varrho : T_0 T \to H^1(\mathscr{A})$ is an isomorphism. This means that the composition of the differential map at zero of the natural map $\iota : T \to \Xi$ with the projection ${}^r\mathscr{A}^1 \to \mathbb{H}^1$ is an isomorphism. In particular $\iota(T)$ is smooth in a neighbourhood of the origin. The above argument remains valid if we substitute the space S by $\iota(T)$ and this proves the last part of the theorem.

A stronger version of the above theorem can be proved for Lie foliations. Recall that a Lie $\mathfrak{g}$-foliation $\mathscr{F}$ is determined by a $\mathfrak{g}$-valued 1-form $\omega = (\omega^1, \ldots, \omega^q)$ on M fulfilling the Maurer–Cartan equation of $\mathfrak{g}$:

$$d\omega^k = -\frac{1}{2} \sum_{i,j=1}^{q} K^k_{ij} \omega^i \wedge \omega^j$$

and such that $\omega^1, \ldots, \omega^q$ are linearly independent at each point of M. As it was already remarked an isomorphism between Lie foliations determined by a $\mathfrak{g}$-valued 1-form ω and a $\mathfrak{g}'$-valued 1-form ω' is just given by a couple (f,v), where f is a diffeomorphism between the manifolds and v is an isomorphism between the Lie algebras such that $f^*\omega = v \circ \omega'$.

Let us consider an element $(\sigma,\psi)\in\mathscr{A}^1$ not necessarily integrable and let K denote the element of $\bigwedge^2\mathfrak{g}^*\otimes\mathfrak{g}$ defined as

$$K = -\frac{1}{2}\sum_{i,j,k} K^k_{ij}\theta^i\wedge\theta^j\otimes e_k.$$

Given a pair (f,v) where f is a diffeomorphism of M and v is a linear isomorphism of $\mathfrak{g}$, as vector space, we can associate to (σ,ψ) a new element $(\sigma',\psi')\in\mathscr{A}^1$ defined in the following way:

$$\sigma' = v^{-1}\circ f^*(\sigma+\omega)-\omega$$

and

$$\psi' = v(\psi+K)-K.$$

Here v stands for the linear isomorphism of $\bigwedge^2\mathfrak{g}^*\otimes\mathfrak{g}$ induced by $v:\mathfrak{g}\to\mathfrak{g}$. If (σ,ψ) is integrable defining a Lie foliation $\mathscr{F}'$ then (σ',ψ') represents the Lie foliation $\mathscr{F}'\circ(f,v)$. Therefore, when we are dealing with Lie foliations the map $\rho:S\times{}^{r+1}\mathscr{A}^0\to{}^r\Xi\subset{}^r\mathscr{A}^1$ can be extended to a map

$$\rho:{}^r\mathscr{A}^1\times{}^{r+1}\mathscr{A}^0\to{}^r\mathscr{A}^1$$

which is properly defined in a neighbourhood of $(0,0)$.

Given the elements $(\sigma,\psi)\in\mathscr{A}^1$ and $(\kappa,v)\in\mathscr{A}^0$ close to zero one has

$$\rho((\sigma,\psi),(\kappa,v)) = (\sigma,\psi)+D(\kappa,v)+Q((\sigma,\psi),(\kappa,v)),$$

where $Q((\sigma,\psi),(\kappa,v))$ denotes the terms of order two or more in (σ,ψ) and (κ,v). Using this relation one can reason as in [11] or [13] and show that for each $(\sigma,\psi)\in\mathscr{A}^1$ close to zero there is $(\kappa,v)\in\mathscr{A}^0$ close to zero such that

$$D^*\rho((\sigma,\psi),(\kappa,v)) = 0.$$

It follows that the restriction of ρ to $S\times A$ is surjective onto a neighbourhood of zero in Ξ and one obtains the following weak versality property for the family $\mathscr{F}_s$ parametrized by S.

Theorem 3. *Let $\mathscr{F}$ be a Lie foliation on a compact manifold M defined by a $\mathfrak{g}$-valued form ω. Let $(S,0)$ denote the germ at the origin of the analytic subset*

$$S = \Sigma\cap\Xi = \{\tau\in{}^r\mathscr{A}^1\,|\,D^*\tau = 0 \text{ and } D\tau+P\tau = 0\}.$$

The elements of S are smooth and determine a family of Lie foliations $\mathscr{F}_s$. This family is weakly versal in the following sense.

Let (σ,ψ) be an integrable element of $\mathscr{A}^1$, i.e. fulfilling $D(\sigma,\psi)+P(\sigma,\psi)=0$. Then it defines a Lie foliation $\mathscr{F}'$ on M. If the Sobolev norm $\|(\sigma,\psi)\|_r$ is small enough then $\mathscr{F}'$ is isomorphic to $\mathscr{F}_s$ for a certain $s\in S$.

In the case the analytic space S is smooth the family $\mathscr{F}_s$ is versal.

Remark 5. The deformations of the pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$ can be described by means of the complex $\mathscr{V}$ defined in (16). More precisely, the space of infinitesimal deformations of the pair $(\mathfrak{g},\mathfrak{h})$ is naturally identified to $H^2(\mathscr{V})$ and one can construct a versal space of deformations following the approach of Kodaira–Spencer–Kuranishi. For the general theory of deformations of a single algebra we refer to Gerstenhaber [5].

One could also develop a deformation theory for $\mathfrak{g}/\mathfrak{h}$-foliations, when the transverse model is fixed, by means of the complex $\mathscr{A}_0$ defined in (15). Nevertheless, the deformations of a foliation modeled on a fixed homogeneous space G/H is well understood by a completely different and general argument given by Thurston [16] (cf. also [6]).

6. Some examples

There exist different structures of homogeneous space on the circle $\mathbb{S}^1$. All of them are $\mathfrak{g}/\mathfrak{h}$-foliations because each principal bundle over $\mathbb{S}^1$ is trivial. In this section, we calculate the deformations of several of these homogeneous structures. We also consider the deformations of Abelian Lie foliations and study two concrete examples. Finally, we study the deformations of the homogeneous foliations, that is foliations on a manifold $M = \Gamma\backslash G$, where Γ is a discrete and cocompact subgroup of the Lie group G, whose leaves are the left cosets defined by a subgroup H of G. We compute in particular the infinitesimal deformations of the Roussarie foliation on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ and of the homogeneous flows on $S^3\times S^3$.

6.1. $\mathbb{S}^1$ as an Abelian homogeneous space

Let $\mathbb{S}^1$ be the circle obtained as the quotient $\mathbb{R}^n/\mathbb{R}^{n-1}\times\mathbb{Z}$. Then $\mathbb{S}^1$ inherits a structure of homogeneous space determined by the family of 1-forms $\omega_1 = \mathrm{d}t$, $\omega_2 = \cdots = \omega_n = 0$ and structure constants $K_{ij}^k = 0$. The spaces $\mathscr{A}^0$ and $\mathscr{A}^1$ can be described as follows. If f^k,σ^k denote functions over $\mathbb{S}^1$ and a_j^k, b_{ij}^k denote constants, with $1 \leqslant i,j,k \leqslant n$, one has

$$\mathscr{A}^0 = \{((f^1,\ \dots, f^n),(a_j^k))\,|\,a_j^1 = 0 \text{ for } j \geqslant 2\},$$

$$\mathscr{A}^1 = \{((\sigma^1\,\mathrm{d}t,\dots,\sigma^n\,\mathrm{d}t),(b_{ij}^k))\,|\,b_{ij}^k = -b_{ji}^k,\ b_{ij}^1 = 0 \text{ for } i,j \geqslant 2\}. \tag{23}$$

Moreover, the differential map $D:\mathscr{A}^0 \to \mathscr{A}^1$ is given by

$$D(f^k, a_j^k) = \left(\left(\frac{\mathrm{d}f^k}{\mathrm{d}t} - a_1^k\right)\mathrm{d}t, 0\right)$$

and since the manifold is one-dimensional and the group G is Abelian the differential map $D:\mathscr{A}^1 \to \mathscr{A}^2$ vanishes. We fix on $\mathbb{S}^1$ the Riemannian metric for which $\mathrm{d}t$ is the volume element. With this choice the adjoint $D^*:\mathscr{A}^1 \to \mathscr{A}^0$ is given by $D^*(\sigma^k\,\mathrm{d}t, b_{ij}^k) = (f^k, a_j^k)$ where

$$f^k = -\frac{\mathrm{d}\sigma^k}{\mathrm{d}t} \quad \text{and} \quad a_1^k = -\int \sigma^k\,\mathrm{d}t, \quad a_j^k = 0 \quad \forall j \geqslant 2$$

and $D^*:\mathscr{A}^2 \to \mathscr{A}^1$ also vanishes. This last fact implies that

$$\begin{aligned}\Sigma &= \{\tau \in \mathscr{A}^1 \mid D^*\tau = 0, D^*(D\tau + P\tau) = 0\}\\ &= \{\tau \in \mathscr{A}^1 \mid D^*\tau = 0\}\\ &= \mathbb{H}^1.\end{aligned}$$

One can see that in fact

$$\Sigma = \mathbb{H}^1 = \{(0, b_{ij}^k) \in \mathscr{A}^1 \mid b_{ij}^k = -b_{ji}^k,\ b_{ij}^1 = 0 \text{ for } i,j \geqslant 2\}.$$

In particular, the space $\mathbb{H}^1$ has dimension $n\binom{n}{2} - \binom{n-1}{2}$. Therefore, the space S of integrable elements of Σ is given by the elements $(0, b_{ij}^k) \in \mathbb{H}^1$ fulfilling the integrability condition

$$\sum_{j=1}^{n} b_{jk}^i b_{lm}^j + b_{jl}^i b_{mk}^j + b_{jm}^i b_{kl}^j = 0 \quad \text{for } 1 \leqslant k, l, m \leqslant n. \tag{24}$$

For $n = 1$ one has $\mathbb{H}^1 = 0$ and the structure is rigid. For $n = 2$ the space S has dimension 2 and it is smooth, therefore it is versal. For $n \geqslant 3$, S is the singular space defined by the intersection of quadrics (24). Notice that in any case S is a cone. Moreover, S is just the versal space of the deformations of the pair $(\mathfrak{r}_n, \mathfrak{r}_{n-1})$, where $\mathfrak{r}_n$ denotes the Abelian Lie algebra of dimension n. Since the structure constants of $\mathfrak{r}_n$ are all zero, the elements of S of the form $(0, \lambda b_{ij}^k)$ define the same pair of algebras for each $\lambda \in \mathbb{R}\setminus\{0\}$. Therefore, by deformation of the original structure one obtains all the couples $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is an arbitrary Lie algebra of dimension n admitting a Lie subalgebra $\mathfrak{h}$ of codimension one.

6.2. $\mathbb{S}^1$ as homogeneous space of the affine group GA

We consider the circle $\mathbb{S}^1$ as the homogeneous space $G/H = GA/\mathbb{R} \rtimes \mathbb{Z}$ where $G = GA$ is the group of the affine transformations of the real line which preserve the orientation. If we think GA as the matrix group

$$GA = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}_+^*,\ y \in \mathbb{R} \right\}$$

then $H = \mathbb{R} \rtimes \mathbb{Z}$ is a semidirect product included in GA as the subgroup

$$\left\{ \begin{pmatrix} \lambda^n & y \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}, y \in \mathbb{R} \right\}$$

for a given positive number $\lambda \neq 1$. We fix the basis of left-invariant vector fields on GA given by

$$e_1 = x\frac{\partial}{\partial x}, \qquad e_2 = x\frac{\partial}{\partial y}.$$

The dual basis is given by $\theta^1 = \mathrm{d}x/x$ and $\theta^2 = \mathrm{d}y/x$. With these choices the $\mathfrak{g}/\mathfrak{h}$-foliation on $\mathbb{S}^1$ is determined by the 1-forms $\omega^1 = \mathrm{d}t$, and $\omega_2 = 0$, where $t = \log x$, and the set of structure constants

K_{ij}^k fulfilling $K_{12}^1 = 0$ and $K_{12}^2 = 1$. One can compute the space $\mathbb{H}^1$ of harmonic elements of $\mathscr{A}^1$ as in the previous example obtaining

$$\mathbb{H}^1 = \{((a\,\mathrm{d}t,0),(b,-a))\,|\,a,b\in\mathbb{R}\}.$$

Moreover, since the integrability condition is always satisfied one has $S = \Sigma = \mathbb{H}^1$ and $\mathbb{H}^1$ parametrizes a versal family of deformations of the given structure. Notice that the group G remains unchanged and that the deformations correspond to move the embedding of $\mathbb{R}\rtimes\mathbb{Z}$ inside GA. The new structures can also be viewed as the homogeneous structures on $\mathbb{S}^1$ given by the quotients $GA/H_{\mu,c}$ where $H_{\mu,c}$ is the subgroup of GA given by

$$H_{\mu,c} = \left\{\begin{pmatrix} \mathrm{e}^{ct+n\log\mu} & \frac{1}{c}(\mathrm{e}^{ct+n\log\mu}-1) \\ 0 & 1 \end{pmatrix}\,|\,n\in\mathbb{Z},\ t\in\mathbb{R}\right\}.$$

We remark that $H_{\mu,c}$ is not a continuous deformation, at $c=0$, of H.

6.3. $\mathbb{S}^1$ *as homogeneous space of the group of projectivities* SL(2,$\mathbb{R}$)

Now we consider $\mathbb{S}^1$ as the homogeneous space SL(2,$\mathbb{R}$)/GA where SL(2,$\mathbb{R}$) is the special linear group and the affine group GA is included in SL(2,$\mathbb{R}$) as the subgroup

$$GA\cong H = \left\{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\middle|\, x,y\in\mathbb{R},\ x>0\right\}.$$

We fix the basis of left-invariant vector fields on SL(2,$\mathbb{R}$) corresponding to the matrices of $\mathfrak{sl}(2)$

$$X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\qquad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\qquad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we denote by $\theta^1,\theta^2,\theta^3$ the corresponding dual basis. Notice that X_2 and X_3 generate the Lie algebra of H and that the Lie brackets of vector fields X_i are

$$[X_1,X_2] = -X_3,\qquad [X_1,X_3] = 2X_1,\qquad [X_2,X_3] = -2X_2.$$

We can think the elements of $\mathbb{S}^1$ as the linear rotations of $\mathbb{R}^2$. Then the restriction of the forms θ^i to $\mathbb{S}^1$ are

$$\omega^1 = \mathrm{d}t,\quad \omega^2 = -\mathrm{d}t,\quad \omega^3 = 0,$$

where t denotes the rotation angle. This family of 1-forms and the structure constants K_{ij}^k, which are all zero except

$$K_{12}^3 = -K_{21}^3 = -1,\qquad K_{13}^1 = -K_{31}^1 = 2,\qquad K_{23}^2 = -K_{32}^2 = -2,$$

define the structure of $\mathfrak{g}/\mathfrak{h}$-foliation on $\mathbb{S}^1$.

The spaces $\mathscr{A}^0$ and $\mathscr{A}^1$ are also described by (23) and

$$\mathscr{A}^2 = \{(0,(c^1,c^2,c^3))\,|\,c^k\in\mathbb{R}\}.$$

The operator $D^*: \mathscr{A}^2 \to \mathscr{A}^1$ is given by $D^*(0, c^k) = (0, b^k_{ij})$ where

$$b^1_{12} = -b^3_{23} = 2c^1,$$
$$b^2_{12} = -b^3_{13} = -2c^2,$$
$$b^2_{13} = -b^2_{23} = c^3,$$
$$b^3_{12} = b^2_{13} = b^1_{23} = 0.$$

It follows that this operator is injective and therefore

$$\begin{aligned}\Sigma &= \{\tau \in \mathscr{A}^1 \mid D^*\tau = 0, D^*(D\tau + P\tau) = 0\}\\ &= \{\tau \in \mathscr{A}^1 \mid D^*\tau = 0, D\tau + P\tau = 0\}\\ &= S.\end{aligned}$$

Similar computations to those in the previous examples show that, also in this case, $\Sigma = \mathbb{H}^1$ and in fact that $\mathbb{H}^1$ is the subspace of elements $(\sigma^k \mathrm{d}t, b^k_{ij}) \in \mathscr{A}^1$ with

$$\sigma^1(t) = a\cos 2t + b\sin 2t + 4c, \qquad \sigma^2(t) = \sigma^1(t) - 8c, \qquad \sigma^3(t) = -\frac{\mathrm{d}\sigma^1}{\mathrm{d}t},$$
$$b^1_{12} = -\tfrac{1}{2}b^1_{13} = -b^2_{13} = \tfrac{1}{2}b^2_{23} = \tfrac{1}{4}b^3_{12} = b^3_{23} = c,$$
$$b^2_{12} = b^3_{13} = b^1_{23} = 0,$$

where a, b, c are real parameters. All this implies that $S = \mathbb{H}^1$ is a smooth versal space and of dimension 3. The family of pairs of Lie algebras associated to the versal deformation is parametrized by c. For each value of c different from the unique real root c_0 of the polynomial $P(c) = -17c^3 + 36c^2 - 24c + 4$ the Lie algebra $\mathfrak{g}_c$ is isomorphic to $\mathfrak{sl}(2)$ while $\mathfrak{g}_{c_0}$ is not semisimple. In fact, $\mathfrak{g}_{c_0}$ is the Lie algebra $\mathfrak{sol}$, the unique solvable nonnilpotent Lie algebra of dimension 3. In any case $\mathfrak{h}_c$ is the Lie algebra of the affine group GA.

6.4. Abelian Lie foliations

Let M be a compact manifold endowed with an Abelian Lie foliation $\mathscr{F}$ of codimension $m = q$. That is, $\mathscr{F}$ is modeled on the group $G = \mathbb{R}^m$ and is defined by a collection of m closed 1-forms $\omega^1, \ldots, \omega^m$ on M which are linearly independent at each point.

Each 1-form ω^i defines a nontrivial cohomology class $[\omega^i]$ in the first de Rham's cohomology group $H^1_{\mathrm{DR}}(M)$ of the manifold M. Let Z^1 be the subspace of $H^1_{\mathrm{DR}}(M)$ generated by the classes $[\omega^1], \ldots, [\omega^m]$. More generally, we denote by Z^* the subalgebra of $H^*_{\mathrm{DR}}(M)$ generated by Z^1.

In the case of an Abelian Lie foliation the complex $\mathscr{A}$ has the following particular form

$$\mathscr{A}^k = (\Omega^k(M))^m \oplus V^{k+1} = (\Omega^k(M))^m \oplus \left(\bigwedge^{k+1} \mathfrak{g}^* \otimes \mathfrak{g} \right)$$

and

$$D : \mathscr{A}^k \to \mathscr{A}^{k+1},$$
$$(\sigma, \psi) \mapsto ((\mathrm{d}\sigma^i - \mathscr{R}^i(\psi)), 0).$$

Recall that $\mathscr{R} = (\mathscr{R}^1, \ldots, \mathscr{R}^m)$ is the realization map defined in (7). In our situation $\mathscr{R}$ induces a linear map

$$\hat{\mathscr{R}}_k : \left(\bigwedge^k \mathfrak{g}^* \otimes \mathfrak{g} \right) \to H^k_{\mathrm{DR}}(M).$$

Notice that Z^k is the image of this map. We denote by B^{k+1} the kernel of $\hat{\mathscr{R}}_{k+1}$. In this situation, we have

Proposition 9. $H^k(\mathscr{A}) \cong (H^k_{\mathrm{DR}}(M)/Z^k)^m \oplus B^{k+1}$.

We apply this result to the following particular examples.

Linear foliations on the torus $\mathbb{T}^n$: We suppose now that the manifold M is the torus $\mathbb{T}^n$. A set $\omega^1, \ldots, \omega^m$ of independent linear 1-forms on $\mathbb{T}^n$ defines an Abelian Lie foliation $\mathscr{F}$ of codimension m. The deformations of $\mathscr{F}$ inside the space of linear foliations on $\mathbb{T}^n$ are still Abelian Lie foliations and are parametrized by a smooth space of dimension $m(n-m)$. The Kodaira–Spencer map of the family of deformations so obtained is an isomorphism and we deduce from Theorem 2 that this family is actually versal.

An example of deformation of an Abelian Lie flow on a nilmanifold: We construct here an example of Abelian Lie foliation with nonabelian deformations. Let K be the nilpotent Lie group of matrices

$$\begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{e}^t \end{pmatrix}.$$

The vector fields

$$Z = \frac{\partial}{\partial z}, \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial t}$$

is a basis of left-invariant vector fields and the corresponding dual basis is

$$\beta = \mathrm{d}z - x\,\mathrm{d}y, \quad \omega^1 = \mathrm{d}x, \quad \omega^2 = \mathrm{d}y, \quad \omega^3 = \mathrm{d}t.$$

Let Γ be the discrete subgroup of K whose elements are the matrices with $x, y, z, t \in \mathbb{Z}$ and denote by M the compact nilmanifold $\Gamma \backslash K$. We still denote by Z, X_i and β, ω^i the projection onto M of the above vector fields and 1-forms. The vector field Z defines on M an Abelian Lie foliation $\mathscr{F}$ of codimension 3 which is also defined by the differential system $\omega^1 = \omega^2 = \omega^3 = 0$. This foliation can be deformed into the family $\mathscr{F}_s$ defined by the vector fields $Z + s \cdot X_3$. For $s \neq 0$, $\mathscr{F}_s$ is a Lie foliation modeled on the three-dimensional Heisenberg group.

The de Rham cohomology $H^*_{\mathrm{DR}}(M)$ is isomorphic to the cohomology of the Lie algebra Lie(K) of K (cf. [14]). Therefore H^1_{DR} is generated by the cohomology classes of $\omega^1, \omega^2, \omega^3$ and H^2_{DR} by the cohomology classes of $\beta \wedge \omega^1, \beta \wedge \omega^2, \omega^1 \wedge \omega^3, \omega^2 \wedge \omega^3$. Using the above proposition and by means of an easy computation one can see that $\dim H^1(\mathscr{A}) = 3$, more precisely $H^1(\mathscr{A})$ is a direct sum of three copies of the one-dimensional space generated by $(-\beta, \omega^1 \wedge \omega^2)$.

The computation of the space S in this case is much more involved than in the previous examples because now the system of equations defining the space Σ is nonlinear. We suspect that the only nonobstructed deformations of $\mathscr{F}$ correspond to the family $\mathscr{F}_s$ exhibited above.

6.5. Homogeneous foliations

Let G be a connected and simply-connected Lie group, H a connected subgroup of G and Γ a discrete and cocompact subgroup of G. Then $M = \Gamma\backslash G$ is a compact manifold endowed with the foliation $\mathscr{F}$ whose leaves are the left cosets defined by H. We say that $\mathscr{F}$ is a homogeneous foliation. The basis of left-invariant 1-forms $\theta^1,\dots,\theta^m$ on G project onto a basis of 1-forms $\omega^1,\dots,\omega^m$ on M. These forms define $\mathscr{F}$ as a $\mathfrak{g}/\mathfrak{h}$-foliation. In this paragraph, we describe the space of infinitesimal deformations $H^1(\mathscr{A})$ of a homogeneous foliation in two different situations.

Any deformation of the representation of groups $\vartheta: \pi_1(M) \cong \Gamma \to G$ induces a deformation of the $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}$. The following theorem states that, under certain conditions implying that the pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$ is rigid, the infinitesimal deformations of $\mathscr{F}$ coincides with the infinitesimal deformations of the representation ϑ. We recall that the space of infinitesimal deformations of ϑ is the first cohomology group $H^1(\Gamma,\mathfrak{g})$ of the group Γ with values in $\mathfrak{g}$ viewed as a Γ-module via the adjoint representation. We also recall that the deformations of the pair $(\mathfrak{g},\mathfrak{h})$ are described by the complex $\mathscr{V}$ introduced in (16).

Theorem 4. *Assume that G is contractible and that* $H^k(\mathscr{V}) = 0$ *for* $k = 1,2$. *Then* $H^1(\mathscr{A}) \cong H^1(\Gamma,\mathfrak{g})$.

Proof. Besides the complexes $\mathscr{A}$ and $\mathscr{V}$ we also consider the complex $\mathscr{A}_0$ which is the kernel of the complex morphism defined by the projections $\mathscr{A}^r \to V^{r+1}$ given by $(\sigma,\psi) \mapsto \psi$. This complex morphism induces the long exact sequence of cohomology

$$\cdots \to H^0(\mathscr{A}) \to H^1(\mathscr{V}) \to H^1(\mathscr{A}_0) \to H^1(\mathscr{A}) \to H^2(\mathscr{V}) \cdots .$$

The hypothesis made on $\mathscr{V}$ imply that $H^1(\mathscr{A}_0) \cong H^1(\mathscr{A})$.

In order to compute $H^1(\mathscr{A}_0)$ we consider now the system of linear partial differential equations

$$0 = (\hat{d}_M f)^k = \mathrm{d}f^k + \sum_{ij} K^k_{ij} f^j \omega^i \tag{25}$$

associated to the differential operator $\hat{d}_M$ introduced in (8) and where the unknown is the vector-valued function $f = (f^1,\dots,f^m)$. The space of global solutions of (25) is the kernel of $\hat{d}_M : \mathscr{A}^0_0 \to \mathscr{A}^1_0$. Using Jacobi's identity one can easily check that Eq. (25) fulfills the conditions of Frobenius integrability theorem. Therefore, the set of germs of solutions of (25) is a sheaf Θ over M which is locally isomorphic to a constant sheaf. It follows that the cohomology $H^*(\mathscr{A}_0)$ is naturally identified to the cohomology of M with values in Θ.

Let us consider the fibered product $M_{\mathfrak{g}} = G \times_\Gamma \mathfrak{g}$, where Γ acts on G by left translations and on $\mathfrak{g}$ by means of the adjoint representation. The projection onto the first factor

$$G \times_\Gamma \mathfrak{g} \to \Gamma\backslash G \tag{26}$$

defines $M_{\mathfrak{g}}$ as a flat vector bundle over $M = \Gamma \backslash G$. By a flat (local) section of (26) we shall mean a section s which is locally constant in the following sense. If U is a local chart of G mapped homeomorphically by $\pi : G \to M$ onto an open subset of M then U can also be viewed as a local chart of M on which fibration (26) is trivial. In such a chart s is given by $s(g) = v$ with v a fixed vector in $\mathfrak{g}$. Notice that in the chart $\gamma \cdot U$, where $\gamma \in \Gamma$, the section s is written $s(\gamma g) = \mathrm{Ad}_\gamma\, v$.

Given a flat local section s of (26), which in the local coordinates considered above will be written $s(g) = v$, we define the local function given by

$$f_s(g) = \mathrm{Ad}_{g^{-1}}\, v. \tag{27}$$

If we consider local coordinates where s is written $s(\gamma g) = \mathrm{Ad}_\gamma\, v$ then $f_s(\gamma g) = \mathrm{Ad}_{(\gamma g)^{-1}}\, \mathrm{Ad}_\gamma\, v = \mathrm{Ad}_{g^{-1}}\, v$. This means that f_s is a well-defined local function on M with values in $\mathfrak{g}$.

The key point in the proof of the theorem is the following lemma.

Lemma 2. *The vector-valued function $f_s(g) = \mathrm{Ad}_{g^{-1}}\, v$ is a solution of the differential equation $\hat{d}_M f = 0$. Moreover, every local solution of this equation is a function f_s for a suitable flat local section s of the bundle* (26).

Proof. Since the question is local we can suppose that the function f_s is defined in an open subset of G taking values in $\mathfrak{g}$. The elements of the basis $e_1, \ldots, e_m$ of the Lie algebra $\mathfrak{g}$ will be also thought as left-invariant vector fields on G. Moreover, since we are working on G we have $\omega^k = \theta^k$. With these identifications we have

$$\begin{aligned} d_g f_s(e_i) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_s(g \exp(t\, e_i)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(-t\, e_i)} \mathrm{Ad}_{g^{-1}}(v) \\ &= [-e_i, f_s(g)] = -\sum_{kj} K^k_{ij} f^j_s(g) e_k. \end{aligned}$$

Therefore, f_s fulfills the system of differential equations

$$\mathrm{d} f^k_s + \sum_{ij} K^k_{ij} f^j_s \theta^i = 0. \tag{28}$$

With the identifications made this is just Eq. (25). On the other hand, and since system (28) fulfills the conditions of Frobenius integrability theorem each local solution is of the form $f_s(g) = \mathrm{Ad}_{g^{-1}}\, v$ for a suitable $v \in \mathfrak{g}$. □

End of the Proof. The above lemma implies that the correspondence $s \mapsto f_s$ induces a sheaf isomorphism $\mathscr{G} \to \Theta$, where $\mathscr{G}$ denotes the sheaf of germs of flat local sections of (26). Therefore there are natural isomorphisms

$$H^*(M, \Theta) \cong H^*(M, \mathscr{G}).$$

Finally, and since the group G is contractible we are in the hypothesis of a theorem due to Eilenberg [3] stating that $H^*(\Gamma \backslash G, \mathscr{G})$ is canonically isomorphic to $H^*(\Gamma, \mathfrak{g})$. This ends the proof. □

This result can be applied to the following example of homogeneous foliation due to Roussarie.

Roussarie's foliation: Let Γ be a discrete and cocompact subgroup of SL(2,$\mathbb{R}$). We identify the affine group GA with the subgroup H of SL(2,$\mathbb{R}$) defined by

$$H = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \colon a, b \in \mathbb{R},\ a > 0 \right\}.$$

The foliation of SL(2,$\mathbb{R}$) whose leaves are the left cosets of H induces a homogeneous foliation $\mathscr{F}$ on $M = \Gamma \backslash \mathrm{SL}(2,\mathbb{R})$. The fundamental group $\tilde{\Gamma} = \pi_1(M)$ is a subgroup of the universal covering $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of SL(2,$\mathbb{R}$) in a natural way. Notice that $\tilde{\Gamma}$ is an extension of Γ by $\mathbb{Z}$. A direct computation shows that for this foliation the cohomology $H^*(\mathscr{V})$ vanishes. Since $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is contractible we are in the hypothesis of the theorem and we can conclude that the space of infinitesimal deformations of $\mathscr{F}$ is identified to $H^1(\tilde{\Gamma},\mathfrak{sl}(2))$.

We end this section by considering the case in which the Lie group G is compact and connected, $\Gamma = 1$, $M = G$ and the homogeneous foliation is defined by the left cosets of a subgroup H of G. In this situation $\omega^k = \theta^k$. The action of G onto itself by left translations induce an action of G on each space $\mathscr{A}^k$: if $((\sigma^1,\ldots,\sigma^m),\psi) \in \mathscr{A}^k$ then $g \cdot (\sigma,\psi) = ((L_g^*\sigma^1,\ldots,L_g^*\sigma^m),\psi)$. This action commutes with the differential map D. Hence the subspaces $\mathscr{A}^k_G$ of those elements of $\mathscr{A}^k$ which are invariant by G define a subcomplex $\mathscr{A}_G$ of $\mathscr{A}$. In this situation, we can prove the following version of a classical theorem of E. Cartan.

Proposition 10. *The inclusion* $\imath \colon \mathscr{A}_G \to \mathscr{A}$ *induces an isomorphism in cohomology* $H^*(\mathscr{A}_G) \cong H^*(\mathscr{A})$.

Proof. We consider on the product manifold $M \times \mathbb{R}$ the $\mathfrak{g}/\mathfrak{h}$-foliation $\mathscr{F}_{M\times\mathbb{R}}$ whose leaves are the product of the leaves of $\mathscr{F}$ with $\mathbb{R}$. We denote by $\mathscr{A}(M)$ and $\mathscr{A}(M\times\mathbb{R})$ complexes (14) corresponding, respectively, to the foliations $\mathscr{F}$ and $\mathscr{F}_{M\times\mathbb{R}}$. Notice that $\mathscr{A}^p(M\times\mathbb{R}) = (\Omega^p(M\times\mathbb{R}))^m \oplus V^{p+1}$. We define the retraction morphism $R \colon \mathscr{A}^p(M\times\mathbb{R}) \to \mathscr{A}^{p-1}(M)$ by $R((\sigma^1,\ldots,\sigma^m),\psi) = ((\int_0^1 \sigma^1\,\mathrm{d}t,\ldots,\int_0^1\sigma^m\,\mathrm{d}t),0)$. Here $\int \sigma^k\,\mathrm{d}t$ stands for the integration of σ^k along the fibers of the fibration $M\times\mathbb{R} \to M$. Let $j_t \colon M \to M\times\mathbb{R}$ be the inclusion defined by $j_t(x) = (x,t)$ and define $J_t^* \colon \mathscr{A}^p(M\times\mathbb{R}) \to \mathscr{A}^p(M)$ by $J_t^*(\sigma,\psi) = (j_t^*\sigma,\psi)$. Then one can easily check that

$$DR + RD = J_1^* - J_0^*. \tag{29}$$

Using this identity one can complete the proof as in the classical theorem: it follows from (29) and the connectedness of the Lie group G that $g\cdot(\sigma,\psi)$ is cohomologous to (σ,ψ) for each $g \in G$ and each cocycle $(\sigma,\psi) \in \mathscr{A}(M)$. Now integration with respect to the Haar measure of the compact Lie group G defines a morphism $\mu \colon \mathscr{A} \to \mathscr{A}_G$ such that the corresponding morphism in cohomology μ^* is just the inverse map of $\imath^*$. □

Homogeneous flows on $S^3\times S^3$: The Lie algebra $\mathfrak{so}(3)$ of S^3 is generated by left-invariant vector fields v_1, v_2, v_3 on S^3 such that

$$[v_1,v_2] = v_2, \quad [v_2,v_3] = v_1, \quad [v_3,v_1] = v_2.$$

Let $M = G = S^3 \times S^3$ be the Lie group obtained as a product of two copies of S^3 and set $\tilde{v}_i = (v_i, 0)$ for $i = 1,2,3$ and $\tilde{v}_i = (0, v_{i-3})$ for $i = 4,5,6$. The vector fields $\tilde{v}_1, \ldots, \tilde{v}_6$ determine a basis of $\mathfrak{g} = \mathfrak{so}(3) \times \mathfrak{so}(3)$. Define $e_i = \tilde{v}_i$ for $i = 1, \ldots, 5$ and $e_6 = \tilde{v}_6 + \lambda \tilde{v}_3$ where $\lambda \in \mathbb{R}$ is a fixed parameter. Let H be the one-parameter subgroup of G associated to e_6. The dual basis $\omega^1 = \theta^1, \ldots, \omega^6 = \theta^6$ of $e_1, \ldots, e_6$ determines a homogeneous foliation $\mathscr{F}$ on M whose leaves are the left cosets defined by H. Using the above proposition one can easily compute $H^1(\mathscr{A}) \cong H^1(\mathscr{A}_G)$. It is the vector space of dimension 5 generated by the cohomology classes of

$$(\sigma, \hat{d}_{\mathfrak{g}} \sigma)$$

with $\sigma = (\sigma^1, \ldots, \sigma^5, 0)$, $\sigma^k = \alpha^k \omega^6$ and $\alpha^k \in \mathbb{R}$. Moreover, all the vector-valued forms $\tau = \omega + \sigma$ are integrable and the family of deformations of $\mathscr{F}$ defined by them is versal. This family corresponds to the family of one parameter subgroups of G generated by the vector fields $e_6 - \sum_{i=1}^{5} \alpha^i e_i$.

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