FOLIATIONS

by

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HANDBOOK OF DIFFERENTIAL GEOMETRY
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0. Foreword

Foliation Theory is the qualitative study of Differential Equations. It was initiated by the works of H. Poincaré, I. Bendixon and developped later by C. Ehresmann, G. Reeb and many other people. Since then the subject has been a wide field in mathematical research. Actually it is almost impossible to describe all the results and the different steps of its development. So the purpose of this chapter is to give definitions, some examples and the fundamental concepts like holonomy, transverse structures *etc.* Some themes in the point of view of Differential Geometry are discussed: characteristic classes, basic Hodge theory, deformations *etc.* A complete account on Foliation Theory can be found in the book [God2] by C. Godbillon. The bibliography is not complete. It is motivated by two reasons: the first one is to indicate references for the reader who wants to learn much more on foliations; the second one is to mention people who make contributions to the subject; for most of them the selected list is non exhaustive. All foliations considered are *regular* that is, all leaves have the same dimension. The theory of *singular foliations* and specially *holomorphic singular foliations* is well developped with a plentiful litterature. It merits to be presented independantly. References on the subject can be found on the paper [Cev] by D. Cerveau.

Unless otherwise stated, all the objects (manifolds, maps, functions *etc.*) are assumed to be of class C^∞. Moreover, for simplicity, we will suppose that all the manifolds are orientable. For any manifold M, we denote by A the algebra of functions on M. If $E \longrightarrow M$ is a vector bundle, $C^\infty(E)$ will denote the space of its global sections; this is an A-module and, equipped with the C^∞-topology, it is a Fréchet space. In case E is the tangent bundle TM of M, we denote $C^\infty(TM)$ simply by $\chi(M)$ (the space of vector fields tangent to M). For $r \in \mathbf{N}$, $\Omega^r(M)$ is the space of differential forms of degree r on M which is by definition $C^\infty(\Lambda^r T^*M)$ where $\Lambda^r T^*M \longrightarrow M$ is the vector bundle with fibre at $x \in M$ the vector space of skew-symmetric forms of degree r on T_xM; $\Omega^0(M)$ is just A. The other notations will be introduced at need.

We would like to thank Cédric Rousseau for drawing all the pictures in this paper.

1. Definitions and examples

Let M be the Euclidean space $\mathbf{R}^{m+n} = \mathbf{R}^m \times \mathbf{R}^n$ with canonical coordinates denoted $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and consider the family of affine subspaces F_y of M where

$y \in \mathbf{R}^n$, defined by the differential system: $dy_1 = \ldots = dy_n = 0$. Then M, considered as a disjoint union of these spaces, is a non connected manifold of dimension m. Its topology is the product of the usual topology on $\mathbf{R}^m$ and the discrete one on $\mathbf{R}^n$. We say that M, with this structure, is a *foliated manifold* of *dimension* m and *codimension* n. It constitutes the *local model* of a *foliation* of codimension n on a manifold of dimension $m+n$. Let $\mathcal{O}$ be an open set of $\mathbf{R}^{m+n}$; let us call a *plaque* of $\mathcal{O}$ any intersection of $\mathcal{O}$ with a horizontal space F_y.

Definition 1 *Let M be a manifold of dimension $m+n$. A codimension n foliation $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = (U_i)_{i \in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbf{R}^{m+n} \longrightarrow U_i$ such that, on each non empty intersection $U_i \cap U_j$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : (x, y) \in \varphi_i^{-1}(U_i \cap U_j) \longrightarrow (x', y') \in \varphi_j^{-1}(U_i \cap U_j)$ has the form:*

$$x' = \varphi_{ij}(x, y) \quad \text{and} \quad y' = \gamma_{ij}(y). \tag{1}$$

This means that the diffeomorphism $\varphi_j^{-1} \circ \varphi_i$ sends a plaque of $\varphi_i^{-1}(U_i \cap U_j)$ into a plaque of $\varphi_j^{-1}(U_i \cap U_j)$. The manifold M is decomposed into connected submanifolds of dimension m. Each of these submanifolds is called a *leaf* of $\mathcal{F}$. A subset U of M is *saturated* for $\mathcal{F}$ if it is union of leaves that is, if $x \in U$ then the leaf passing through x is contained in U.

Coordinate patches (U_i, φ_i) satisfying conditions of definition 1 are said to be *distinguished* for the foliation $\mathcal{F}$.

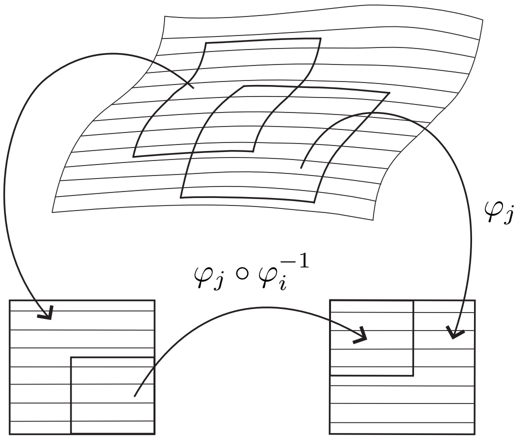

Fig.1

Let $\mathcal{F}$ be a codimension n foliation on M defined by a maximal atlas $(U_i, \varphi_i)_{i \in I}$ like in definition 1. Let $\pi : \mathbf{R}^{m+n} = \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be the second projection. Then the map

$f_i : U_i \overset{\pi\circ\varphi_i^{-1}}{\longrightarrow} \mathbf{R}^n$ is a submersion. On $U_i \cap U_j \neq \emptyset$ we have $f_j = \gamma_{ij} \circ f_i$. The submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbf{R}^n$ give a complete characterization of $\mathcal{F}$.

Definition 2 *A codimension n foliation on M is given by an open cover $(U_i)_{i\in I}$, submersions $f_i : U_i \longrightarrow T$ over a n dimensional transverse manifold T and, for $U_i \cap U_j \neq \emptyset$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$ satisfying:*

$$f_j(x) = \gamma_{ij} \circ f_i(x) \ \textit{for} \ x \in U_i \cap U_j. \tag{2}$$

We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a foliated cocycle defining $\mathcal{F}$.

The proof of the equivalence of definitions 1 and 2 is not difficult; it is left to the reader.

The foliation $\mathcal{F}$ is said to be *transversely orientable* if T can be given an orientation preserved by all the local diffeomorphisms γ_{ij}.

1.1. Morphisms of foliations

Let M and M' be two manifolds endowed respectively with two foliations $\mathcal{F}$ and $\mathcal{F}'$. A map $f : M \longrightarrow M'$ will be called *foliated* or a *morphism* between $\mathcal{F}$ and $\mathcal{F}'$ if, for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf of $\mathcal{F}'$; we say that f is an *isomorphism* if, in addition, f is a diffeomorphism; in this case the restriction of f to any leaf $L \in \mathcal{F}$ is a diffeomorphism on the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M with a codimension n foliation $\mathcal{F}$. Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation $\mathcal{F}'$ on M; we say that $\mathcal{F}'$ is the *image* of $\mathcal{F}$ by the diffeomorphism f and we write $\mathcal{F} = f^*(\mathcal{F}')$. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are said to be C^r-*conjugated* (*topologically* if $r = 0$, *differentiably* if $r = \infty$ and *analytically* in the case $r = \omega$) if there exists a C^r-homeomorphism $f : M \longrightarrow M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

1.2. The concept of holonomy

This is a very important notion in Foliation Theory. In many situations it determines completely the structure of the foliation. In this subsection, we will introduce this concept and give the statement of the local and global *stability theorems.*

Let $\mathcal{F}$ be a codimension n foliation on M, let L be a leaf of $\mathcal{F}$ and $x \in L$. Let T be a small transversal to $\mathcal{F}$ passing through x. Let $\sigma : [0,1] \longrightarrow L$ be a continuous path such that $\sigma(0) = \sigma(1) = x$. Then there exist a finite open cover U_i, $i = 0, 1, \ldots, k$ of M with $U_0 = U_k$ and a subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1]$ such that:

– $\sigma([t_{i-1}, t_i]) \subset U_i$,

– if $U_i \cap U_j \neq \emptyset$ then $U_i \cup U_j$ is contained in a distinguished chart of $\mathcal{F}$. We say that U_i is a *subordinated chain* to σ. For $i = 0, 1, \ldots, k$ let T_i be a small transversal to $\mathcal{F}$ passing through $\sigma_i(t)$ with $T_0 = T_k = T$. For every point $z \in T_i$, sufficiently close to $\sigma(t_i)$, the plaque of $\mathcal{F}$ passing through z intersects T_{i+1} in a unique point $f_i(z)$. The domain of f_i contains a transversal T_i' passing through $\sigma(t_i)$ and homeomorphic to an open ball of $\mathbf{R}^n$. Then, it is clear that the map: $f_\sigma = f_{k-1} \circ f_{k-2} \circ \ldots \circ f_0$ is well defined on an open neighbourhood of x; it is called the *holonomy map* associated to σ. We can prove (see [CL] for instance) that the germ of f_σ:

– does not depend on the chain U_i, $i = 1, \ldots, k$ and in the choice of σ in its homotopy class in the group $\pi_1(L, x)$ of the homotopy classes of loops based at x,

– satisfies $f_\sigma(x) = x$.

So we get a homomorphism $h : [\sigma] \in \pi_1(L, x) \longrightarrow f_\sigma \in G(T, x)$ where $G(T, x)$ is the group of germs of diffeomorphisms of T fixing the point x. This representation h is called the *holonomy* of the leaf L at x. It is trivial if L is simply connected. The foliation $\mathcal{F}$ is said to be *without holonomy* if this representation is trivial for every leaf L of $\mathcal{F}$ and every point $x \in L$.

Theorem 1. (Local stability) *Suppose that $\mathcal{F}$ admits a compact leaf L with finite fundamental group. Then L admits a saturated neighbourhood V such that every leaf contained in V is compact with finite fundamental group.*

Theorem 2. (Global stability) *Suppose that M is compact, the codimension of $\mathcal{F}$ is one and that $\mathcal{F}$ admits a compact leaf with finite fundamental group. Then all leaves of $\mathcal{F}$ are compact with finite fundamental group.*

The proof can be found in the original paper of G. Reeb [Ree1] or in the book [CL] by C. Camacho and A. Lins Neto.

1.3. Examples of foliations

i) <u>Simple foliations</u>. On every manifold M we have a foliation by taking points as leaves. Its codimension is equal to the dimension of M. Also M can be equipped with a codimension zero foliation with only one leaf, namely, M itself.

In general, every submersion $M \xrightarrow{\pi} B$ with connected fibres defines a foliation. The leaves being the fibres $\pi^{-1}(b)$, $b \in B$. In particular, every product $F \times B$ is a foliation with leaves $F \times \{b\}$, $b \in B$. These foliations are transversely orientable if, and only if, the manifold B is orientable.

These are *simple foliations.* We shall give more interesting examples in different situations.

ii) One dimensional foliations. Let us begin by surfaces. Let $\widetilde{M} = \mathbf{R}^2$ and consider the differential equation $dy - \alpha dx = 0$ where α is a real number. This equation has $y = \alpha x + c$, $c \in \mathbf{R}$ as general solution. When c varies, we obtain a family of parallel lines which defines a foliation $\widetilde{\mathcal{F}}$ in $\widetilde{M}$.

The natural action of $\mathbf{Z}^2$ on $\widetilde{M}$ preserves the foliation $\widetilde{\mathcal{F}}$ (*i.e.* the image of any leaf of $\widetilde{\mathcal{F}}$ by an integer translation is a leaf of $\widetilde{\mathcal{F}}$). Then $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. The leaves are all diffeomorphic to the circle $\mathbf{S}^1$ if α is rational and to the real line if α is not rational (Fig. 2). In fact, if α is not rational, every leaf of $\mathcal{F}$ is dense; this shows that even if locally a foliation is simple, globally it can be complicated.

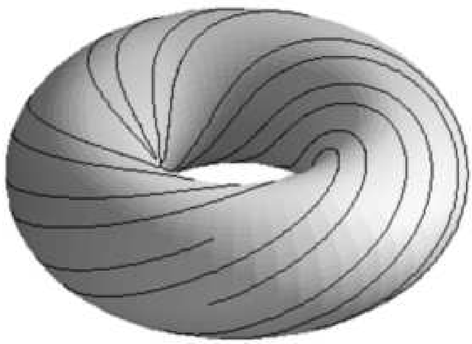

Fig. 2

Let M be a closed orientable surface. The fact that M admits a one dimensional foliation depends on the topology of M, which is described by the *Euler-Poincaré number* $\chi(M)$; this number can be defined as follows: take a triangulation of M *i.e.* a decomposition of M into triangles such as shown for the 2-sphere $\mathbf{S}^2$ (Fig. 3).

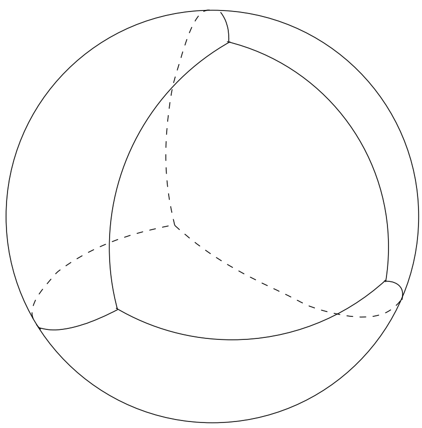

Fig. 3

Let b_0, b_1 and b_2 be the numbers respectively of vertices, edges and triangles. Then $\chi(M) = b_0 - b_1 + b_2$ is independant of the triangulation; it is called the *Euler-Poincaré number* of M. (There are many books on Algebraic or Differential Topology where we can find the proof of this fact.) It classifies completely the topology of closed orientable surfaces *i.e.* M and M' are homeomorphic if, and only if, $\chi(M) = \chi(M')$. For the triangulation of $\mathbf{S}^2$ in Fig. 3 we have $b_0 = 4$, $b_1 = 6$ and $b_2 = 4$. So $\chi(\mathbf{S}^2) = 2$.

The Euler-Poincaré number of M is the only obstruction to the existence of dimension one foliation on M: M admits such foliation if, and only if, $\chi(M) = 0$. For example $\mathbf{S}^2$ cannot support a one dimensional foliation. In fact, $\mathbf{T}^2$ is the only one compact orientable surface which admits a foliation of dimension one. The reader can prove, by using an adequate triangulation, that a closed orientable surface M_g of genus g (see in Fig. 4 the case $g = 2$) has $\chi(M_g) = 2 - 2g$ as Euler-Poincaré number. Then M_g admits a foliation $\mathcal{F}$ of dimension one if, and only if, $g = 1$, *i.e.*, M_g is $\mathbf{T}^2$.

Suppose M is compact of dimension n. For each $r = 0, 1, \ldots, n$, let $H^r(M, \mathbf{R})$ denote the real r-th *cohomology space* of M which is finite dimensional. Then the number

$$\chi(M) = \sum_{r=0}^{n} (-1)^r \dim\, H^r(M, \mathbf{R})$$

is a *topological invariant* called the *Euler-Poincaré number* of M. For a surface, it is exactly the number defined above by using a triangulation. The manifold M admits a one dimensional foliation if, and only if, $\chi(M) = 0$.

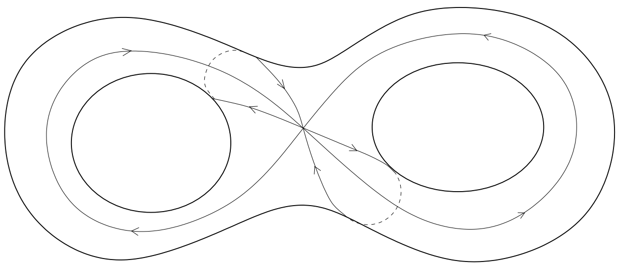

Fig. 4

iii) <u>Reeb Foliation on the 3-sphere $\mathbf{S}^3$</u>. Let M be the 3 dimensional sphere $\mathbf{S}^3 = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Denote by $\mathbf{D}$ the open unit disc in $\mathbf{C}$ and $\overline{\mathbf{D}}$ its closure which is the closed unit disc $\{z \in \mathbf{C} : |z| \leq 1\}$. The two subsets:

$$M_+ = \left\{(z_1, z_2) \in \mathbf{S}^3 : |z_1|^2 \leq \frac{1}{2}\right\} \text{ and } M_- = \left\{(z_1, z_2) \in \mathbf{S}^3 : |z_2|^2 \leq \frac{1}{2}\right\}$$

are diffeomorphic to $\overline{\mathbf{D}} \times \mathbf{S}^1$. They have $\mathbf{T}^2$ as common boundary:

$$\partial M_+ = \partial M_- = \left\{ (z_1, z_2) \in \mathbf{S}^3 : |z_1|^2 = |z_2|^2 = \frac{1}{2} \right\}$$

and their union is equal to $\mathbf{S}^3$. Then $\mathbf{S}^3$ can be obtained by gluing M_+ and M_- along their boundaries by the diffeomorphism $(z_1, z_2) \in \partial M_+ \longrightarrow (z_2, z_1) \in \partial M_-$, *i.e.* we identify (z_1, z_2) with (z_2, z_1) in the disjoint union $M_+ \coprod M_-$. Let $f : \mathbf{D} \longrightarrow \mathbf{R}$ be the function defined by:

$$f(z) = \exp\left(\frac{1}{1 - |z|^2}\right).$$

Let t denote the second coordinate in $\mathbf{D} \times \mathbf{R}$. The family of surfaces $(S_t)_{t\in\mathbf{R}}$ obtained by translating the graph S of f along the t-axis defines a foliation on $\mathbf{D} \times \mathbf{R}$. If we add the cylinder $\mathbf{S}^1 \times \mathbf{R}$, where $\mathbf{S}^1$ is viewed as the boundary of $\overline{\mathbf{D}}$, we obtain a codimension one foliation $\widetilde{\mathcal{F}}$ on $\overline{\mathbf{D}} \times \mathbf{R}$. By construction, $\widetilde{\mathcal{F}}$ is invariant by the transformation

$$(z, t) \in \overline{\mathbf{D}} \times \mathbf{R} \longrightarrow (z, t + 1) \in \overline{\mathbf{D}} \times \mathbf{R};$$

so it induces a foliation $\mathcal{F}_0$ on the quotient:

$$\overline{\mathbf{D}} \times \mathbf{R}/(z, t) \sim (z, t + 1) \simeq \overline{\mathbf{D}} \times \mathbf{S}^1.$$

It has the boundary $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ as a closed leaf. The others are diffeomorphic to $\mathbf{R}^2$ (see Fig. 5).

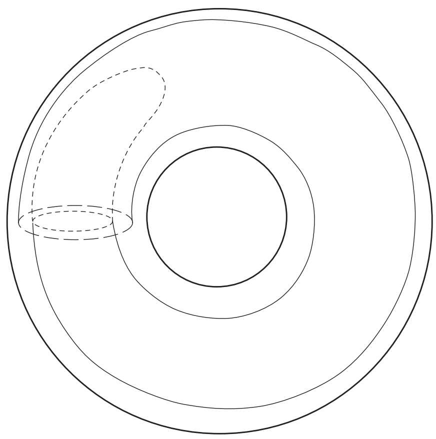

Fig. 5

Because M_+ and M_- are diffeomorphic to $\overline{\mathbf{D}} \times \mathbf{S}^1$, $\mathcal{F}_0$ defines on M_+ and M_- respectively two foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ which give a codimension one foliation $\mathcal{F}$ on $\mathbf{S}^3$ called the *Reeb foliation*.

iv) Lie group actions. Let M be a manifold of dimension $m+n$ and G a connected Lie group of dimension m. An *action* of G on M is a map $G \times M \xrightarrow{\Phi} M$ such that:

– $\Phi(e,x) = x$ for every $x \in M$ (where e is the unit element of G),

– $\Phi(g', \Phi(g,x)) = \Phi(g'g, x)$ for every $x \in M$ and every $g, g' \in G$.

Suppose that, for every point $x \in M$, the dimension of the *isotropy subgroup*:

$$G_x = \{g \in G : \Phi(g,x) = x\}$$

is independant of x. Then the action Φ defines a foliation $\mathcal{F}$ of dimension $= m - \dim G_x$; its leaves are the orbits $\{\Phi(g,x) : g \in G\}$. In particular this is the case if Φ is *locally free i.e.* if, for every $x \in M$, the isotropy subgroup G_x is discrete. An explicit example is given when M is the quotient H/Γ of a Lie group H by a discrete subgroup Γ and G is a connected Lie subgroup of H; the action of G on M being induced by the left action of G on H. We say that $\mathcal{F}$ is a *homogeneous foliation.* Let us give an explicit example (for more details see [EN3]).

Let $A \in \mathrm{SL}(m+n-1, \mathbf{Z})$, where $m+n \geq 3$, be a matrix diagonalizable and having all its eigenvalues $\mu_1, \ldots, \mu_{m-1}, \lambda_1, \ldots, \lambda_n$ real and positive. We can think of A as a diffeomorphism of the $(m+n-1)$-torus $\mathbf{T}^{m+n-1}$. Let $X_1, \ldots, X_{m-1}$, $Y_1, \ldots, Y_n$ be linear vector fields on $\mathbf{T}^{m+n-1}$ such that:

$$A_* X_j = \mu_j X_j, \quad A_* Y_k = \lambda_k Y_k \quad \text{for} \quad j = 1, \ldots, m-1 \quad \text{and} \quad k = 1, \ldots, n$$

and denote by $\mathcal{F}_0$ the foliation on $\mathbf{T}^{m+n-1}$ defined by the vector fields $X_1, \ldots, X_{m-1}$. The product of $\mathcal{F}_0$ by $\mathbf{R}$ gives a codimension n foliation on $\mathbf{T}^{m+n-1} \times \mathbf{R}$ which is invariant by the diffeomorphism ϕ of $\mathbf{T}^{m+n-1} \times \mathbf{R}$ sending (z,t) to $(A(z), t+1)$. So, it induces a codimension n foliation $\mathcal{F}$ on the quotient manifold $\mathbf{T}_A^{m+n} = \mathbf{T}^{m+n-1} \times \mathbf{R}/\phi$. Notice that $\mathbf{T}_A^{m+n}$ is a flat bundle over the circle $\mathbf{S}^1$ with fibre $\mathbf{T}^{m+n-1}$. In fact $\mathbf{T}_A^{m+n}$ is the homogeneous space H/Γ where H is the semi-direct product of $\mathbf{R}^{m+n-1}$ by $\mathbf{R}$ given by the action:

$$(t,z) \in \mathbf{R} \times \mathbf{R}^{m+n-1} \longrightarrow A^t z \in \mathbf{R}^{m+n-1}$$

and Γ is the subgroup:

$$\{(\mathbf{m}, k) \in H \ \mid \mathbf{m} \in \mathbf{Z}^{m+n-1}, k \in \mathbf{Z}\}.$$

If $v_1, \ldots, v_{m-1} \in \mathbf{R}^{m+n-1}$ are eigenvectors of A corresponding respectively to the eigenvalues $\mu_1, \ldots, \mu_{m-1}$ then the subgroup:

$$G = \left\{ \left(\sum_{i=1}^{m-1} a_i v_i, b \right) \in H \;\middle|\; a_1, \ldots, a_{m-1}, b \in \mathbf{R} \right\}$$

is isomorphic to the semi-direct product of $\mathbf{R}^{m-1}$ by $\mathbf{R}_+^*$ where $\mathbf{R}_+^*$ acts on $\mathbf{R}^{m-1}$ by homotheties on each factor. The action of G on $\mathbf{T}_A^{m+n}$, induced by this identification, is a locally free action whose orbits define the foliation $\mathcal{F}$.

v) Foliations obtained by suspension. Let B and F be two manifolds, respectively of dimensions m and n. Suppose that the fundamental group $\pi_1(B)$ of B is finitely generated. Let $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ be an injective representation, where $\mathrm{Diff}(F)$ is the diffeomorphism group of F. Denote by $\widetilde{B}$ the universal covering of B and $\widetilde{\mathcal{F}}$ the horizontal foliation on $\widetilde{M} = \widetilde{B} \times F$, *i.e.*, the foliation whose leaves are the subsets $\widetilde{B} \times \{y\}$, $y \in F$. This foliation is invariant by all the transformations $T_\gamma : \widetilde{M} \longrightarrow \widetilde{M}$ defined by $T_\gamma(\widetilde{x}, y) = (\gamma \cdot \widetilde{x}, \rho(\gamma)(y))$ where $\gamma \cdot \widetilde{x}$ is the natural action of $\gamma \in \pi_1(B)$ on $\widetilde{B}$; then $\widetilde{\mathcal{F}}$ induces a codimension n foliation $\mathcal{F}_\rho$ on the quotient manifold:

$$M = \widetilde{M}/(\widetilde{x}, y) \sim (\gamma \cdot \widetilde{x}, \rho(\gamma)(y)).$$

We say that $\mathcal{F}_\rho$ is the *suspension* of the diffeomorphism group $\Gamma = \rho(\pi_1(B))$. The leaves of $\mathcal{F}_\rho$ are transverse to the fibres of the natural fibration induced by the first projection $\widetilde{B} \times F \longrightarrow \widetilde{B}$.

Conversely, suppose that $F \longrightarrow M \stackrel{\pi}{\longrightarrow} B$ is a fibration with compact fibre F and that $\mathcal{F}$ is a codimension n foliation ($n =$ dimension of F) transverse to the fibres of π. Then there exists a representation $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ such that $\mathcal{F} = \mathcal{F}_\rho$.

Concrete example: let B be the circle $\mathbf{S}^1$ and $F = \mathbf{R}_+ = [0, +\infty[$. Let ρ be the representation of $\mathbf{Z} = \pi_1(\mathbf{S}^1)$ in $\mathrm{Diff}([0, +\infty[)$ defined by $\rho(1) = \varphi$ where $\varphi(y) = \lambda y$ with $\lambda \in]0, 1[$. Because φ is isotopic to the identity map of F, the manifold M is diffeomorphic to $\mathbf{S}^1 \times \mathbf{R}_+$ and the foliation $\mathcal{F}_\rho$ has one closed leaf diffeomorphic to the circle $\mathbf{S}^1$, corresponding to the fixed point $\varphi(0) = 0$ (see Fig. 6).

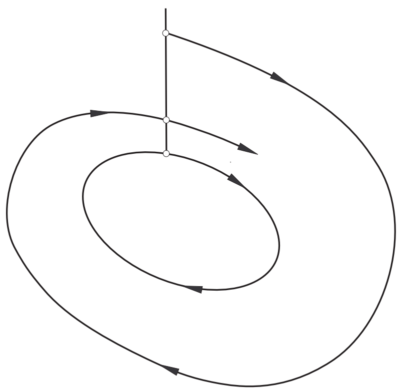

Fig. 6

1.4. Foliations and Differential Systems

Let M be a manifold of dimension $m+n$. Denote by TM the tangent bundle of M and let E be a subbundle of rank m. Let U be an open set of M such that on U, TM is equivalent to the product $U \times \mathbf{R}^{m+n}$. At each point $x \in U$, the fibre E_x can be considered as the kernel of n differential 1-forms $\omega_1, \ldots, \omega_n$ linearly independant:

$$E_x = \bigcap_{j=1}^{n} \ker\, \omega_j(x). \tag{S}$$

The subbundle E is called an *m-plane field* on M. We say that E is *involutive*, if for every vector field X and Y tangent to E (*i.e.* sections of E), the bracket $[X,Y]$ is also tangent to E. We say that E is *completely integrable* if, through each point $x \in M$, there exists a submanifold P_x of dimension m which admits $E_{|P_x}$ (the restriction of E to P_x) as tangent bundle. The maximal connected submanifolds satisfying this property are called the *integral submanifolds* of the differential system (S). They define a partition of M *i.e.* a codimension n foliation. We have the:

Theorem 3. (Frobenius) *Let E be a subbundle of rank m given locally by a differential system like in* (S). *Then the following assertions are equivalent:*

– *E is involutive,*

– *E is completely integrable,*

– *there exist differential 1-forms (defined locally) (β_{ij}) , $i,j = 1,\ldots,n$ such that $d\omega_i = \sum_{j=1}^{n} \beta_{ij} \wedge \omega_j \quad i = 1,\ldots,n$.*

For example, let ω be a non singular 1-form. The corresponding subbundle E has fibre $E_x = \ker(\omega_x)$. It defines a codimension one foliation if, and only if, there exists a 1-form β

such that $d\omega = \beta\wedge\omega$; this condition is equivalent to $d\omega\wedge\omega = 0$. In particular, if ω is closed it defines a codimension one foliation $\mathcal{F}$. If M is compact, all leaves are diffeomorphic and integration of ω over loops of M gives rise to a morphism $h : \pi_1(M) \longrightarrow \mathbf{R}$. The range $\Gamma = h(\pi_1(M))$ of h is a subgroup of $\mathbf{R}$ called the *holonomy group* of $\mathcal{F}$. Example ii) is of this type: $M = \mathbf{T}^2$, $\omega = dy - \alpha dx$ which is closed. The fundamental group of $\mathbf{T}^2$ is $\mathbf{Z}\oplus\mathbf{Z}$ and it is easy to see that $\Gamma = \{p + q\alpha : p, q \in \mathbf{Z}\}$.

1.5. Notations

Let $\mathcal{F}$ be a codimension n foliation on M. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\nu\mathcal{F}$ the quotient $TM/T\mathcal{F}$ which is the *normal bundle* to $\mathcal{F}$; $\chi(\mathcal{F})$ will denote the space of sections of $T\mathcal{F}$ (elements of $\chi(\mathcal{F})$ are vector fields $X \in \chi(M)$ tangent to $\mathcal{F}$).

A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \chi(\mathcal{F})$. (Here i_X and L_X denote respectively the inner product and the Lie derivative with respect to the vector field X.) For a function $f : M \longrightarrow \mathbf{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \chi(\mathcal{F})$ *i.e.* f is constant on the leaves of $\mathcal{F}$; we denote by $\Omega^r(M/\mathcal{F})$ the space of basic forms of degree r on the foliated manifold $(M, \mathcal{F})$; this is a module over the algebra A_b of basic functions. A vector field $Y \in \chi(M)$ is said to be *foliated*, if for every $X \in \chi(\mathcal{F})$, the bracket $[X, Y] \in \chi(\mathcal{F})$. We can easily see that the set $\chi(M, \mathcal{F})$ of foliated vector fields is a Lie algebra and an A_b-module; by definition $\chi(\mathcal{F})$ is an ideal of $\chi(M, \mathcal{F})$ and the quotient

$$\chi(M/\mathcal{F}) = \chi(M, \mathcal{F})/\chi(\mathcal{F})$$

is called the Lie algebra of *transverse* (or *basic*) *vector fields* on the foliated manifold $(M, \mathcal{F})$. Also, it has a module structure over the algebra A_b.

2. Transverse structures

Let M be a manifold of dimension $m+n$ endowed with a codimension n foliation $\mathcal{F}$ defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$ like in definition 2.

Definition 3. *A transverse structure to $\mathcal{F}$ is a geometric structure on T invariant by the local diffeomorphisms γ_{ij}.*

This is a very important notion in Foliation Theory. To make it clear, let us give the main examples.

2.1. Measurable foliations

Let $\mathcal{B}_T$ denote the family of Borel sets on T. A *transverse invariant measure* to $\mathcal{F}$ is a measure μ on $\mathcal{B}_T$ such that, for any $A \in \mathcal{B}_T$ in the domain of definition of γ_{ij}, we have:

$$\mu(\gamma_{ij}(A)) = \mu(A).$$

We say that $\mathcal{F}$ is a *measurable foliation* if it admits a transverse measure. The notion of measurable foliation was introduced firstly by J. F. Plante; he obtained many intersting results on the qualitative behaviour of codimension one measurable foliations on compact manifolds (cf. [Pla2]).

2.3. Lie foliations

We say that $\mathcal{F}$ is a *Lie foliation*, if T is a Lie group G and γ_{ij} are restrictions of left translations on G.

Such foliation can also be defined by a 1-form ω on M with values in the Lie algebra $\mathcal{G}$ such that:

i) $\omega_x : T_xM \to \mathcal{G}$ is surjective for every $x \in M$,

ii) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

If $\mathcal{G}$ is Abelian, ω is given by n linearly independant closed scalar 1-forms $\omega_1, \ldots, \omega_n$. In particular if $n = 1$, an important topological property of compact manifolds supporting such foliation is given by the following theorem due to Tischler [Tic].

Theorem 4. *If a compact manifold admits a closed non singular* 1*-form, then it is a locally trivial fibration over the circle* $\mathbf{S}^1$*.*

The hypothesis $\mathcal{G}$ is Abelian is important: D. Lehmann [Leh] proved that, in general, the result is false even if $\mathcal{G}$ is nilpotent.

Foliations defined by non singular closed 1-forms can be considered as topological prototype of codimension one foliations without holonomy as it is illustrated by Sacksteder's Theorem [Sac1]:

Theorem 5. *Let $\mathcal{F}$ be a C^r ($r \geq 2$) codimension one foliation on a connected compact manifold. If $\mathcal{F}$ has no holonomy, then it is topologically conjugated to a foliation defined by a non singular closed* 1*-form.*

In the general case, the structure of a Lie foliation on a compact manifold, is given by the following theorem due to E. Fédida [Féd]:

Theorem 6. *Let $\mathcal{F}$ be a Lie G-foliation on a compact manifold M. Let $\widetilde{M}$ be the universal covering of M and $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\widetilde{M}$. Then there exist a homomorphism $h : \pi_1(M) \longrightarrow G$ and a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ whose fibres are the leaves of $\widetilde{\mathcal{F}}$ and such that, for every $\gamma \in \pi_1(M)$, the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{M} & \stackrel{\gamma}{\longrightarrow} & \widetilde{M} \\ D\downarrow & & \downarrow D \\ G & \stackrel{h(\gamma)}{\longrightarrow} & G \end{array}$$

where the first line denotes the deck transformation of $\gamma \in \pi_1(M)$ on $\widetilde{M}$.

The subgroup $\Gamma = h(\pi_1(M)) \subset G$ is called the *holonomy group* of $\mathcal{F}$ although the holonomy of each leaf is trivial. The fibration $D : \widetilde{M} \longrightarrow G$ is called the *developing map* of $\mathcal{F}$.

2.3. Transversely parallelizable foliations

We say that $\mathcal{F}$ is *transversely parallelizable* if there exist on M, foliated vector fields $Y_1, \ldots, Y_n$, transverse to $\mathcal{F}$ and everywhere linearly independant. This means that the manifold T admits a parallelism $(Y_1, \ldots, Y_n)$ invariant by all the local diffeomorphisms γ_{ij} or, equivalently, that the A_b-module $\chi(M/\mathcal{F})$ is free of rank n. The structure of a transversely parallelizable foliation on a compact manifold is given by the following theorem due to L. Conlon [Col] for $n = 2$ and in general to P. Molino [Mol4].

Theorem 7. *Let $\mathcal{F}$ be a transversely parallelizable foliation of codimension n on a compact manifold M. Then:*

(1) *the closures of the leaves are submanifolds which are fibres of a locally trivial fibration $\pi : M \longrightarrow W$ where W is a compact manifold,*

(2) *there exists a simply connected Lie group G_0 such that the restriction $\mathcal{F}_0$ of $\mathcal{F}$ to any leaf closure F is a G_0-Lie foliation,*

(3) *the cocycle of the fibration $\pi : M \longrightarrow W$ has values in the group of diffeomorphisms of F preserving $\mathcal{F}_0$.*

The fibration $\pi : M \longrightarrow W$ and the manifold W are called respectively the *basic fibration* and the *basic manifold* associated to $\mathcal{F}$. Theorem 7 says that if, in particular, the leaves of $\mathcal{F}$ are closed, then the foliation is just a fibration over W. This is still true even if the leaves are not closed: the manifold M is a fibration over the leaf space $M/\mathcal{F}$ which is, in this case, a *Q-manifold* in the sense of [Bar]. Theorem 7 is still valid for *transversely complete foliations* on non compact manifolds (cf. [Mol3]).

It is not difficult to see that any Lie foliation is transversely parallelizable. This is a consequence of the fact that a Lie group is parallelizable and that the parallelism can be chosen invariant by left translations.

2.4. Riemannian foliations

The foliation $\mathcal{F}$ is said to be *Riemannian* if there exists on T a Riemannian metric such that the local diffeomorphisms γ_{ij} are isometries. Using the submersions $f_i : U_i \longrightarrow T$ one can construct on M a Riemannian metric which can be written in local coordinates:

$$ds^2 = \sum_{i,j=1}^{m} \theta_i \otimes \theta_j + \sum_{k,\ell=1}^{n} g_{k\ell}(y) dy_k \otimes dy_\ell.$$

Equivalently, $\mathcal{F}$ is Riemannian, if any geodesic orthogonal to the leaves at a point is orthogonal to the leaves everywhere [Rei1].

Let $\mathcal{F}$ be Riemannian. Then there exists a Levi-Civita connection, transverse to the leaves which, by unicity argument, coincides on any distinguished open set, with the pullback of the Levi-Civita connection on the Riemannian manifold T. This connection is said to be *projectable.* Let $\mathrm{O}(n) \longrightarrow M^{\#} \stackrel{\tau}{\longrightarrow} M$ be the principal bundle of orthonormal frames transverse to $\mathcal{F}$; this is an $\mathcal{F}$-bundle, in the sense of subsection 7.1. The following theorem is due to P. Molino [Mol4].

Theorem 8. *Suppose M is compact. Then, the foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}^{\#}$ on $M^{\#}$ of the same dimension and such that:*

(1) *$\mathcal{F}^{\#}$ is transversely parallelizable,*

(2) *$\mathcal{F}^{\#}$ is invariant under the action of $O(n)$ on $M^{\#}$ and projects, by τ, on $\mathcal{F}$.*

The basic manifold $W^{\#}$ and the basic fibration $F^{\#} \longrightarrow M^{\#} \stackrel{\pi^{\#}}{\longrightarrow} W^{\#}$ are called respectively the *basic manifold* and the *basic fibration* of $\mathcal{F}$.

We have the following properties:

– the restriction of τ to a leaf of $\mathcal{F}^{\#}$ is a covering over a leaf of $\mathcal{F}$. So all leaves of $\mathcal{F}$ have the same universal covering,

– the closure of any leaf of $\mathcal{F}$ is a submanifold of M and the leaf closures define a *singular foliation* (the leaves have different dimensions) on M. (For more details about this notion see [Mol4].)

Another interesting result for Riemannian foliations is the Global Reeb Stability Theorem which is valid even if the codimension is greater than 1.

Theorem 9. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. If there exists a compact leaf with finite fundamental group, then all leaves are compact with finite fundamental group.*

The property $\mathcal{F}$ is Riemannian means that the leaf space $Q = M/\mathcal{F}$ is a Riemannian manifold even if Q does not support any differentiable structure !

2.5. Transversely holomorphic foliations

The foliation $\mathcal{F}$ is said to be *transversely holomorphic* if T is a complex manifold and the γ_{ij} are local biholomorphisms. Particular case is a *holomorphic foliation*: the manifolds M and T are complex, all the f_i are holomorphic and all γ_{ij} are local biholomorphisms.

If T is Kählerian and γ_{ij} biholomorphisms which preserve the Kähler form on T we say that $\mathcal{F}$ is *transversely Kählerian.* For example, any codimension 2 Riemannian foliation which is transversely orientable is transversely Kählerian.

Let us give concrete examples of such foliations. Let M be the unit sphere in the Hermitian space $\mathbf{C}^{n+1}$:

$$M = \mathbf{S}^{2n+1} = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1} : \sum_{k=1}^{n+1} |z_k|^2 = 1 \right\}.$$

Let Z be the holomorphic vector field on $\mathbf{C}^{n+1}$ given by the formula:

$$Z = \sum_{k=1}^{n+1} a_k z_k \frac{\partial}{\partial z_k}$$

where $a_k = \alpha_k + i\beta_k \in \mathbf{C}$. There exists a good choice of the numbers a_k such that the orbits of Z intersect transversely the sphere M; then Z induces on M a real vector field X which defines a foliation $\mathcal{F}$. It is not difficult to see that $\mathcal{F}$ is transversely holomorphic. It is tansversely Kählerian if we choose in addition $\alpha_k = 0$ for any $k = 1, \ldots, n+1$.

3. Codimension one foliations

Codimension one foliations constitute a rich thema which was studied extensively by many people. The richness comes from the existence, for such foliations, of non singular transverse vector fields which give a way to go from a leaf to an other. Most of the results in Foliation Theory were first obtained in the codimension one case; this section is devoted to summarize some of them.

Let $\mathcal{F}$ be a codimension one foliation on a manifold M and ν a transverse vector bundle to $\mathcal{F}$. Because ν is of rank one, it is integrable and defines a foliation $\mathcal{V}$ transverse to $\mathcal{F}$. So we have clearly $\chi(M) = 0$. It is natural to ask if this condition is sufficient for the existence of a codimension one foliation on M; this was conjectured by E. Thomas [Tho]. The reader can see the paper [Law1] by B. Lawson about the history of the different steps for solving this conjecture. The final solution was given by W. Thurston [Thu7] who proved the:

Theorem 10. (Thurston) *Let M be a compact manifold. Then M admits a codimension one foliation if, and only if, the Euler-Poincaré number $\chi(M)$ of M is zero.*

Recall that two vector bundles $E \longrightarrow M$ and $E' \longrightarrow M$ are said to be *homotopic* if there exists a continuous family $E_t \longrightarrow M$, $t \in [0,1]$ of vector bundles such that $E_0 = E$ and $E_1 = E'$. So we can formulate the question of existence of codimension one foliations, in general, in the following:

Let M be a compact manifold. Then any codimension one plane field on M is homotopic to an integrable one.

The first results solving (in some particular cases), this conjecture were obtained by J. Wood (see [Woo1]) and also by P. Schweitzer and W. Thurston in the C^0-case. As far as we know this conjecture is still open.

Notice that the compacity of the manifold is a big constraint. Indeed on open manifolds the answer to this conjecture is positive [Phi].

The regularity property seems to be very important in the existence of foliations on compact manifolds. In particular there is a big difference in the treatment between the C^∞ case and the real analytic one. In this direction A. Haefliger proved in [Hae1] the following important theorem.

Theorem 11. (Haefliger) *Let M be a compact manifold with a finite fundamental group. Then M has no real analytic codimension one foliation.*

Let us end this section with one of the most important results obtained in codimension one foliation theory on 3-manifolds [Nov1].

Theorem 12. (Novikov) *Let M be a compact 3-manifold with a finite fundamental group. Then any codimension one foliation on M has a compact leaf diffeomorphic to the torus $\mathbf{T}^2$.*

4. Γ-structures

This notion was introduced by A. Haefliger and became a key ingredient in studying characteristic classes of foliations.

Definition 4. *A groupoid is given by a set Γ, a subset $\Gamma^{(2)}$ of $\Gamma \times \Gamma$ with a law*

$$(\gamma, \sigma) \in \Gamma^{(2)} \longrightarrow \gamma\sigma \in \Gamma$$

and an inverse map $\gamma \in \Gamma \longrightarrow \gamma^{-1} \in \Gamma$ satisfying the following properties:

i) $(\gamma^{-1})^{-1} = \gamma$,

ii) if (γ, σ), $(\sigma, \tau) \in \Gamma^{(2)}$, then $(\gamma\sigma, \tau)$, $(\gamma, \sigma\tau) \in \Gamma^{(2)}$ and $(\gamma\sigma)\tau = \gamma(\sigma\tau)$,

iii) if $(\gamma^{-1}, \gamma) \in \Gamma^{(2)}$ and $(\gamma, \sigma) \in \Gamma^{(2)}$, then $\gamma^{-1}(\gamma\sigma) = \sigma$, *iv) if* $(\gamma, \gamma^{-1}) \in \Gamma^{(2)}$ *and* $(\tau, \gamma) \in \Gamma^{(2)}$*, then* $(\tau\gamma)\gamma^{-1} = \tau$.

For $\gamma \in \Gamma$, $s(\gamma) = \gamma^{-1}\gamma$ is called the *source* of γ and $r(\gamma) = \gamma\gamma^{-1}$ the *range* of γ.

Then, there are two projections s, r (or α, β): $\Gamma \longrightarrow \Gamma^{(0)} = \mathrm{Im} r$. The subset $\Gamma^{(0)}$ of Γ is called the *unit space* of Γ.

A *topological groupoid* is a groupoid with a topology compatible with the composition and inverse maps. As a consequence, the two projections s, r on the unit space are also continuous.

A *differentiable structure* on Γ is given by a manifold structure on Γ and $\Gamma^{(0)}$ compatible with the composition and inverse maps and such that:

– $s : \Gamma \longrightarrow \Gamma^{(0)}$ is a submersion,

– the canonical injection $\Gamma^{(0)} \longrightarrow \Gamma$ is an embedding.

The differentiable (or topological) groupoid Γ is *étale* if s is étale.

Let M be a manifold, Γ a topological groupoid and $\{U_i\}$ an open cover of M; a 1-*cocycle* on M with values in Γ is given as follows: for each pair (i, j), let

$$\gamma_{ij} : U_i \cap U_j \longrightarrow \Gamma$$

be a continuous map such that, if $x \in U_i \cap U_j \cap U_k$, then $(\gamma_{ij}(x), \gamma_{jk}(x)) \in \Gamma^{(2)}$ and

$$\gamma_{ik}(x) = \gamma_{ij}(x)\gamma_{jk}(x).$$

Two 1-cocycles are said to be *cohomologous* if they are restrictions of the same cocyle on the union of their coverings. A Γ-*structure* on M, or an element of $H^1(M, \Gamma)$, is an equivalence class of 1-cocycles.

Let Γ be the groupoid of germs of local diffeomorphisms of $\mathbf{R}^n$; then the unit space $\Gamma^{(0)}$ may be identified to $\mathbf{R}^n$. A codimension n foliation $\mathcal{F}$ on M may be viewed as a particular Γ-structure for which a representative is a 1-cocycle on an open covering $\{U_i\}$ such that the following maps $f_i = \gamma_{ii} : U_i \longrightarrow \Gamma^{(0)} = \mathbf{R}^q$ are submersions.

5. The leaf space

Let $\mathcal{F}$ be a codimension n foliation on M. Let U be a subset of M and denote by $\widehat{U}$ the union of the leaves intersecting U. Recall that U is *saturated* if $U = \widehat{U}$. It is easy to see that if U is open, so is $\widehat{U}$. Then, the equivalence relation on M, $x \sim y$ if, and only if, x and y are in the same leaf, is open. The set of equivalence classes $M/\sim$, endowed with the quotient topology, is called the *leaf space* of $\mathcal{F}$ and usually denoted by $M/\mathcal{F}$.

We can think of $M/\mathcal{F}$ as follows. The foliation $\mathcal{F}$ is the geometric realization of a completely integrable differential system (S) on M. Each integral submanifold is a leaf of $\mathcal{F}$ and corresponds to an initial condition of (S). So we can consider Q as a parameter space of the initial conditions of this differential system. In general Q is not a manifold, but we can define on this space many geometrical objects like functions, differential forms, differential operators *etc.* (cf. for instance section 7). They correspond to their analogues on M invariant along the leaves (in a sense to be determined following the context).

There were many attempts to give the leaf space of a foliation a differentiable structure, even if its topology is, generally poor.

A first one was from Satake, whose point of view was recovered by W. Thurston. In other domains, let us cite G.W. Mackey [Mac] who introduced the virtual group notion in *Ergodic Theory* and M. Artin [Art] the algebraic space notion. The former corresponds to the measurable version of the S-atlas of W.T. Van Est, the latter suggested the definition of a Q-manifold.

In fact, there is no uniform definition. Each corresponds to a given situation or a particular problem. Nevertheless, the point of view of *Noncommutative Geometry*, by A. Connes, using the C^*-algebra of the groupoid of a foliation, or the crossed-product of the C^*-algebra of a manifold by a group acting on it, is attractive and efficient too. For example, there are Longitudinal and Transversal Index Theorems for foliations; one gets also Godbillon-Vey classes *etc.*

5.1. V-manifolds

Let Ω be an open set in $\mathbf{R}^n$ and let Σ be a finite group of diffeomorphisms of Ω. Denote by Ω/Σ the orbit space with quotient topology and p the canonical projection $\Omega \longrightarrow \Omega/\Sigma$. If Ω' is another open set of $\mathbf{R}^n$, Σ' a finite group of diffeomorphisms of Ω' and p' the canonical projection $\Omega' \longrightarrow \Omega'/\Sigma'$, then a *morphism* from Ω/Σ to Ω'/Σ' is a continuous map f from Ω/Σ to Ω'/Σ', which admits local coverings by smooth local maps from Ω to Ω'. An *isomorphism* is a bijective morphism, the inverse of which is a morphism.

If V is a second countable Hausdorff space, a *Satake atlas* of dimension n is a family $\mathcal{A} = (U_i, \Phi_i)$ where (U_i) is an open covering of V and $\Phi_i : U_i \longrightarrow \Omega_i/\Sigma_i$ is a homeomorphism of U_i on the quotient of an open subset Ω_i of $\mathbf{R}^n$ by a finite group of diffeomorphisms, with following coherence condition: for all i, j such that $U_i \cap U_j \neq \emptyset$ the map $\Phi_j \circ \Phi_i^{-1} : \Phi_i(U_i \cap U_j) \longrightarrow \Phi_j(U_i \cap U_j)$ is a morphism as previously defined.

A *V-manifold* (or a *Satake manifold* or an *orbifold*) of dimension n is a space V with a maximal Satake atlas of dimension n. The following are simple examples illustrating the notion of a V-manifold.

i) Let Γ be a finite group of isometries of a Riemannian manifold M of dimension n. Then the quotient space M/Γ is a V-manifold of dimension n.

ii) A waterdrop obtained by gluing two open discs along their boundaries, one of them being implemented with a rotation of $\frac{2\pi}{3}$ around its center.

It is proved in [Mol4] that *every leaf space of a Riemannian foliation with compact leaves on a compact manifold is a V-manifold.*

Conversely : *every compact V-manifold is the leaf space of a Riemannian foliation with compact leaves on a compact manifold* (cf. [GHS]).

5.2. QF-manifolds

Let (X, p, S) be a triple where X is a manifold, S a set and p a surjective map from X to S; this is an *étale QF-atlas* of S if it satisfies the following conditions:

(H) for every pair (x, y) in X^2 such that $p(x) = p(y)$, there are open neighbourhoods U and V respectively of x and y and a diffeomorphism h from U to V such that $h(x) = y$ and $p \circ h(t) = p(t)$ for every $t \in U$,

(QF) every morphism from a manifold Z to X such that $p \circ f$ is constant is locally constant.

As usual two étale QF-atlases (X_1, p_1, S) and (X_2, p_2, S) are *equivalent* if (X, p, S) is an étale QF-atlas where X is the disjoint union of X_1 and X_2 and p is p_1 on X_1 and p_2 on X_2. A *QF-manifold structure* on S is an equivalence class of étale QF-atlases on S. All the leaf spaces of foliated secound countable manifolds are in this category.

5.3. Q-manifolds

Let (X, p, S) be a triple where X is a manifold, S a set and p a surjective map from X to S; this is a *Q-atlas* of S if it satisfies the following conditions:

(H) is as in the definition of an étale QF-atlas,

(Q) let $f = (f_1, f_2)$ be a morphism from a manifold Z to X^2 such that $p \circ f_1 = p \circ f_2$; then the subset $T = \{z \in Z : f_1(z) = f_2(z)\}$ is open in Z.

A *Q-manifold structure* on S is an equivalence class of Q-atlases of S. The following are examples of Q-manifolds:

i) the leaf space of foliated torus with geodesics having irrational slope,

ii) more generally, the leaf space of a transversely parallelizable foliation on a compact manifold.

It was first tried to generalize to leaf spaces the classical theorems and tools (Gauss-Bonnet, de Rham cohomology, Poincaré duality, Leray-Serre spectral sequence, fundamental group *etc.*) to get results on the transverse structure.

The V-manifolds are met in natural way and there exist many examples of them. They appear also with ramified coverings.

The Q-manifolds permitted to restore the third Lie theorem for Banach Lie algebras (cf. [Plt]); they appear also in the structure theorem of P. Molino. Recently, G. Meigniez got a characterization of Godbillon Homotopy Extension Property for foliations, where they play a role (*cf.* [Mei1]).

6. Characteristic classes

We follow here the lectures of R. Bott as written by L. Conlon in [Bot1]. We will restrict ourself to only one result : Bott vanishing theorem. The reader can find more material in [Bot1] or in [Hae12].

6.1. The classifying space and the universal bundle

Let $\mathcal{H}$ be a separable real Hilbert space with norm $|| \quad ||$. If u and v are non zero vectors in $\mathcal{H}$, (u, v) will be the angle defined by u and v; it is immediate to see that for every positive number λ and μ we have $(\lambda u, \mu v) = (u, v)$. Denote by BGL_n the set of n-dimensional subspaces of $\mathcal{H}$. Let $\tau, \sigma \in \mathrm{BGL}_n$ and set $\delta(\tau, \sigma) = \inf(u, v)$ where the infinimum is taken over all the vector $u \in \tau$ and $v \in \sigma$ with $||u|| = ||v|| = 1$. It is not difficult to see that δ defines a distance on BGL_n. The topological space BGL_n is called the *classifying space* of the group $\mathrm{GL}(n, \mathbf{R})$ of linear transformations of the vector space $\mathbf{R}^n$.

The cohomology $H^*(\mathrm{BGL}_n, \mathbf{R})$ of BGL_n is a polynomial ring $\mathbf{R}\left[p_1, \ldots, p_{[n/2]}\right]$ where the $p_i \in H^{4i}(\mathrm{BGL}_n, \mathbf{R})$ are the universal Pontryagin classes (cf. [BH]).

On BGL_n we have a canonical real vector bundle $S \longrightarrow \mathrm{BGL}_n$ of rank n whose fibre at each τ is the space τ itself; it is called the *universal bundle* on BGL_n.

6.2. Classification of real vector bundles

As M is paracompact, it admits a countable locally finite open cover $\mathcal{U} = \{U_i\}$ which, in addition, can be chosen such that each finite intersection $U_{i_1} \cap \ldots \cap U_{i_\ell}$ is contractible. Such an open cover is called a *good cover*; it always exists: take a Riemannian metric on M and a countable family of geodesically convex open balls which covers M. If $E \stackrel{\pi}{\longrightarrow} M$ is a real vector bundle of rank n, its restriction $E_{|U_i}$ to any U_i is trivial *i.e.* there exists a diffeomorphism $\varphi_i : E_{|U_i} \longrightarrow U_i \times \mathbf{R}^n$ which sends the fibre E_x isomorphically on $\{x\} \times \mathbf{R}^n$. Let $(s_i^1, \ldots, s_i^n)$ be a basis of the free module $C^\infty(E_{|U_i})$ over the algebra $A(U_i)$ of real valued C^∞-functions on U_i. Let $\{\rho_i\}$ be a partition of the unity subordinated to $\{U_i\}$ and let V_i be the real vector space spanned by $(\rho_i s_i^1, \ldots, \rho_i s_i^n)$. For each i we set

$\psi_i = q_i \circ \varphi_i$ where $q_i : U_i \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is the second projection. Now express $\mathcal{H}$ as orthogonal direct sum of the $V_i \simeq \mathbf{R}^n$ and denote by $\zeta_i : V_i \longrightarrow \mathcal{H}$ the inclusion of the i^{th} summand. Define $\Phi : E \longrightarrow \mathcal{H}$ by

$$\Phi(x,\xi) = \sum_{i=1}^{\infty} \rho_i(x) \cdot \zeta_i(\psi_i(x,\xi)).$$

Then Φ is continuous and sends each fibre $\pi^{-1}(x)$ of E isomorphically on an n-dimensional subspace of $\mathcal{H}$. Thus $f(x) = \Phi(\pi^{-1}(x))$ defines a continuous map $f : M \longrightarrow \mathrm{BGL}_n$ called the *classifying map* for the vector bundle E namely E is the pullback by f of the universal bundle $S \longrightarrow \mathrm{BGL}_n$. In fact there is a natural one-one correspondance between the set $\mathrm{Vect}_n(M)$ of isomorphism classes of real vector bundles of rank n on M and the set $[M, \mathrm{BGL}_n]$ of homotopy classes of maps $M \longrightarrow \mathrm{BGL}_n$. The i^{th}-Pontryagin class of the real vector bundle $E \longrightarrow M$ is by definition $p_i(E) = f^*(p_i)$. The graded subring

$$\mathrm{Pont}^*(E) = f^*(H^*(\mathrm{BGL}_n, \mathbf{R})) \subset H^*(M, \mathbf{R})$$

is called the Pontryagin ring of E. The first important result obtained in the theory of characteristic classes of foliations is the following.

Theorem 13. (Bott vanishing theorem) *Let $\mathcal{F}$ be a foliation of codimension n with normal bundle $\nu\mathcal{F}$. Then* $\mathrm{Pont}^i(\nu\mathcal{F}) = 0$ *for $i > 2n$.*

As a non trivial example of characteristic class of a foliation $\mathcal{F}$, we have the *Godbillon-Vey invariant $GV(\mathcal{F})$* (discovered by C. Godbillon and J. Vey [GoV]) which is, in general, non zero as shown by R. Roussarie. An elementary construction of this invariant in the codimension one case is as follows.

Let M a compact manifold endowed with a codimension one foliation defined by a differential 1-form ω. Then the integrability condition implies the equality $\omega \wedge d\omega = 0$ *i.e.* $d\omega = \alpha \wedge \omega$. It is easy to see that $\alpha \wedge d\alpha$ is closed and that its cohomology class in $H^3(M, \mathbf{R})$, which is by definition $GV(\mathcal{F})$, is independant of the choice of α.

One of the most important results in the study of the Gobillon-Vey invariant for codimension one foliations was obtained by G. Duminy in [Dum1]. Let us describe it briefly; a complete account is given in [Ghy16]. Let $\mathcal{F}$ be a codimension one foliation on a compact manifold M. A leaf L of $\mathcal{F}$ is called *resilient* if there exist a loop $\sigma : [0,1] \longrightarrow L$ and a transversal T to $\mathcal{F}$ passing through $\sigma(0)$ such that the following conditions are satisfied: i) there exists a point $x \in L \cap T$ in the domain of holonomy h_σ of σ and different from $\sigma(0)$; ii) the sequence $h_\sigma^n(x)$ converges to $\sigma(0)$ as $n \to +\infty$. G. Duminy proved that *if $\mathcal{F}$ has no resilient leaf then the Godbillon-Vey invariant of $\mathcal{F}$ is zero.*

Recently, A. Connes and H. Moscovici have discovered a universal Hopf algebra with cohomology from which one is able to recover the characteristic classes of a foliation without use of Chern-Weil homomorphism or connections (*cf.* [CM1]).

7. Basic global analysis

Let M be a manifold endowed with a foliation $\mathcal{F}$ of codimension n. We suppose for simplicity that $\mathcal{F}$ is transversely orientable.

7.1. Foliated vector bundles and basic sections

Let $\mathcal{P} : G \hookrightarrow P \overset{\iota}{\longrightarrow} M$ be a principal bundle with structural group $G \subset \mathrm{GL}(N, \mathbf{C})$. The group G acts on P on the right and on its Lie algebra $\mathcal{G}$ by the adjoint representation. Denote by $\mathcal{V}$ the vector bundle whose fibre V_z at a point $z \in P$ is the tangent space at z of the fibre of $\mathcal{P}$. A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ of TP such that:

– for every $z \in \mathcal{P}$, $T_zP = V_z \oplus H_z$,

– for every $g \in G$ and every $z \in P$, $H_{zg} = (R_g)_*(H_z)$ where R_g is the right action of g on P.

As is well known the subbundle $\mathcal{H}$ is also the kernel of an invariant (under the action of G) 1-form ω on P (called the *connection form*) with values in $\mathcal{G}$.

It is easy to see that the restriction of ι_* (the derivative of ι) to H_z is an isomorphism onto $T_{\iota(z)}M$. Let $\tau = \iota_*^{-1}(T\mathcal{F})$. We say that $\mathcal{P}$ is *foliated* if τ is integrable. In this case, τ defines a foliation $\widetilde{\mathcal{F}}$ on P such that

– $\dim(\widetilde{\mathcal{F}}) = \dim(\mathcal{F})$,

– $\widetilde{\mathcal{F}}$ is invariant under the action of G.

We say that the connection $\mathcal{H}$ is *basic*, if the ω is basic (cf. subsection 1.5). A foliated bundle $\mathcal{E}$ is said to be an *$\mathcal{F}$-bundle*, if it admits a basic connection.

Let $E \longrightarrow M$ be a complex vector bundle defined by a cocycle $\{U_i, \gamma_{ij}, G\}$ where U_i is an open cover of M and $\gamma_{ij} : U_i \cap U_j \longrightarrow G \subset \mathrm{GL}(N, \mathbf{C})$ are the transition functions. We say that E is an *$\mathcal{F}$-bundle*, if the associated principal bundle $G \longrightarrow P \longrightarrow M$ is an $\mathcal{F}$-bundle. Because $E = P \times_G \mathbf{C}^N$, $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}_E$ on E. An *$\mathcal{F}$-morphism* $\varphi : (E, \omega) \longrightarrow (E', \omega')$ between two $\mathcal{F}$-bundles is a morphism of vector bundles which sends leaves of $\mathcal{F}_E$ into leaves of $\mathcal{F}_{E'}$.

(Notice that the collection of $\mathcal{F}$-bundles and $\mathcal{F}$-morphisms is a category. So we can define the group $K(M, \mathcal{F})$ of *foliated K-theory* as in the classical case.)

Let $E \longrightarrow M$ be an $\mathcal{F}$-bundle. Then the dual bundle E^* and all its exterior powers $\Lambda^* E^*$ are $\mathcal{F}$-bundles; also $\mathcal{H}^2 E = \{$Hermitian forms on $E\}$ is an $\mathcal{F}$-bundle.

7.2. Transversely elliptic operators

Let $E \longrightarrow M$ be a $\mathcal{F}$-foliated vector bundle. Denote by ∇ the covariant derivative $\chi(M) \times C^\infty(E) \xrightarrow{\nabla} C^\infty(E)$ associated to the connection $\mathcal{H}$. We say that a section $\alpha \in C^\infty(E)$ is *basic*, if it satisfies the condition $\nabla_X \alpha = 0$ for every $X \in \chi(\mathcal{F})$. The space $C^\infty(E/\mathcal{F})$ of basic sections of E is an A_b-module.

Let E and E' two $\mathcal{F}$-bundles (with the same rank N for simplicity). A *basic differential operator of order* ℓ from E to E' is a linear map $C^\infty(E/\mathcal{F}) \xrightarrow{D} C^\infty(E'/\mathcal{F})$ such that on local coordinates $(x_1, \ldots, x_m, y_1, \ldots y_n)$ for which $\mathcal{F}$ is defined by the differential equations $dy_1 = \ldots = dy_n = 0$, D has the expression:

$$D = \sum_{|s| \leq \ell} a_s(y) \frac{\partial^{|s|}}{\partial y_1^{s_1} \ldots \partial y_n^{s_n}}$$

where $s = (s_1, \ldots, s_n) \in \mathbf{N}^n$, $|s| = s_1 + \ldots + s_n$ and a_s are $N \times N$-matrices whose coefficients are basic functions. The *principal symbol* of D at the point $z = (x, y)$ and the basic covector $\xi \in \nu_z^* \mathcal{F}$ is the linear map $\sigma(D)(z, \xi) : E_z \longrightarrow E'_z$ defined by

$$\sigma(D)(z, \xi)(\eta) = \sum_{|s| = \ell} \xi_1^{s_1} \cdot \ldots \cdot \xi_n^{s_n} a_s(y)(\eta).$$

We say that D is *transversely elliptic* if $\sigma(D)(z, \xi)$ is an isomorphism for every $z \in M$ and every basic covector ξ different from 0. If $\mathcal{F}$ is Riemannian, its conormal bundle $\nu^* \mathcal{F}$ is an $\mathcal{F}$-bundle and is equipped with a foliation $\mathcal{F}^*$. Then the principal symbol $\sigma(D)(z, \xi)$ of a transversely elliptic operator D defines an element $[D]$ in the group $K(\nu^* \mathcal{F}, \mathcal{F}^*)$.

A *Hermitian metric* on E is a positive definite section h of $\mathcal{H}^2 E$. If h is basic we say that E is a *Hermitian $\mathcal{F}$-bundle.*

Let $E \longrightarrow M$ be a Hermitian $\mathcal{F}$-foliated bundle with Hermitian metric h and let D be a basic differential operator of order $\ell = 2\ell'$ on $C^\infty(E/\mathcal{F})$. For every $z \in M$ and every basic covector $\xi \in \nu_z^* \mathcal{F}$ we define a quadratic form $A(D)(z, \xi) : E_z \longrightarrow \mathbf{C}$ by

$$A(D)(z, \xi)(\eta) = (-1)^{\ell'} \langle \sigma(D)(z, \xi)(\eta), \eta \rangle.$$

We say that D is *strongly transversely elliptic*, if $A(D)(z, \xi)$ is positive definite for every $z \in M$ and every non zero ξ. Obviously every strongly transversely elliptic operator is transversely elliptic.

From now on we suppose that M is compact and connected. Assume that the foliation $\mathcal{F}$ is Riemannian tansversely oriented. Let $E^\#$ be the pullback of E to the principal bundle $\mathrm{SO}(n) \longrightarrow M^\# \xrightarrow{p} M$ of the orthonormal direct frames transverse to $\mathcal{F}$ (cf. Theorem

8). Then $E^{\#}$ is a SO(n)-bundle and a Hermitian $\mathcal{F}^{\#}$-bundle equipped with a Hermitian metric $h^{\#}$. Let $W^{\#}$ be the basic manifold associated to the transversely parallelizable foliation $\mathcal{F}^{\#}$ on $M^{\#}$. The basic sections of E are canonically identified to basic sections of $E^{\#}$ which are invariant under the action of SO(n). In particular if $f : M \longrightarrow \mathbf{C}$ is a basic function, $f \circ p$ is a basic function on $M^{\#}$ (with respect to $\mathcal{F}^{\#}$); moreover $f \circ p$ is invariant by the action of SO(n). Because $f \circ p$ is continuous, it is constant on the leaf closures of $\mathcal{F}^{\#}$ so it induces an SO(n)-invariant C^{∞} function on the basic manifold $W^{\#}$. We can prove, by the converse process, that any SO(n)-invariant C^{∞} function on the basic manifold $W^{\#}$ defines a C^{∞} basic function on M; in other words, the algebra A_b of basic functions on M is canonically isomorphic to the algebra $A_{\mathrm{SO}(n)}(W^{\#})$ of functions on $W^{\#}$ invariant by SO(n). The bundle like metric on $M^{\#}$ induces a Riemannian metric on $W^{\#}$ for which SO(n) acts by isometries. Let μ be the measure on $W^{\#}$ associated to this metric.

On $C^{\infty}(E/\mathcal{F})$ we define an inner product as follows. Let α and β be two elements of $C^{\infty}(E/\mathcal{F})$. The function $\Theta(\alpha,\beta) : z \in M \longrightarrow h_z(\alpha(z),\beta(z)) \in \mathbf{C}$ is basic; so it defines an SO(n)-invariant function $\Theta^{\#}(\alpha,\beta)$ on $W^{\#}$. We set

$$\langle \alpha, \beta \rangle = \int_W \Theta^{\#}(\alpha,\beta)(w) d\mu(w).$$

For any basic differential operator D from a Hermitian $\mathcal{F}$-bundle E to a Hermitian $\mathcal{F}$-bundle E', denote by $N(D)$ the kernel of D and $R(D)$ its range.

Theorem 15. *Let E and E' be two Hermitian $\mathcal{F}$-bundles on M and let D be a transversely elliptic operator from $C^{\infty}(E/\mathcal{F})$ to $C^{\infty}(E'/\mathcal{F})$. Denote by D^* the formal adjoint of D which is also a basic transversely elliptic operator from $C^{\infty}(E'/\mathcal{F})$ to $C^{\infty}(E/\mathcal{F})$. Then $N(D)$ and $N(D^*)$ are finite dimensional and we have an orthogonal decomposition:*

$$C^{\infty}(E/\mathcal{F}) = N(D) \oplus R(D^*).$$

In particular D has an index : $\mathrm{ind}(D/\mathcal{F}) = \dim N(D) - \dim N(D^*)$.

All the details of the proof of this theorem can be found in [Elk].

7.3. Transversely elliptic complexes

Let $(E^r, D_r)_{r=0,1,\ldots,n}$ be a family of Hermitian $\mathcal{F}$-bundles and basic differential operators of order one $D_r : C^{\infty}(E^r/\mathcal{F}) \longrightarrow C^{\infty}(E^{r+1}/\mathcal{F})$ (by convention $D_n = 0$) such that the sequence

$$(*) \qquad \cdots \xrightarrow{D_{r-1}} C^{\infty}(E^r/\mathcal{F}) \xrightarrow{D_r} C^{\infty}(E^{r+1}/\mathcal{F}) \xrightarrow{D_{r+1}} \cdots$$

is a differential complex, that is, $D_{r+1} \circ D_r = 0$ for $r = 0, 1, \dots, n-1$. Let $z \in M$ and $\xi \in \nu_z^* \mathcal{F}$; denote by $\sigma(D_r)(z, \xi)$ the principal symbol of D_r at (z, ξ) which is a linear map $\sigma(D_r)(z, \xi) : E_z^r \longrightarrow E_z^{r+1}$. Set $\sigma_r = \sigma(D_r)(z, \xi)$; we say that the complex $(*)$ is *transversely elliptic* if its symbol sequence

$$(*') \qquad \cdots \xrightarrow{\sigma_{r-1}} E_z^r \xrightarrow{\sigma_r} E_z^{r+1} \xrightarrow{\sigma_{r+1}} \cdots$$

is exact for every z and every non zero ξ. Let $D_r^* : C^\infty(E^{r+1}/\mathcal{F}) \longrightarrow C^\infty(E^r/\mathcal{F})$ be the formal adjoint of D_r (with respect to the inner product defined in subsection 7.2). Then it is easy to see that the complex $(*)$ is transversely elliptic if and only if the basic operator of order 2: $L_r : C^\infty(E^r/\mathcal{F}) \longrightarrow C^\infty(E^r/\mathcal{F})$ defined by $L_r = D_r^* D_r + D_{r-1} D_{r-1}^*$ is strongly transversely elliptic.

Let (E^r, D_r), $r = 0, 1, \dots, n$ be a transversely elliptic complex with cohomology $H_b^r(E^*)$. Then applying Theorem 15, we have the

Theorem 16. *i) For each $r = 0, 1, \dots, n$, the kernel $\mathcal{H}_b^r(E^*)$ of L_r is equal to the space $N(D_r) \cap N(D_{r-1}^*)$.*

ii) The space $\mathcal{H}_b^r(E^*)$ is finite dimensional and we have an orthogonal decomposition

$$C^\infty(E^r/\mathcal{F}) = \mathcal{H}_b^r(E^*) \oplus R(D_{r-1}) \oplus R(D_r^*).$$

iii) The orthogonal projection $C^\infty(E^r/\mathcal{F}) \longrightarrow \mathcal{H}_b^r(E^*)$ induces an isomorphism from $H_b^r(E^*)$ to $\mathcal{H}_b^r(E^*)$.

We will give two concrete examples to illustrate this result: the basic de Rham complex and the basic Dolbeault complex.

Let $r \in \{0, \dots, n\}$ and denote by E^r the vector bundle of exterior r-forms on the normal bundle $\nu\mathcal{F}$. As it was pointed out, E^r is a Hermitian $\mathcal{F}$-bundle; its basic sections are exactly basic differential forms $\Omega^r(M/\mathcal{F})$ of degree r on M. The de Rham exterior differential d restricted to $\Omega^r(M/\mathcal{F}) = C^\infty(E^r/\mathcal{F})$ is a basic differential operator $d : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^{r+1}(M/\mathcal{F})$. Thus we obtain a differential complex

$$(**) \qquad \cdots \xrightarrow{d} \Omega^r(M/\mathcal{F}) \xrightarrow{d} \Omega^{r+1}(M/\mathcal{F}) \xrightarrow{d} \cdots$$

called the *basic de Rham complex* of $\mathcal{F}$; its homology $H^r(M/\mathcal{F})$ is called the *basic cohomology* of $\mathcal{F}$ and depends only on the transverse structure of $\mathcal{F}$.

Let $\delta_b : \Omega^{r+1}(M/\mathcal{F}) \longrightarrow \Omega^r(M/\mathcal{F})$ be the formal adjoint of d; this operator can be described explicitely in terms of coefficients of the transverse metric on $\nu\mathcal{F}$ and the Hermitian metrics on the bundles E^r (cf. for instance [Alv4], [Ton], [PR], [Rei], [EH], [KT], [Elk]). Let $\Delta_b = d\delta_b + \delta_b d$; this is a basic differential operator of order 2 on $\Omega^r(M/\mathcal{F})$ called the *basic Laplacian.* A basic form $\alpha \in \Omega^r(M/\mathcal{F})$ which sastisfies the equation $\Delta_b\alpha = 0$, or equivalently $d\alpha = 0$ and $\delta_b\alpha = 0$, is called a *basic harmonic* form; denote by $\mathcal{H}^r(M/\mathcal{F})$ the space of such forms. Applying Theorem 15 we obtain the following

Theorem 17. *i) The space $\mathcal{H}^r(M/\mathcal{F})$ is finite dimensional and we have an orthogonal decomposition*

$$\Omega^r(M/\mathcal{F}) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(d) \oplus R(\delta_b).$$

ii) The orthogonal projection $\Omega^r(M/\mathcal{F}) \longrightarrow \mathcal{H}^r(M/\mathcal{F})$ induces an isomorphism from $H^r(M/\mathcal{F})$ to $\mathcal{H}^r(M/\mathcal{F})$.

iii) Suppose that the vector space $H^n(M/\mathcal{F})$ is non zero; then there exists a natural non degenerate pairing $\Phi : ([\alpha],[\beta]) \in H^r(M/\mathcal{F}) \times H^{n-r}(M/\mathcal{F}) \longrightarrow \Phi([\alpha],[\beta]) \in \mathbf{C}$. So the basic cohomology satisfies Poincaré duality.

During the last decades, many people contributed to the proof of this theorem. It was first proved by B.L. Reinhart in [Rei3]. But Y. Carrière [Car1] discoverd a mistake which makes assertion iii) false: B.L. Reinhart does not suppose $H^n(M/\mathcal{F})$ different from $\{0\}$ to obtain Poincaré duality; he was probably thinking that this hypothesis is automatically satisfied. Later on F.W. Kamber and P. Tondeur [KT2] have shown that the Reinhart's proof is still valid if we suppose the leaves minimal (cf. subsection 9.3). Finally the theorem was proved in full generality (without any assumption on the minimality of the leaves) in [EH].

Now suppose that $\mathcal{F}$ is Hermitian. Let ν be the complexified normal bundle $\nu\mathcal{F} \otimes_{\mathbf{R}} \mathbf{C}$ of $\nu\mathcal{F}$. Let J be the automorphism of ν associated to the complex structure; J satisfies the relation $J^2 = -\mathrm{id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles respectively denoted ν^{10} and ν^{01}. We have a splitting $\nu = \nu^{10} \oplus \nu^{01}$ which gives rise to a decomposition

$$\Lambda^r\nu^* = \bigoplus_{p+q=r} \Lambda^{p,q}$$

where $\Lambda^{p,q} = \Lambda^p\nu^{10^*} \otimes \Lambda^q\nu^{01^*}$. Basic sections of $\Lambda^{p,q}$ are called *basic forms of type* (p,q). They form a vector space denoted $\Omega^{p,q}(M/\mathcal{F})$. We have

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F}).$$

The exterior differential decomposes into a sum of two operators

$$\partial : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p+1,q}(M/\mathcal{F}) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p,q+1}(M/\mathcal{F})$$

as in the classical case of a complex manifold. We have $\overline{\partial}^2 = 0$; so we obtain, for p fixed, a differential complex

$$(***) \qquad \cdots \xrightarrow{\overline{\partial}} \Omega^{p,q}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \cdots$$

called the *basic Dolbeault complex* of $\mathcal{F}$; its homology $H^{p,q}(M/\mathcal{F})$ is the *basic Dolbeault cohomology* of the foliation $\mathcal{F}$: even though the leaf space is topologically bad, it can be considered as a "complex manifold" whose Dolbeault cohomology is $H^{p,*}(M/\mathcal{F})$!

Let δ_b'' denote the formal adjoint of $\overline{\partial}$; this is an operator of type $(0,-1)$. The operator $\Delta_b'' = \delta_b''\overline{\partial} + \overline{\partial}\delta_b''$ is selfadjoint; a simple computation in local coordinates, like for the basic Laplacian, shows that Δ_b'' is strongly transversely elliptic. Therefore the complex $(***)$ is transversely elliptic. Let

$$\mathcal{H}^{p,q}(M/\mathcal{F}) = \mathrm{Ker}\Delta_b'' = \{\alpha \in \Omega^{p,q}(M/\mathcal{F}) : \overline{\partial}\alpha = 0 \text{ and } \delta_b''\alpha = 0\}.$$

Applying Theorem 15, we obtain

Theorem 18. *i) The space $\mathcal{H}^{p,q}(M/\mathcal{F})$ is finite dimensional and we have an orthogonal decomposition*

$$\Omega^{p,q}(M/\mathcal{F}) = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\Delta_b'') = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\overline{\partial}) \oplus R(\delta_b'').$$

ii) The orthogonal projection $\Omega^{p,q}(M/\mathcal{F}) \longrightarrow \mathcal{H}^{p,q}(M/\mathcal{F})$ induces an isomorphism from $H^{p,q}(M/\mathcal{F})$ to $\mathcal{H}^{p,q}(M/\mathcal{F})$.

iii) Suppose that the vector space $H^n(M/\mathcal{F})$ is non zero; then there exists a natural non degenerate pairing $\Psi : ([\alpha],[\beta]) \in H^{p,q}(M/\mathcal{F}) \times H^{n-p,n-q}(M/\mathcal{F}) \longrightarrow \Psi([\alpha],[\beta]) \in \mathbf{C}$. So the basic Dolbeault cohomology satisfies Serre duality.

Suppose now that $\mathcal{F}$ is transversely Kählerian with Kähler form ω (it is a basic differential form of degree 2; it is closed and non degenerate). In this case, we can prove that $\Delta_b = 2\Delta_b''$. Because of the decomposition

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F}),$$

every basic differential r-form can be uniquely writen as a sum $\alpha = \sum_{p+q=r} \alpha_{pq}$ where $\alpha_{pq} \in \Omega^{p,q}(M/\mathcal{F})$. Then we have the following assertions.

iv) *α is Δ_b-harmonic if, and only if, each component α_{pq} is Δ_b''-harmonic. So we have a direct decomposition*

$$H^r(M/\mathcal{F}) = \bigoplus_{p+q=r} H^{p,q}(M/\mathcal{F}).$$

v) *The complex conjugacy induces an isomorphism (of real vector spaces)*

$$H^{p,q}(M/\mathcal{F}) \simeq H^{q,p}(M/\mathcal{F}).$$

vi) *For every odd $r \in \{0, \cdots, 2n\}$, the dimension of the space $H^r(M/\mathcal{F})$ is even. In particular if $n = 1$ we have $b_1(M/\mathcal{F}) = 2\text{dim } H^{01}(M/\mathcal{F})$.*

The integer dim $H^{01}(M/\mathcal{F})$ will be denoted $g(\mathcal{F})$ and called the *genus* of the foliation $\mathcal{F}$. It is similar to the genus of a compact Riemann surface; it counts the number of linearly independant basic holomorphic 1-forms.

vii) *Por every $p \in \{0, \cdots, n\}$ the differential form $\omega^p = \omega \wedge \cdots \wedge \omega$ (wedge product p times) is harmonic. So, the space $H^{p,p}(M/\mathcal{F})$ is non zero.*

Notice that this theorem is also a particular case of Theorem 16. It can be used to establish more properties : (basic Hodge structures for transversely Kählerian foliations, basic Calabi-Yau theorem [Elk] and deformation of transversely holomorphic foliations with a fixed differentiable type [EN4].

8. Deformation theory of foliations

We will describe only the real case following Hamilton's paper [Ham1]. Deformation theory of holomorphic or generally transversely holomorphic foliations is more rich. The reader can find a good account of the subject in [GHS].

Let M be a manifold of dimension $m + n$. For each $x \in M$, let $G(x, m)$ be the Grassmanian manifold of m-planes in T_xM. Then:

$$\mathcal{G}(m) = \bigcup_{x \in M} G(x, m)$$

can be given a structure of a differentiable manifold such that the canonical projection $(x, \tau) \in \mathcal{G}(m) \longrightarrow x \in M$ is a locally trivial fibration, the fibre being the Grassmanian $G(m)$ of m-planes in the space $\mathbf{R}^{m+n}$. Then a subbundle of rank m of TM is just a section of the bundle $\mathcal{G}(m) \longrightarrow M$. Denote by $C^\infty(\mathcal{G}(m))$ the space of sections of this bundle.

Let $\tau \in C^\infty(\mathcal{G}(m))$. By Frobenius Theorem, τ is tangent to a foliation if, and only if for any pair (U, V) of (global) sections of τ, the Lie bracket $[U, V]$ is also a section of τ. Let $(X_1, \ldots, X_m)$ be a local basis of τ. Then

$$U = \sum_{i=1}^{m} a^i X_i \quad \text{and} \quad V = \sum_{j=1}^{m} b^j X_j.$$

So the bracket $[U, V]$ can be expressed as

$$[U, V] = \sum_{i,j=1}^{m} \left\{ a^i b^j [X_i, X_j] + \left(a^i X_i(b^j) X_j - b^j X_j(a^i) X_i \right) \right\}.$$

Therefore the value of $[U, V]$ in $\nu\tau = TM/\tau$ at a point $x \in M$ depends only on the value of U and V at x. Hence $Q_\tau(U, V) = \pi([U, V])$ is a skew-symmetric bilinear map $Q_\tau : \tau \times \tau \longrightarrow \nu\tau$ where $\pi : TM \longrightarrow \nu\tau$ is the canonical projection. In other words, Q_τ is a global section of the vector bundle $\Lambda^2(\tau, \nu\tau)$ of skew-symmetric bilinear forms on the bundle τ. The integrability condition of τ is equivalent to Q_τ identically equal to 0. So we get a map $Q : C^\infty(\mathcal{G}(m)) \longrightarrow \Sigma$ where Σ is a fibre bundle over $\mathcal{G}(m)$ whose fibre over a point $\sigma \in \mathcal{G}(m)$ is the infinite dimensional space $\Omega^2(\sigma, \nu\sigma)$ of global sections of the bundle $\Lambda^2(\tau, \nu\tau)$. The space $\mathcal{F}ol(M, m)$ of dimension m foliations on M is exactly the set $\{Q = 0\}$. It will be equipped with the C^∞-topology induced by the topology of the Fréchet manifod $C^\infty(\mathcal{G}(m)$ (cf. [Ham2]). Let $\mathcal{D}$ be the diffeomorphism group of M; then $\mathcal{D}$ acts on $C^\infty(\mathcal{G}(m))$ and the action preserves $\mathcal{F}ol(M, m)$. Two foliations $\mathcal{F}, \mathcal{F}' \in \mathcal{F}ol(M, m)$ are conjugated, if they are in the same orbit of the action of $\mathcal{D}$, that is, there exists $\varphi \in \mathcal{D}$ such that $\mathcal{F}' = \varphi^*(\mathcal{F})$.

Now fix τ in $C^\infty(\mathcal{G}(m))$ and suppose that it is tangent to a foliation $\mathcal{F}$. Then the map $P_\tau : \varphi \in \mathcal{D} \longrightarrow \varphi^*(\mathcal{F}) \in C^\infty(\mathcal{G}(m))$ takes its values in $\mathcal{F}ol(M, m)$. So we get a sequence of Fréchet manifolds and differentiable maps

$$\mathcal{D} \xrightarrow{P_\tau} C^\infty(\mathcal{G}(m)) \xrightarrow{Q} \Sigma.$$

Following R. Hamilton, this sequence is called the *non linear deformation complex* of the foliation [Ham1]. $\mathcal{F}$.

Definition 4. *We say that $\mathcal{F}$ is C^∞-stable if there exist an open neighbourhood $\mathcal{O}$ of the identity in $\mathcal{D}$ and an open neighborhood $\mathcal{U}$ of $\mathcal{F}$ in $\mathcal{F}ol(M, m)$ such that the sequence $\mathcal{O} \xrightarrow{P_\tau} \mathcal{U} \xrightarrow{Q} \Sigma$ is exact, that is, every dimension m foliation $\mathcal{F}'$ on M, close enough to $\mathcal{F}$ in the C^∞-topology, is conjugated to $\mathcal{F}$ by an element of $\mathcal{O}$.*

An important tool to prove the C^∞-stability of a foliation is Hamilton's criterion (cf. [Ham1], p. 47) that we shall describe. This criterion is based on the implicit function theorem of Nash-Moser which is nicely explained in Hamilton's paper [Ham2].

Given a foliation $\mathcal{F}$ of dimension m on a compact manifold M, let $A^k_{\mathcal{F}}$ denote the space of differentiable sections of $\Lambda^k T\mathcal{F}^* \otimes \nu\mathcal{F}$. Since $\nu\mathcal{F}$ is a foliated bundle there is a well defined "*exterior derivative along the leaves*" $d_{\mathcal{F}} : A^k_{\mathcal{F}} \longrightarrow A^{k+1}_{\mathcal{F}}$ given by:

$$d_{\mathcal{F}}\eta(X_1, \ldots, X_{k+1}) = \sum_i (-1)^i X_i \eta\left(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}\right)$$

$$+ \sum_{i<j} (-1)^{i+j} \eta\left([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\right).$$

An easy computation shows that $d^2_{\mathcal{F}} = 0$ and thus we obtain a differential complex

$$0 \longrightarrow A^0_{\mathcal{F}} \xrightarrow{d_{\mathcal{F}}} A^1_{\mathcal{F}} \xrightarrow{d_{\mathcal{F}}} A^2_{\mathcal{F}} \xrightarrow{d_{\mathcal{F}}} \ldots \xrightarrow{d_{\mathcal{F}}} A^m_{\mathcal{F}} \longrightarrow 0$$

which is only elliptic along the leaves. Let $|| \quad ||_0 \leq || \quad ||_1 \leq \ldots \leq || \quad ||_s \leq \ldots$ be an increasing collection of norms (of Sobolev or Hölder type) on the Fréchet space

$$A^*_{\mathcal{F}} = \bigoplus_{k \geq 0} A^k_{\mathcal{F}}.$$

With this notation one has

Theorem 19. (Hamilton) *Assume that there exist continuous linear operators* $H : A^1_{\mathcal{F}} \longrightarrow A^0_{\mathcal{F}}$ *and* $K : A^2_{\mathcal{F}} \longrightarrow A^1_{\mathcal{F}}$ *fulfilling the following conditions*

(i) $d_{\mathcal{F}} \circ H + K \circ d_{\mathcal{F}} = id$,

(ii) there is a fixed number $r \in \mathbf{N}$ *for which we have tame estimates for all* s

$$||H(\beta)||_s \leq C_s ||\beta||_{s+r} \quad \text{and} \quad ||K(\gamma)||_s \leq C_s ||\gamma||_{s+r}$$

where C_s *are positive constants depending only on* s.

Then the foliation $\mathcal{F}$ *is* C^∞*-stable.*

Unfortunately Hamilton's paper is still unpublished. In [EN3], the authors constructed a class of foliations and, using Hamilton's criterion, they proved that these foliations are C^∞-stable. Example 1.3 iv), with some assumptions on the matrix A, is in this class.

9. Some other themes

As we have pointed out in the foreword, Foliation Theory is a wide field in Mathematics and so huge to discuss completely here. For this reason we have chosen only some of the themes related to Differential Geometry which is the main topic to which this book is devoted. The non warned reader may be inclined to believe that the theory is reduced to this part. Fortunately this is not the case. We devote this section to other themes which are no less important than the above ones.

9.1. Compact leaves

Let M be a connected compact orientable manifold of dimension $m+n$ and $\mathcal{F}$ a codimension n foliation on M. A *compact leaf* of $\mathcal{F}$ is a leaf L which is compact as a subset of M. If $m=1$ such leaf is a periodic orbit and it describes a stationary state of the dynamical system defined by $\mathcal{F}$. The problem of the existence of compact leaves is highly non trivial. It was first introduced by H. Poincaré in his studies on *limit cycles* for ordinary differential equations. One of the famous problems was the Seifert conjecture: *Every continuous vector field on the* 3*-dimensional sphere* $\mathbf{S}^3$ *has a periodic orbit.*

In 1974, Using Denjoy's example of a vector field with exceptional minimal set on the 2-torus, P. Schweitzer [Sch1] constructed a counterexample in class C^1. In 1988, J. Harrison [Har] gave a C^2 counterexample. Finally in 1993, K. Kuperberg [Kup] solved completely the problem by constructing in any compact 3-manifold a real analytic vector field without periodic orbit. However, M. Brunella [Bru1] proved that the conjecture is true if the flow is transversely holomorphic; in fact, he established a complete classification of these flows on compact 3-manifolds.

The most important result concerning the problem of existence of compact leaves was Novikov's theorem stated above (theorem 12). Nothing is known in higher dimensions and the following question is still open: *is it true that every codimension one foliation on the odd sphere* $\mathbf{S}^{2p+1}$ *(where* $p \geq 2$*) admits a compact leaf?*

We say that $\mathcal{F}$ is a *compact foliation* if all leaves are compact. For example, every foliation defined by a locally free action of a connected compact Lie group is a compact foliation. Compact foliations was a thema which interested many people (R. Edwards, K. Millet, D. Sullivan, D. Epstein, E. Vogt, H. Rummler *etc.*).

9.2. When is a manifold a leaf?

Let L be a non compact connected manifold. *Does there exist a compact manifold* M *endowed with a foliation* $\mathcal{F}$ *with a leaf diffeomorphic to* L *?* This question was asked by J. Sondow in [Son] where he gave some sufficient conditions on L to be a leaf. J. Cantwell and L. Conlon proved in [CC5] that every surface is a leaf. Along the same lines, G. Hector and W. Bouma proved in [HB] that every non compact surface can be a leaf of a simple foliation of $\mathbf{R}^3$ *i.e.* a foliation defined by a submersion $\mathbf{R}^3 \longrightarrow \mathbf{R}$.

In [Ghy2] E. Ghys observed that the topology of a leaf of a foliation on a compact manifold has to be, in some sense, "recurrent"; then he constructed, for any positive integer

d, a non compact manifold L of dimension d which can not be homeomorphic to any leaf of any foliation on a compact manifold. In [Ghy13] he also studies the topology of the generic leaves of a *lamination* by surfaces on a compact metric space and proved that there exist only six non compact surfaces which can be realized as leaves:

a) the plane $\mathbf{R}^2$,

b) the cylinder $\mathbf{S}^1 \times \mathbf{R}$,

c) the "Loch-Ness monster" *i.e.* the plane with infinitely many handles attached,

d) the "Jacob ladder",

e) the "Cantor tree" *i.e.* the sphere $\mathbf{S}^2$ with a Cantor set removed,

f) the "flowered Cantor tree" *i.e.* the Cantor tree with infinitely many handles attached in all directions.

9.3. Minimal leaves

Let M be a Riemannian manifold. Denote by ∇ the covariant derivative associated to the Levi-Civita connection. Let L be a submanifold of M (not necessarily properly embedded). Let $x \in L$ and ν a vector field defined on a neighbourhood of x and orthogonal to L. For $X \in T_xL$, we set:

$$W_x^\nu(X) = -p_x(\nabla_X \nu)$$

where $p_x : T_xM \longrightarrow T_xL$ is the orthogonal projection. Then W_x^ν is an endomorphism of the vector space T_xL, symmetric with respect to the induced metric on T_xL; it is called the *Weingarten map* associated to ν. The trace of W_x^ν describes the variation at x of the volume element when L moves in the direction of ν. We say that L is *minimal*, if the trace of W_x^ν is zero for all vector fields ν orthogonal to L. A foliation $\mathcal{F}$ on M is said to be with *minimal leaves*, if all leaves of $\mathcal{F}$ are minimal submanifolds.

Given an m-dimensional foliation on a compact manifold M, does there exist a Riemannian metric on M for which the leaves are minimal?

This question was discussed by H. Rummler [Rum] and D. Sullivan [Sul3]. They proved the following criterion : *such a metric exists if, and only if, there exists an m-form χ positive on the leaves and relatively closed, namely $d\chi(X_1, \ldots, X_m, Y) = 0$ whenever the vector fields $X_1, \ldots, X_m$ are tangent to $\mathcal{F}$.*

In [Hae3] A. Haefliger proved that the property for $\mathcal{F}$ to be with minimal leaves depends only on the transverse structure. He also gave a criterion in terms of transverse invariant currents and used it to give many examples of minimal foliations and non minimal ones.

Suppose now that $\mathcal{F}$ is a Riemannian codimension n foliation and denote by v the volume basic form associated to the metric. If $\mathcal{F}$ is with minimal leaves then v defines a non zero class in the basic cohomology $H^n(M/\mathcal{F})$. Indeed, let χ the be m-form given by the Rummler-Sullivan criterion. Suppose that $v = d\beta$ where $\beta \in \Omega^{n-1}(M/\mathcal{F})$. Then:

$$\chi \wedge v = \chi \wedge d\beta = (-1)^m \{d(\chi \wedge \beta) - d\chi \wedge \beta\}.$$

But $d\chi \wedge \beta = 0$ because χ is relatively closed. So $\chi \wedge v$ is an exact form. But this is impossible because it is a volume form on the compact orientable manifold M. The converse of this assertion was conjectured by Y. Carrière [Car1] and proved by X. Masa in [Mas1].

In [Ghy7] E. Ghys proved that any Riemannian foliation on a simply connected compact manifold admits a bundle-like metric for which the leaves are minimal.

Now let $\mathcal{F}$ be a foliation on a compact Riemannian manifold M. We say that $\mathcal{F}$ is *totally geodesic* if every geodesic tangent to a leaf L at a point is tangent to L everywhere. This is a special class of foliations with minimal leaves which was studied for instance by Y. Carrière, G. Cairns, E. Ghys (see [Cai1], [Cai2], [Cai3], [CG], [CrG]). In particular E. Ghys, in [Ghy6], has completely classified all the totally geodesic foliations of codimension one on compact manifolds.

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