On leafwise meromorphic functions with prescribed poles

by

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(Avril 2012)

Abstract. Let $\mathcal{F}$ be a complex foliation by Riemann surfaces defined by a trivial (in the differentiable sense) fibration $\pi : M \longrightarrow B$ but for which the complex structure on each fibre $\pi^{-1}(t)$ may depend on t. Let $\sigma : B \longrightarrow M$ be a section of π contained in a $\mathcal{F}$-relatively compact subset of M. We prove: for any relatively compact open set U containing $\Sigma = \sigma(B)$ and any integer $s \geq 0$, there exists a function $U \longrightarrow \mathbb{C}$ of class C^s nonconstant on any leaf of $(U, \mathcal{F})$, meromorphic along the leaves and whose set of poles is exactly Σ.

1. Preliminaries

Let M be a differentiable manifold of dimension $2m + n$ endowed with a dimension $2m$ orientable foliation $\mathcal{F}$.

1.1. Definition. *We say that $\mathcal{F}$ is* **complex** *if it can be defined by an open cover $\mathcal{U} = \{U_i\}$ of M and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \longrightarrow U_i$ (where Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$) such that, for any pair (i, j) with $U_i \cap U_j \neq \emptyset$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \longrightarrow \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', t') = \big(\phi_{ij}^1(z, t), \phi_{ij}^2(t)\big)$ with $\phi_{ij}^1(z, t)$ holomorphic in z for t fixed.*

An open set U of M like one of the cover $\mathcal{U}$ in Definition 1.1 is called *adapted* to the foliation. Any leaf of $\mathcal{F}$ is a complex manifold of dimension m. The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold. A manifold M with a complex foliation $\mathcal{F}$ will be denoted $(M, \mathcal{F})$.

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two complex foliations. A *morphism* from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$ is a differentiable map $f : M \longrightarrow M'$ which sends every leaf F of $\mathcal{F}$ into a leaf F' of $\mathcal{F}'$ such that the restriction map $F \xrightarrow{f} F'$ is holomorphic.

We say that a morphism $f : (M, \mathcal{F}) \longrightarrow (M', \mathcal{F}')$ is an *isomorphism* of complex foliations (*automorphism* of $(M, \mathcal{F})$ if $(M, \mathcal{F}) = (M', \mathcal{F}')$) if f is a diffeomorphism whose restriction to any leaf $F \longrightarrow F'$ (where $F' = f(F)$) is a biholomorphism. We say that two complex foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are *conjugated* if there exists an isomorphism $f : (M, \mathcal{F}) \longrightarrow (M, \mathcal{F}')$. Automorphisms of $\mathcal{F}$ form a group denoted $G(\mathcal{F})$.

1.2. Examples

i) Any complex manifold M of dimension m is a complex foliation of dimension m. Its automorphism group is exactly the automorphism group of the complex manifold M.

ii) Any holomorphic foliation (on a complex manifold M) is a complex foliation.

iii) Let B be a differentiable manifold and M an open set of $\mathbb{C}^m \times B$. For $t \in B$, $M_t = \{z \in \mathbb{C}^m : (z,t) \in M\}$ is an open set of $\mathbb{C}^m$ called the *section* of M along t. The connected components of the sections of M are leaves of a complex foliation $\mathcal{F}$ of dimension m called the complex *canonical* foliation of M.

iv) Let F be a complex manifold and B a differentiable one. Any locally trivial fibration $F \hookrightarrow M \longrightarrow B$ whose cocycle takes values in the complex automorphism group $\mathrm{Aut}(F)$ of F is a complex foliation, the fibres being the leaves. If the fibration is trivial *i.e.* $M = F \times B$, we say that $\mathcal{F}$ is a *complex product foliation.* In that case all the leaves are holomorphically equivalent. Suppose that $\mathcal{F}$ is a complex foliation on $M = F \times B$ whose leaves are the factors $F \times \{t\}$ but the complex structure may depend on t; then we say that $\mathcal{F}$ is a *differentiable product.*

v) Let $\rho_1 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\rho_2 : \mathbb{R}^* \longrightarrow \mathbb{R}$ be functions of class C^1 satisfying the following conditions:

- $\rho_1(-t) = \rho_1(t)$ and $\rho_2(-t) = \rho_2(t)$;
- $\rho_1(1) = 0$ and $\rho_1 < 0$ on $]-1, +1[$;
- ρ_1 is strictly increasing on $[1, +\infty[$ and $\lim\limits_{t \to +\infty} \rho_1(t) = 1$;
- ρ_2 is strictly decreasing on $]0, +\infty[$, $\lim\limits_{t \to +\infty} \rho_2(t) = 1$ and $\lim\limits_{t \to +0^+} \rho_2(t) = +\infty$.

Let M be the open set of $\mathbb{C} \times \mathbb{R}$ defined by $M = \{(z,t) \in \mathbb{C} \times \mathbb{R} : \rho_1(t) < |z| < \rho_2(t)\}$ equipped with its canonical complex foliation $\mathcal{F}$. Then the leaves are: $\mathbb{C}$ if $t = 0$, open discs for $t \neq 0$ and $|t| < 1$, two punctured discs if $|t| = 1$ and the others are annulus.

On the open set $N = \{(z,t) \in M : t > 1\}$ the complex foliation $\mathcal{F}_N$ is a differentiable product. Two leaves are never isomorphic; each one has a complex structure coded by the ratio $\varepsilon(t) = \frac{\rho_2(t)}{\rho_1(t)}$. Since $\varepsilon(t) \neq \varepsilon(t')$ for $t \neq t'$, any automorphism of $\mathcal{F}_N$ must be the identity on the transversal. Then the automorphism group $G(\mathcal{F}_N)$ of $\mathcal{F}$ is generated by the group $C^\infty(]1, +\infty[, \mathbb{S}^1)$ and the map $(z,t) \longmapsto \left(\frac{\rho_1(t)\rho_2(t)}{z}, t\right)$ which preserves each annulus.

1.3. Question. *Does the odd sphere $\mathbb{S}^{2n+1}$ support a codimension one complex foliation?*

Of course, yes for $\mathbb{S}^3$ (any orientable foliation by surfaces is a complex one). In higher dimension I already asked this question in 1995 during a lecture I gave in the seminar *Géométrie dynamique* at Université de Lille 1. A construction of such foliation on the sphere $\mathbb{S}^5$ was given by L. Meersseman and A. Verjovsky in [MV1]. But recently they have discovered that the manifold supporting this foliation is in fact a bundle over the circle with fibre a projective Fermat surface (*cf.* [MV2]). Even the authors have failed to answer the question for $\mathbb{S}^5$ their example is highly non trivial and interesting. But the question now remains open.

2. The $\overline{\partial}_{\mathcal{F}}$-cohomology

Let $(M, \mathcal{F})$ be a complex foliation of dimension m. Let $A^{pq}(\mathcal{F})$ be the space of foliated differential forms of type (p,q) that is, differential forms on M which can be written in local coordinates adapted to the foliation $(z,t) = (z_1, \dots, z_m, t_1, \dots, t_n)$ (the foliation is defined by the differential system $dt_1 = \cdots = dt_n = 0$):

$$\alpha = \sum \alpha_{j_1 \dots j_p k_1 \dots k_q}(z,t) dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \dots \wedge d\overline{z}_{k_q}$$

where the coefficients $\alpha_{j_1 \dots j_p k_1 \dots k_q}$ are functions of class C^s and C^∞ along the leaves (with $s \in \mathbb{N} \cup \{\infty\}$. Let $\overline{\partial}_{\mathcal{F}} : A^{pq}(\mathcal{F}) \longrightarrow A^{p,q+1}(\mathcal{F})$ be the Cauchy-Riemann operator along the leaves defined by:

$$\overline{\partial}_{\mathcal{F}} \alpha = \sum \left(\sum_{k=1}^{m} \frac{\partial \alpha_{j_1 \dots j_p k_1 \dots k_q}}{\partial \overline{z}_k}(z,t) d\overline{z}_k \wedge dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \dots \wedge d\overline{z}_{k_q} \right)$$

where $\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left\{ \frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right\}$ with $z_k = x_k + i y_k$. It satisfies $\overline{\partial}_{\mathcal{F}}^2 = 0$, hence we have a differential complex $0 \longrightarrow A^{p0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^{p1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^{p,m-1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^{pm}(\mathcal{F}) \longrightarrow 0$ called the *$\overline{\partial}_{\mathcal{F}}$-complex* of $(M, \mathcal{F})$; its homology $H^{pq}_{\mathcal{F}}(M)$ is called the *foliated Dolbeault cohomology* (or the *$\overline{\partial}_{\mathcal{F}}$-cohomology*) of the complex foliation $(M, \mathcal{F})$. It is locally trivial *i.e.* we have a:

2.1. Foliated Dolbeault-Grothendieck Lemma. *Let $x \in M$. Then there exists an open neighborhood U of x adapted to the foliation such that, for every $p = 0, \dots, m$, $H^{pq}_{\mathcal{F}}(U) = 0$ for $q \geq 1$.*

The proof is a straightforward adaptation to the parametric case of the classical one.

One can describe the cohomology $H^{p*}_{\mathcal{F}}(M)$ by using a sheaf which is analogous to the sheaf of germs of holomorphic p-forms on a complex manifold. A p-form α is said to be *$\mathcal{F}$-holomorphic*, if it is foliated, of type $(p,0)$ and satisfies $\overline{\partial}_{\mathcal{F}} \alpha = 0$. Locally, a *$\mathcal{F}$-holomorphic* p-form can be written: $\alpha = \sum \alpha_{j_1 \cdots j_p}(z,t) dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ with $\alpha_{j_1 \cdots j_p}$ holomorphic on z.

Let $\mathcal{H}^p_{\mathcal{F}}$ be the sheaf of germs of $\mathcal{F}$-holomorphic p-forms on M and $\mathcal{A}^{pq}(\mathcal{F})$ be the sheaf of germs of differential forms of type (p,q) on $\mathcal{F}$; $\mathcal{A}^{pq}(\mathcal{F})$ is a fine sheaf. Lemma 2.1 implies the:

2.2. Proposition. *The sequence $0 \longrightarrow \mathcal{H}^p_{\mathcal{F}} \hookrightarrow \mathcal{A}^{p0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} \mathcal{A}^{pm}(\mathcal{F}) \longrightarrow 0$ is a fine resolution of $\mathcal{H}^p_{\mathcal{F}}$. So we have $H^q(M, \mathcal{H}^p_{\mathcal{F}}) = H^{pq}_{\mathcal{F}}(M)$, for $p, q = 0, 1, \dots m$.*

If $n \geq 1$, this resolution is not elliptic; it is only elliptic along the leaves. Hence the cohomology $H^*(M, \mathcal{H}^p_{\mathcal{F}})$ is not necessarily finite dimensional even if the manifold M is compact.

Any isomorphism of complex foliations $(M, \mathcal{F}) \xrightarrow{f} (M', \mathcal{F}')$ induces an isomorphism $f^* : H^*(M', \mathcal{H}^p_{\mathcal{F}'}) \longrightarrow H^*(M, \mathcal{H}^p_{\mathcal{F}})$. In particular $H^*(M, \mathcal{H}^p_{\mathcal{F}})$ depends only on the complex conjugacy class of $\mathcal{F}$.

For $p = 0$, we denote $\mathcal{H}_{\mathcal{F}}$ the sheaf $\mathcal{H}^0_{\mathcal{F}}$; its sections over an open set U of M are $\mathcal{F}$-holomorphic functions on U; they form a complex vector space which we will denote by

$\mathcal{H}^0_{\mathcal{F}}(U)$ and simply $\mathcal{H}(U)$ in case the codimension of $\mathcal{F}$ is zero, that is, M is a complex manifold and the foliation has just one leaf, M itself.

Let $p \in \mathbb{N}$. An open set U of M (with the induced foliation) is said to be *p-acyclic*, if $H^q(U, \mathcal{H}^p_{\mathcal{F}}) = 0$ for any $q \geq 1$. An open cover $\mathcal{U} = \{U_i\}$ is *p-acyclic* if, for any multi-index $(i_0, \cdots, i_k)$ of I, the open set $U_{i_0 \cdots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$ is p-acyclic. We can easily see by Lemma 2.1 that such open cover exists and, in addition, can be chosen locally finite. By Leray's Theorem (*cf.* [Gm]), $H^*(M, \mathcal{H}^p_{\mathcal{F}}) = H^*(\mathcal{U}, \mathcal{H}^p_{\mathcal{F}})$ for any locally finite p-acyclic open cover $\mathcal{U}$.

We have two ways for computing the $\overline{\partial}_{\mathcal{F}}$-cohomology of $\mathcal{F}$: using foliated differential forms of type (p, q) and the $\overline{\partial}_{\mathcal{F}}$ operator or a locally finite p-acyclic open cover $\mathcal{U}$ adapted to the foliation and Cěch method. Both of the two points of view will be interesting for our purpose.

Let us start with a simple example. Let F be a complex manifold of dimension m and B a differentiable manifold. We denote by $C^s(B)$ the complex vector space of complex C^s (with $s \in \mathbb{N} \cup \{\infty\}$) functions on B. The following proposition is easy to prove.

2.3. Proposition. *Suppose that $\mathcal{F}$ is defined by a locally trivial fibration $F \xrightarrow{\pi} M \longrightarrow B$ (the cocycle is with values in the biholomorphism group of the complex manifold F). Then: $H^{p*}_{\mathcal{F}}(M) = H^{p*}(F) \otimes C^s(B)$ where $H^{p*}(F)$ is the Dolbeault cohomology of the complex manifold F. In particular, $H^{p*}_{\mathcal{F}}(M) = 0$ for $* \geq 1$ if the the fibre F is a Stein manifold.*

2.4. Open questions

Some questions inspired by the classical complex analysis are natural. Let M be a differentiable manifold with a complex foliation $\mathcal{F}$ of dimension m.

Question 1. *Suppose that every leaf is closed (in the topological sense as a subset of M) and Stein that is, it can be embedded in some $\mathbb{C}^N$). Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?*

A weak version of this question is obtained by imposing an extra hypothesis on the foliated manifold $(M, \mathcal{F})$.

Question 2. *Suppose that every leaf is closed and Stein and that $\mathcal{F}$ is a complete Riemannian foliation (the normal bundle $TM/T\mathcal{F}$ admits a Riemannian metric invariant alog the leaves). Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?*

In fact, by a localization procedure, question 2 can be reduced essentially to the following one.

Question 3. *Suppose that M is a diffentiable product $F \times B$ where F is a Stein manifold and B is a ball of $\mathbb{R}^n$; each leaf $F \times \{t\}$ is diffeomorphic to F but has a complex structure which may depend on $t \in B$ and is Stein. Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?*

For a study of foliated Dolbeault cohomology and its explicit calculus on some complex foliations with a more complicated dynamics see [ES].

2.5. Zeros and poles

Suppose that the dimension of $\mathcal{F}$ is 1 that is, the leaves are Riemann surfaces. Let U be an open set of M with the induced complex foliation $\mathcal{F}$.

Let $f : U \longrightarrow \mathbb{C}$ be a $\mathcal{F}$-holomorphic function and let Z be the set of its zeros. The restriction of f to any leaf F is a holomorphic function; then, if $f : F \longrightarrow \mathbb{C}$ is not identically zero, $Z \cap F$ is a discrete set of F. So in a neighborhood of a point of $Z \cap F$ where f does not vanish identically, $Z \cap F$ is 'transverse' to F.

We say that a function $f : U \longrightarrow \mathbb{C}$ is *$\mathcal{F}$-meromorphic*, if its restriction to any leaf is a meromorphic function. Let $\mathcal{P}$ be the set of poles of f; then, similarly to the case of zeros, the intersection of $\mathcal{P}$ with any leaf is a discrete set of F (see [ElK]).

2.6. Statement of the main result

From now on $\mathcal{F}$ will be a complex foliation by Riemann surfaces on a differentiable manifold M.

Main Theorem. *Suppose that $\mathcal{F}$ is defined by a differentiable trivial fibration $\pi : M \longrightarrow B$. Let $\Sigma : B \longrightarrow$ be a section of π contained in a $\mathcal{F}$-relatively compact subset of M. Then for any relatively compact open set U containing $\Sigma = \sigma(B)$ and any integer $s \geq 0$, there exists a function $U \longrightarrow \mathbb{C}$ of class C^s nonconstant on any leaf of $(U, \mathcal{F})$, meromorphic along the leaves and whose set of poles is exactly Σ.*

This result is a weak parametric version of Mittag-Leffler Theorem. A strong version was already established in [ElK] in case the leaves are noncompact, simply connected or with $\mathbb{Z}$ as common fundamental group.

The remaining part of the paper will be devoted to the proof of the Main Theorem stated above. It will result in a series of lemmas and propositions for which some of our proofs are inspired from methods developed in [For]. The main difficulty here is the control step by step of the transverse regularity and this is far to be a trivial job.

Without lost of generality we may suppose that $n = 1$ and B is an open interval I of the real line $\mathbb{R}$ (containing the origin) or the circle $\mathbb{S}^1$. We choose to treat the case $B = I$. All leaves are noncompact and diffeomorphic. Let M_0 be a leaf of $\mathcal{F}$; there exists a diffeomorphism $\Phi : M \longrightarrow M_0 \times I$ such that, for any $t \in I$, $M_t = \Phi^{-1}(M_0 \times \{t\})$ is a leaf of $\mathcal{F}$ that is, if $M_0 \times I$ is equipped with the foliation $\mathcal{F}_0$ whose leaves are $M_0 \times \{t\}$ with $t \in I$, the two foliated manifolds $(M, \mathcal{F})$ and $(M_0 \times I, \mathcal{F}_0)$ are differentiably isomorphic.

A *$\mathcal{F}$-open set* (resp. an *$\mathcal{F}$-closed set*) of M is an open set of the type $U_0 \times I$ (resp. a closed set of the type $F_0 \times I$) where U_0 is open in M_0 (resp. F_0 is closed in M_0). A *$\mathcal{F}$-open cover* is a cover of M by $\mathcal{F}$-open sets. The unions and finite intersections of $\mathcal{F}$-open sets are also $\mathcal{F}$-open sets. A subset E of M is *$\mathcal{F}$-connected* if, for any $t \in I$, E_t is connected; it is *$\mathcal{F}$-compact* (resp. *$\mathcal{F}$-relatively compact*) if there exists a compact (resp. relatively compact) set K_0 of M_0 such that $E \subset K_0 \times I$. Let $\mathcal{U} = \{U_i\}$ and $\mathcal{V} = \{V_i\}$ two $\mathcal{F}$-open covers indexed by the same set; the notation $\mathcal{V} \ll \mathcal{U}$ means that, for any i, V_i is contained in U_i and is $\mathcal{F}$-relatively compact in this set.

Let Γ denote the common fundamental group of the leaves. The case where Γ is trivial or isomorphic to $\mathbb{Z}$ was studied in [ElK]. So we will suppose that Γ is non Abelian; then the universal covering of each leaf is the upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$. If E is a subset of M, E_t will be its intersection with M_t.

3. Spaces of $\mathcal{F}$-holomorphic functions

Let $U \subset \mathbb{C} \times I$ be an $\mathcal{F}$-open set and $s \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{H}_{\mathcal{F}}^{s}(U)$ be the space of functions $U \longrightarrow \mathbb{C}$ of class C^s and $\mathcal{F}$-holomorphic; the space of basic functions (constant on the leaves) is a subspace of $\mathcal{H}_{\mathcal{F}}^{s}(U)$ and is canonically isomorphic to the space $C^s(I)$ of functions of class C^s on the interval I. For any function $f \in \mathcal{H}_{\mathcal{F}}^{s}(U)$, any measurable subset $E \subset U$, any $t \in I$ and $k \in \{0, 1, \cdots, s\}$, we set:

$$J_k(f, E_t) = \left(\int_{E_t} \left| \frac{\partial^k f}{\partial t^k} \right|^2 dz d\overline{z} \right)^{\frac{1}{2}}$$

and:

$$N_k(f, E) = \sup_{(z,t) \in E} \left| \frac{\partial^k f}{\partial t^k} \right| .$$

We denote by $\mathcal{H}_{\mathcal{F}}^{2,s}(U)$ the space of functions $f \in \mathcal{H}_{\mathcal{F}}^{s}(U)$ such that, for any $k = 0, \cdots, s$:

$$\sup_{t \in I} J_k(f, U_t) < +\infty$$

We equip this space with the norm:

$$||f||_{2,U}^{s} = \max_{k=0,\cdots,s} \left\{ \sup_{t \in I} J_k(f, U_t) \right\}$$

for which it will be complete as we shall show.

Now, we consider the functions $f \in \mathcal{H}_{\mathcal{F}}^{s}(U)$ satisfying the condition:

$$N_k(f, U) < +\infty$$

for any $k \in \{0, 1, \cdots, s\}$. These functions form a vector space $\mathcal{H}_{\mathcal{F}}^{b,s}(U)$ which can be equipped with the norm:

$$||f||_{\infty,U}^{s} = \max_{k=0,\cdots,s} N_k(f, U).$$

By the usual methods one can easily prove that it is complete.

One can observe that, if the measures of the U_t are uniformly bounded, we have $\mathcal{H}_{\mathcal{F}}^{b,s}(U) \subset \mathcal{H}_{\mathcal{F}}^{2,s}(U)$.

The space $\mathcal{B}^s$ of basic functions of class C^s whose derivatives up to the order s are bounded equipped with the norm:

$$||\phi||_{\infty}^{s} = \max_{k=0,\cdots,s} \left\{ \sup_{t \in I} \left| \frac{d^k \phi}{dt^k} \right| \right\}$$

is a Banach algebra. Note that $\mathcal{B}^s$ is a subspace of $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ while it is not one of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ except if the measures of the sections U_t (with $t \in I$) are uniformly bounded. But both of the spaces $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ and $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ are $\mathcal{B}^s$-modules. Let:

$$\mathcal{H}^{b}_{\mathcal{F}}(U) = \bigcap_{s\in\mathbb{N}} \mathcal{H}^{b,s}_{\mathcal{F}}(U), \qquad \mathcal{H}^{2}_{\mathcal{F}}(U) = \bigcap_{s\in\mathbb{N}} \mathcal{H}^{2,s}_{\mathcal{F}}(U) \quad \text{and} \quad \mathcal{B} = \bigcap_{s\in\mathbb{N}} \mathcal{B}^s.$$

These are Fréchet spaces whose topologies are respectively defined by the countable families of norms considered above:

$$\left\{|| \ ||^s_{\infty,U}\right\}_{s\in\mathbb{N}}, \quad \left\{|| \ ||^s_{2,U}\right\}_{s\in\mathbb{N}} \quad \text{and} \quad \left\{|| \ ||^s_{\infty}\right\}_{s\in\mathbb{N}}.$$

Now we suppose $U = B = B_0 \times I$ where B_0 is the open ball centered at a of radius $r > 0$ of the complex plane $\mathbb{C}$. Any function $f \in \mathcal{H}^s_{\mathcal{F}}(U)$ admits an expansion:

$$f(z,t) = \sum_{n=0}^{\infty} f_n(t)(z-a)^n$$

where the coefficients f_n are functions in $t \in I$ given by the integral Cauchy formula:

$$f_n(t) = \frac{1}{2i\pi} \int_{\gamma_t} \frac{f(z,t)}{(z-a)^{n+1}} dz.$$

Here $\{\gamma_t\}$ is a differentiable family of circles centered at a and $\gamma_t \subset B_t$. This shows that $f_n \in \mathcal{B}^s$ if $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ or $f \in \mathcal{H}^{b,s}_{\mathcal{F}}(U)$ and that the sequence (indexed by N):

$$f_N(z,t) = \sum_{n=0}^{N} f_n(t)(z-a)^n$$

converges to f both in the spaces $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ and $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$. The spaces $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ and $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ are free modules over the Banach algebra $\mathcal{B}^s$ with basis $\{\phi_n(z)\}_{n\in\mathbb{N}}$ where $\phi_n(z) = (z-a)^n$.

A simple computation shows that the L^2-norm $||\phi_n||_2$ of ϕ_n (considered as a function in the ball B_0) is:

$$||\phi_n||_2 = \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}$$

from which we deduce that, for any $s \in \mathbb{N}$, we have:

$$||f_n\phi_n||^s_{2,B} = ||f_n||^s_{\infty} \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}.$$

Then:

$$||f||^s_{2,B} \leq \sum_{n=0}^{\infty} ||f_n||^s_{\infty} \frac{\sqrt{\pi} r^{n+1}}{\sqrt{n+1}}.$$

3.2. Theorem. *Let $D \subset \mathbb{C}$ be an open set and $r > 0$. We set $D_r = \{z \in \mathbb{C} : B_0(z,r) \subset D\}$, $U = D \times I$ and $U_r = D_r \times I$. (Here $B_0(z,r)$ is the open ball centered at z with radius r in $\mathbb{C}$.) Then, for any function $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$, we have*:

$$||f||^s_{\infty,U_r} \leq \frac{1}{\sqrt{\pi}r}||f||^s_{2,U}.$$

Proof: Let $(a,t) \in U_r$. Then, on $B = B_0 \times I$, we have $f(z,t) = \sum_{n=0}^{\infty} f_n(t)(z-a)^n$. So $f(a,t) = f_0(t)$ and then, for any $k = 0, \cdots, s$, we have:

$$\left|\frac{d^k f}{dt^k}(a,t)\right|^2 = \left|\frac{d^k f_0}{dt^k}(t)\right|^2 \leq \frac{1}{\sqrt{\pi}r} J_k(f,B_t)^2 \leq \frac{1}{\sqrt{\pi}r} J_k(f,D_t)^2.$$

Taking the upper bound of this quantity over $t \in I$ and the maximum on $k \in \{0,1,\cdots,s\}$, we obtain the following relations:

$$||f||^s_{\infty,U_r} = \sup_{U_r}\left|\frac{d^k f}{dt^k}(a,t)\right| \leq \frac{1}{\sqrt{\pi}r}||f||^s_{2,U}$$

which are exactly the desired inequalities. ♢

3.3. Corollary. *The space $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ equipped with the norm $||\ ||^s_{2,U}$ is complete.*

4. Proof of the Main Theorem

For brevity, we will agree to the following definition: a submodule A of the $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ is of *finite cotype* ((FC)-*submodule* for short) if the quotient $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)/A$ is finitely generated (or of *finite type*).

4.1. Lemma. *Let D_0 and D_0' be two open sets of $\mathbb{C}$ such that D_0' is contained an relatively compact in D_0. We set $U = D_0 \times I$ and $U' = D_0' \times I$. Let $s \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists a closed* (FC)*-submodule A of the $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ such that:*

$$||f||^s_{2,U'} \leq \varepsilon||f||^s_{2,U} \quad \text{for any function } f \in A.$$

Proof. Since $\overline{U}'$ is $\mathcal{F}$-compact in U, there exist $r > 0$ and finitely many points $a_1, \cdots, a_u$ in D such that:

(i) $B_0(a_j,r) \times I \subset U$ for $j = 1, \cdots, u$. ($B_0(a_j,r)$ is the ball of radius r centered at a_j.)

(ii) $U' \subset \bigcup_{j=1}^{u} B_0\left(a_j, \frac{r}{2}\right) \times I$.

Let n be an integer such that $u \leq 2^{n+1}\varepsilon$. Let A be the set of functions $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ whose restriction to any transversal $\{a_j\} \times I$ is zero up to the order n. Then A is a closed (FC)-submodule of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$; the number of generators of the quotient $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)/A$ is less or equal to $n \cdot u$. Let $f \in A$; in a neighborhood of $\{a_j\} \times I$ we have:

$$f(z,t) = \sum_{\ell=n}^{\infty} f_\ell (z-a_j)^\ell .$$

Let $\rho \leq r$; for any $k = 0, \cdots, s$:

$$J_k(f, B_0(a_j,\rho))^2 = \sum_{\ell=n}^{\infty} \frac{\pi\rho^{2\ell+2}}{\ell+1} \left| \frac{d^k f_\ell}{dt^k} \right|^2$$

thus:

$$\begin{aligned} J_k\left(f, B_0\left(a_j, \frac{r}{2}\right)\right)^2 &= \sum_{\ell=n}^{\infty} \frac{\pi r^{2\ell+2}}{r^{2\ell+2}(\ell+1)} \left| \frac{d^k f_\ell}{dt^k} \right|^2 \\ &\leq 2^{-2(n+1)} \sum_{\ell=n}^{\infty} \frac{\pi r^{2\ell+2}}{\ell+1} \left| \frac{d^k f_\ell}{dt^k} \right|^2 \\ &\leq 2^{-2(n+1)} (J_k(f, B_0(a_j, r))^2 . \end{aligned}$$

So, using the properties (i) and (ii):

$$\begin{aligned} J_k(f, U_t) &\leq \sum_{j=1}^{u} J_k(f, B_0(a_j, r)) \\ &\leq u \cdot 2^{-n-1} J_k(f, U_t) \\ &\leq \varepsilon J_k(f, U_t) \end{aligned}$$

Taking the upper bound on $t \in I$ and the maximum on $k = 0, 1, \cdots, s$ of the two sides of this inequality, we obtain:

$$||f||^s_{2,U'} \leq \varepsilon ||f||^s_{2,U},$$

which gives the desired inequality. ♢

We have already observed that $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ is a module over the Banach algebra $\mathcal{B}^s$. Let $f, g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$; for any $t \in I$, we set:

$$\langle f, g \rangle_t = \sum_{k=0}^{s} \int_{U_t} \frac{\partial^k f}{\partial t^k} \cdot \frac{\partial^k \overline{g}}{\partial t^k} d\mu_t .$$

For fixed $t \in I$, $\langle \ , \ \rangle_t$ is a Hermitian product on $\mathcal{H}^{2,s}_{\mathcal{F}}(U_t)$ for which it is a Hilbert space. The following lemma is almost immediate to establish.

4.2. Lemma. *Let $s \in \mathbb{N}$ and f and g be two functions in $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$. The function $\lambda : I \longrightarrow \mathbb{C}$ which associates to each $t \in I$ the complex number $\langle f, g \rangle_t$ belongs to $\mathcal{B}^0$. Moreover there exists a positive constant C such that, for $f, g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$, we have:*

$$||\lambda||_\infty \leq C ||f||^s_{2,U} \cdot ||g||^s_{2,U}$$

that is, the family of Hermitian forms $(f,g) \longmapsto \langle f,g\rangle_t$ *is continuous.*

We say that two functions $f,g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ are *orthogonal* if $\langle f,g\rangle_t = 0$ for any $t \in I$. Of course, any orthonormal system is a free system over the ring $\mathcal{B}^s$. Let $A \subset \mathcal{H}^{2,s}_{\mathcal{F}}(U)$; the *orthogonal* of A is the subset:

$$A^{\perp} = \left\{ f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U) : \langle f,g\rangle_t = 0 \ \text{ for any } g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U) \text{ and any } t \in I \right\}.$$

Since for any fixed $g \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ the map $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U) \longmapsto \langle f,g\rangle_t \in \mathcal{B}^0$ is $\mathcal{B}^s$-linear and continuous, $A^{\perp}$ is a closed submodule of the $\mathcal{B}^s$-module $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$.

4.3. Orthogonal projections in $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$. *Let V be a closed $\mathcal{B}^s$-submodule of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$. Then there exists a continuous $\mathcal{B}^s$-linear map $P : \mathcal{H}^{2,s}_{\mathcal{F}}(U) \longrightarrow V$ such that:*

(i) *For any $g \in V$, $||f - P(f)||^s_{2,U} \leq ||f-g||^s_{2,U}$ that is, $P(f)$ realizes the minimal "distance" from f to V.*

(ii) *For any $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$ and any $v \in V$ we have: $\langle f - P(f), v\rangle_t = 0$ for any $t \in I$.*

(iii) *If V is non trivial the norm $|||P|||^s_{2,U}$ of P is equal to 1.*

The map P is called the *orthogonal projection* from $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ on V. The proof of its existence is a slight adaptation of the classical one on a Hilbert space.

Proof. (i) Let ε be a positive real number and $f \in \mathcal{H}^{2,s}_{\mathcal{F}}(U)$. Let:

$$\delta_s = \inf_{v\in V} ||f - v||^s_{2,U}.$$

Then there exists a sequence (v_n) in V such that $\lim ||f - v_n||^s_{2,U} = 0$ that is:

$$\left(||f - v_n||^s_{2,U}\right)^2 \leq \delta^2 + \varepsilon^2$$

for n sufficiently large and also $J_k(f - v_n, U_t)^2 \leq \delta^2 + \varepsilon^2$ for any $t \in I$ and any $k = 0, 1, \cdots, s$. Let $t \in I$. By the parallelogram identity we have:

$$J_k((f-v_n)-(f-v_p),U_t)^2 + J_k((f-v_n)+(f-v_p),U_t)^2$$
$$= 2\{J_k((f-v_n),U_t)^2 + J_k((f-v_p),U_t)^2\}.$$

Thus

$$J_k(v_n - v_p, U_t)^2 = 2\left\{ J_k(f-v_n,U_t)^2 + J_k(f-v_p,U_t)^2 - 2J_k\left(f - \frac{v_n+v_p}{2}, U_t\right)^2 \right\}.$$

Since:

$$J_k\left(f - \frac{v_n+v_p}{2}, U_t\right)^2 \geq \delta^2$$

we obtain the inequality $J_k(v_n - v_p, U_t)^2 \leq \varepsilon^2$. Taking the upper bound over all $t \in I$, the maximum over $k = 0, \cdots, s$, and the square roots we get $||v_n - v_p||^s_{2,U} \leq \varepsilon$ which shows that (v_n) is a Cauchy sequence in V with respect to the norm $|| \ ||^s_{2,U}$; since V is complete, this sequence converges to an element $v \in V$.

The uniqueness is a consequence of the fact that the construction of the projection is unique for each fixed $t \in I$ in the Hilbert space $\mathcal{H}^{2,s}_{\mathcal{F}}(U_t)$. We set $P(f) = v$. For the same reason we deduce the property (ii) and the inequality $||P(f)||^s_{2,U_t} \leq 1$ which implies assertion (iii) because P is the identity on V. ♢

4.4. Orthogonal decomposition

This is an important consequence of the existence of orthogonal projections: *For any closed submodule V of $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ we have an orthogonal decomposition*: $\mathcal{H}^{2,s}_{\mathcal{F}}(U) = V \oplus V^{\perp}$.

A $\mathcal{F}$-*presheaf* of vector spaces $\mathcal{E}$ on M is a presheaf which associates to any $\mathcal{F}$-open set U a vector space $\mathcal{E}(U)$ with the same conditions of restriction as for a presheaf in the usual sense. It is a $\mathcal{F}$-*sheaf* if, in addition, it possesses the property of gluing local sections on $\mathcal{F}$-open covers to global ones. A $\mathcal{F}$-*sheaf* is *fine* if it is fine in the usual sense for $\mathcal{F}$-open covers. Let us give some:

4.5. Examples

(i) - For any $\mathcal{F}$-open set U, we associate the space $\mathcal{H}^{b,s}_{\mathcal{F}}(U)$ of functions on U of class C^s which are $\mathcal{F}$-holomorphic with locally bounded derivatives $\frac{\partial^k f}{\partial t^k}$ up to the order s. (Locally bounded means bounded on subsets $K_0 \times I$ where K_0 is a compact set of M_0.) Then we obtain a $\mathcal{F}$-sheaf $\mathcal{H}^{b,s}_{\mathcal{F}}$ on M.

(ii) - For any $\mathcal{F}$-open set U, we associate the space $\mathcal{H}^{2,s}_{\mathcal{F}}(U)$ of functions on U which are of class C^s, $\mathcal{F}$-holomorphic and such that the quantities $J_k(f, U_t)$ previously defined are bounded for $k = 0, \cdots, s$. So we obtain a $\mathcal{F}$-presheaf $\mathcal{H}^{2,s}_{\mathcal{F}}$ on M.

(iii) - For any $\mathcal{F}$-open set U, we associate the space $A^0_{b,s}(U)$ (resp. $A^1_{b,s}(U)$) of functions f (resp. foliated $(0,1)$-forms) on U which are of class C^s, C^∞ along the leaves and whose transverse derivatives $\frac{\partial^s f}{\partial t^s}$ up to the order s are locally bounded. Then we obtain a $\mathcal{F}$-sheaf $\mathcal{A}^0_{b,s}$ (resp. $\mathcal{A}^1_{b,s}$).

Let $\mathcal{U} = \{U_i\}$ be a locally finite $\mathcal{F}$-open cover of $M = M_0 \times I$. Let $p : M_0 \times I \longrightarrow M_0$ be the first projection and denote by $\overline{U}_i$ the open set $p(U_i)$. Let $\overline{\rho}_i$ be a C^∞-partition of 1 on M_0 associated to the open cover $\overline{\mathcal{U}} = \{\overline{U}_i\}$ and set $\rho_i = \overline{\rho}_i \circ p$. For any i and any $s \in \mathbb{N}$, the function $\overline{\rho}_i$ is an element of $A^0_{b,s}(M)$ and the family $\{\rho_i\}$ is a C^∞-partition of unity associated to the $\mathcal{F}$-open cover $\mathcal{U} = \{U_i\}$. Using this partition of unity $\{\rho_i\}$ one can easily prove that the two $\mathcal{F}$-sheaves $\mathcal{A}^0_{b,s}$ and $\mathcal{A}^1_{b,s}$ are fine.

4.6. Cohomology with values in a $\mathcal{F}$-sheaf

The definition is the same as in the classical case. Let $\mathcal{E}$ be a $\mathcal{F}$-sheaf on M and $\mathcal{U} = \{U_i\}$ a $\mathcal{F}$-open cover. For any integer $q \in \mathbb{N}$ we denote by $C^q(\mathcal{U}, \mathcal{E})$ the vector space of families $\{c_{i_0 \cdots i_q}\}$ where $c_{i_0 \cdots i_q}$ is an element of $\mathcal{E}(U_{i_0} \cap \cdots \cap U_{i_q})$ (which, by convention, is zero if the intersection $U_{i_0} \cap \cdots \cap U_{i_q}$ is empty). As usual, we define a linear operator

$\delta : C^q(\mathcal{U},\mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U},\mathcal{E})$ by:

$$(\delta c)_{i_0\cdots i_{q+1}} = \sum_{i=0}^{q+1}(-1)^i c_{i_0\cdots\hat{i}\cdots i_{q+1}}.$$

This operator satisfies the relation $\delta^2 = 0$; thus we obtain a differential complex:

$$0 \longrightarrow C^0(\mathcal{U},\mathcal{E}) \xrightarrow{\delta} \cdots\cdots \xrightarrow{\delta} C^q(\mathcal{U},\mathcal{E}) \xrightarrow{\delta} C^{q+1}(\mathcal{U},\mathcal{E}) \xrightarrow{\delta} \cdots$$

whose cohomology is denoted $H^*(\mathcal{U},\mathcal{E})$. If $\mathcal{U}'$ is a $\mathcal{F}$-open cover finer than $\mathcal{U}$ we have a morphism induced by restriction $H^*(\mathcal{U},\mathcal{E}) \longrightarrow H^*(\mathcal{U}',\mathcal{E})$. The cohomology $H^*(M,\mathcal{E})$ of M with values in the $\mathcal{F}$-sheaf will be, by definition, the inductive limit of $H^*(\mathcal{U},\mathcal{E})$ over $\mathcal{F}$-open covers. The cohomology $H^*(M,\mathcal{E})$ satisfies all the usual known properties, for instance we have the:

4.7. Leray's Theorem. *Let $\mathcal{U} = \{U_i\}$ be an acyclic $\mathcal{F}$-open cover that is, for any finite intersection $U_{i_0}\cap\cdots\cap U_{i_q}$, we have $H^q(U_{i_0}\cap\cdots\cap U_{i_q},\mathcal{E}) = 0$ for any $q \geq 1$. Then $H^*(M,\mathcal{E}) = H^*(\mathcal{U},\mathcal{E})$.*

For the proof see [God]. We can easily show that, if $\mathcal{E}$ is $\mathcal{F}$-fine, then $H^q(M,\mathcal{F}) = 0$ for $q \geq 1$. We have also the:

4.8. Abstract de Rham Theorem. *Let $\mathcal{E}$ a $\mathcal{F}$-sheaf on M. Suppose that $\mathcal{E}$ admits a resolution: $0 \longrightarrow \mathcal{E} \hookrightarrow \mathcal{E}^0 \xrightarrow{D_0} \mathcal{E}^1 \xrightarrow{D_1} \cdots$ where each $\mathcal{E}^q$ is a fine $\mathcal{F}$-sheaf. Then the cohomology $H^*(M,\mathcal{E})$ is naturally isomorphic to the cohomology of the differential complex: $0 \longrightarrow \mathcal{E}^0(M) \xrightarrow{D_0} \mathcal{E}^1(M) \xrightarrow{D_1} \cdots$ where, for each q, $\mathcal{E}^q(M)$ is the space of global sections of the $\mathcal{F}$-sheaf $\mathcal{E}^q$.*

These two theorems will be of interest for our purpose. We can first remark that the $\mathcal{F}$-sheaf $\mathcal{H}^{b,s}_{\mathcal{F}}$ admits a fine resolution:

$$0 \longrightarrow \mathcal{H}^{b,s}_{\mathcal{F}} \hookrightarrow \mathcal{A}^0_{b,s} \xrightarrow{\overline{\partial}_{\mathcal{F}}} \mathcal{A}^1_{b,s} \longrightarrow 0.$$

Hence:

$$H^*(M,\mathcal{H}^{b,s}_{\mathcal{F}}) = A^1_{b,s}(M)/\mathrm{Im}\left(A^0_{b,s}(M) \xrightarrow{\overline{\partial}_{\mathcal{F}}} A^1_{b,s}(M)\right)$$

where $A^0_{b,s}(M)$ and $A^1_{b,s}(M)$ are the spaces of global sections respectively of the $\mathcal{F}$-sheaves $\mathcal{A}^0_{b,s}$ and $\mathcal{A}^1_{b,s}$.

Let $\mathcal{U}^* = \{U_i^*\}_{i=1,\cdots,n}$ be a finite family of $\mathcal{F}$-open sets such that each one of them is equivalent to $\mathbb{D}\times I$ (where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$) by a trivialization $\varphi_i : \mathbb{D}\times I \longrightarrow U_i^*$ of the foliation $\mathcal{F}$ restricted to U_i^*. Let $\mathcal{U} = \{U_i\}_{i=1,\cdots,n}$ be an other family of $\mathcal{F}$-open sets such that $U_i \subset U_i^*$ for any $i = 1,\cdots,n$. We shall introduce norms on the spaces $C^*(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$. Let $\eta = \{f_i\} \in C^0(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$ and $\zeta = \{f_{ij}\} =\in C^1(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$; we set:

$$||\eta||^s_{2,\mathcal{U}} = \sum_{i=1}^n ||f_i||^s_{2,U_i} \quad \text{and} \quad ||\zeta||^s_{2,\mathcal{U}} = \sum_{i,j}^n ||f_{ij}||^s_{2,U_{ij}}.$$

The 0-cocyles and the 1-cocycles constitute closed spaces $Z^0(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$ and $Z^1(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$ respectively of $C^0(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$ and $C^1(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$.

For each $i = 1,\cdots,n$ let V_i be a relatively compact $\mathcal{F}$-open set of U_i and denote by $\mathcal{V}$ the $\mathcal{F}$-open cover $\{V_i\}_{i=1,\cdots,n}$. For any $\zeta \in C^q(\mathcal{U},\mathcal{H}^{b,s}_{\mathcal{F}})$, we have $||\zeta||^s_{2,\mathcal{V}} < +\infty$. Applying Lemma 4.1, we easily prove that, for any $\varepsilon > 0$, there exists a closed (FC)-submodule A in the $\mathcal{B}^s$-module $Z^1(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$ such that:

$$||\zeta||^s_{2,\mathcal{V}} \leq \varepsilon||\zeta||^s_{2,\mathcal{U}} \quad \text{for any } \zeta \in A.$$

4.9. Lemma. *We take the same $\mathcal{F}$-open covers $\mathcal{U}^*$, $\mathcal{U}$ and $\mathcal{V}$ as before and we consider a fourth one $\mathcal{W} = \{W_i\}_{i=1,\cdots,n}$. We suppose that $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$. Then, for any $s \in \mathbb{N}$, there exists a constant $C_s > 0$ such that, for any $\xi \in Z^1(\mathcal{V},\mathcal{H}^{2,s}_{\mathcal{F}})$, there exists $\zeta \in Z^1(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$ and $\eta \in C^0(\mathcal{W},\mathcal{H}^{2,s}_{\mathcal{F}})$ with $\zeta = \xi + \delta\eta$ on $\mathcal{W}$ and:*

$$\max\left(||\zeta||^s_{2,\mathcal{V}}, ||\eta||^s_{2,\mathcal{W}}\right) \leq C_s||\xi||^s_{2,\mathcal{U}}.$$

Proof. Let $\xi = \{f_{ij}\} \in Z^1(\mathcal{V},\mathcal{H}^{2,s}_{\mathcal{F}})$. Then, using a same method proving Theorem 3.2, we easily establish that $\xi \in Z^1(\mathcal{V},\mathcal{A}^0_{b,s})$. We have observed in subsection 4.5. iii) that the $\mathcal{F}$-sheaf $\mathcal{A}^0_{b,s}$ of functions of class C^s whose transverse derivatives up to the order s are locally bounded is fine; hence $H^1(\mathcal{V},\mathcal{A}^0_{b,s}) = 0$. Then there exists $\{g_i\} \in C^0(\mathcal{V},\mathcal{A}^0_{b,s})$ such that:

$$f_{ij} = g_j - g_i \quad \text{on} \quad V_i \cap V_j.$$

Since $\overline{\partial}_{\mathcal{F}} f_{ij} = 0$, we have $\overline{\partial}_{\mathcal{F}} g_i = \overline{\partial}_{\mathcal{F}} g_j$ on $V_i \cap V_j$; hence the collection of $(0,1)$-forms $\{\overline{\partial}_{\mathcal{F}} g_i\}$ defines a foliated $(0,1)$-form ω on the union $|\mathcal{V}| = V_1 \cup \cdots \cup V_n$ such that $\omega_{|V_i} = \overline{\partial}_{\mathcal{F}} g_i$. Because $|\mathcal{W}|$ is $\mathcal{F}$-relatively compact in $|\mathcal{V}|$, there exists a function $\psi \in A^0_{b,s}(M)$, basic for the fibration $(z,t) \in M = M_0 \times I \longmapsto z \in M_0$ and such that:

$$\operatorname{supp}(\psi) \subset |\mathcal{V}| \quad \text{and} \quad \psi_{||\mathcal{W}|} = 1.$$

Then $\psi\omega$ can be considered as an element of $A^1_{b,s}(|\mathcal{U}^*|)$. Since, by Ahlfors-Bers Theorem [AB], for each $i \in \{1,\cdots,n\}$, the $\mathcal{F}$-open set U_i^* is isomorphic to the product $\mathbb{D} \times I$, by Proposition 2.3 there exists a function $h_i \in A^0_{b,s}(U_i^*)$ such that $\overline{\partial}_{\mathcal{F}} h_i = \psi\omega_{|U_i^*}$. But:

$$\overline{\partial}_{\mathcal{F}} h_i = \overline{\partial}_{\mathcal{F}} h_j \quad \text{on} \quad U_i^* \cap U_j^*;$$

thus $F_{ij} = h_j - h_i \in \mathcal{H}^{b,s}_{\mathcal{F}}(U_i^* \cap U_j^*)$. Denote by ζ the cocycle $\{F_{ij}\}$; since $\mathcal{U} \ll \mathcal{U}^*$, $\zeta \in Z^1(\mathcal{U},\mathcal{H}^{2,s}_{\mathcal{F}})$. On W_i we have $\overline{\partial}_{\mathcal{F}} h_i = \psi\omega = \omega = \overline{\partial}_{\mathcal{F}} g_i$, $h_i - g_i \in \mathcal{H}^{b,s}_{\mathcal{F}}(W_i)$ and also $h_i - g_i \in \mathcal{H}^{2,s}_{\mathcal{F}}(W_i)$ that is, the 0-cochain $\eta = \{h_i - g_i\}$ is in $C^0(\mathcal{W},\mathcal{H}^{2,s}_{\mathcal{F}})$. So we easily see that, on $W_i \cap W_j$:

$$F_{ij} - f_{ij} = (h_j - g_j) - (h_i - g_i) \quad \textit{i.e.} \quad \zeta - \xi = \delta\eta \text{ on } \mathcal{W}$$

which is exactly the desired relation.

Now, let:

$$E = Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}}) \times Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}}) \times C^0(\mathcal{W}, \mathcal{H}^{2,s}_{\mathcal{F}}).$$

Equipped with norm $||(\zeta, \xi, \eta)||^s_E = ||\zeta||^s_{2,\mathcal{U}} + ||\xi||^s_{2,\mathcal{V}} + ||\eta||^s_{2,\mathcal{W}}$, E is a Banach space. The subspace:

$$L = \{(\zeta, \xi, \eta) \in E : \zeta = \xi + \delta\eta \ \text{ on } \ \mathcal{W}\}$$

is closed and then is a Banach space. Since the map:

$$\pi : (\zeta, \xi, \eta) \in L \longmapsto \xi \in Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}})$$

is continuous and surjective, it is also open (by the Open Map Theorem). Hence there exists a constant $C_s > 0$ such that, for any $\xi \in Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}})$, there exists $(\zeta, \xi, \eta) \in L$ satisfying $\pi((\zeta, \xi, \eta)) = \xi$ and:

$$||(\zeta, \xi, \eta)||^s_E \leq C_s ||\xi||^s_{2,\mathcal{V}}.$$

The constant C_s satisfies the desired inequality. ♢

4.10. Lemma. *Let the hypotheses be like in Lemma 4.9. There exists a finitely generated $\mathcal{B}^s$-submodule $S \subset Z^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}})$ satisfying the following property: for any $\xi \in Z^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}})$, there exists $\sigma \in S$ and $\eta \in C^0(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}})$ such that:*

$$\sigma = \xi + \delta\eta \qquad \text{on } \mathcal{W}.$$

This means that the image of the natural $\mathcal{B}^s$-linear map $H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}})$ is a finitely generated $\mathcal{B}^s$-submodule.

Proof. Let C_s be the constant given in Lemma 4.9 and set $\varepsilon = \frac{1}{2C_s}$. By the Lemma 4.1 there exists a closed (FC)-submodule $A \subset Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}})$ such that:

$$||\xi||^s_{2,\mathcal{V}} \leq \varepsilon ||\xi||^s_{2,\mathcal{U}} \qquad \text{for any } \xi \in A.$$

Let S be the orthogonal of A in $Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}})$ (which is a finitely generated $\mathcal{B}^s$-submodule because A is a (FC)-submodule) that is:

$$Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}}) = A \oplus S.$$

Let $\xi \in Z^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}})$. Since $\mathcal{V} \ll \mathcal{U}$, ξ is in fact in $Z^1(\mathcal{V}, \mathcal{H}^{2,s}_{\mathcal{F}})$; let $\tau = ||\xi||^s_{2,\mathcal{V}} < +\infty$. By Lemma 4.9, there exists $\zeta_0 \in Z^1(\mathcal{U}, \mathcal{H}^{2,s}_{\mathcal{F}})$ and $\eta_0 \in C^0(\mathcal{W}, \mathcal{H}^{2,s}_{\mathcal{F}})$ such that:

$$\zeta_0 = \xi + \delta\eta_0 \qquad \text{on} \qquad \mathcal{W}$$

with the inequalities $||\zeta_0||^s_{2,\mathcal{V}} \leq C_s\tau$ and $||\eta_0||^s_{2,\mathcal{V}} \leq C_s\tau$. On the other hand, ζ_0 decomposes into a sum:

$$\zeta_0 = \xi_0 + \sigma_0 \qquad \text{with} \qquad \xi_0 \in A \qquad \text{and} \qquad \sigma_0 \in S.$$

Now we shall construct elements:

$$\zeta_k \in Z^1(\mathcal{U},\mathcal{H}_{\mathcal{F}}^{2,s}), \quad \eta_k \in C^0(\mathcal{W},\mathcal{H}_{\mathcal{F}}^{2,s}), \quad \xi_k \in A \quad \text{and} \quad \sigma_k \in S$$

with:

(i) $\zeta_k = \xi_{k-1} + \delta\eta_k$ on $\mathcal{W}$;
(ii) $\zeta_k = \xi_k + \sigma_k$ (orthogonal decomposition);
(iii) $||\zeta_k||_{2,\mathcal{U}}^s \leq \frac{C_s\tau}{2^k}$ and $||\eta_k||_{2,\mathcal{U}}^s \leq \frac{C_s\tau}{2^k}$.

Suppose that these elements are constructed up to the rank k. Since $\zeta_k = \xi_k + \sigma_k$ we have by the orthogonal decomposition:

$$||\xi_k||_{2,\mathcal{U}}^s \leq ||\zeta_k||_{2,\mathcal{U}}^s \leq \frac{C_s\tau}{2^k}.$$

This gives:

$$||\xi_k||_{2,\mathcal{V}}^s \leq \varepsilon||\xi_k||_{2,\mathcal{U}}^s \leq \frac{\varepsilon C_s\tau}{2^k} \leq \frac{\tau}{2^{k+1}}.$$

By Lemma 4.9 there exists $\zeta_{k+1} \in Z^1(\mathcal{U},\mathcal{H}_{\mathcal{F}}^{2,s})$ and $\eta_{k+1} \in C^0(\mathcal{W},\mathcal{H}_{\mathcal{F}}^{2,s})$ such that:

$$\zeta_{k+1} = \xi_k + \delta\eta_{k+1} \quad \text{on } \mathcal{W}.$$

and

$$\max\left(||\zeta_{k+1}||_{2,\mathcal{U}}^s, ||\eta_{k+1}||_{2,\mathcal{W}}^s\right) \leq \frac{C_s\tau}{2^{k+1}}.$$

The element ζ_{k+1} admits an orthogonal decomposition:

$$\zeta_{k+1} = \xi_{k+1} + \sigma_{k+1} \text{ with } \xi_{k+1} \in A \text{ and } \sigma_{k+1} \in S.$$

Then we have constructed, up to the rank $k+1$, the sequences (ξ_k), (ζ_k), (η_k) and (σ_k) with the desired properties. By $\zeta_0 = \xi + \delta\eta_0$ and the points (i) and (ii) we have (up to rank k) :

$$(*) \qquad \xi_k + \sum_{\ell=0}^{k} \sigma_\ell = \xi + \delta\left(\sum_{\ell=0}^{k} \eta_\ell\right) \text{ on } \mathcal{W}.$$

From (ii) and (iii) we deduce that:

$$\max\left(||\xi_\ell||_{2,\mathcal{U}}^s, ||\sigma_\ell||_{2,\mathcal{U}}^s, ||\eta_\ell||_{2,\mathcal{W}}^s\right) \leq \frac{C_s\tau}{2^k}.$$

Thus $\lim_k \xi_k = 0$ and the series $\sum_{k=0}^{\infty} \sigma_k$ and $\sum_{k=0}^{\infty} \eta_k$ converge respectively to elements $\sigma \in S$ and $\eta \in C^0(\mathcal{W},\mathcal{H}_{\mathcal{F}}^{2,s})$. In fact, by Theorem 3.2:

$$\sigma \in S \cap Z^1(\mathcal{U},\mathcal{H}_{\mathcal{F}}^{b,s}) \quad \text{and} \quad \eta \in C^0(\mathcal{W},\mathcal{H}_{\mathcal{F}}^{b,s}).$$

From $(*)$ we obtain $\sigma = \xi + \delta\eta$ on $\mathcal{W}$. This ends the proof of the lemma. ♢

4.11. Theorem. *Suppose that $M' = M'_0 \times I$ and $M'' = M''_0 \times I$ are $\mathcal{F}$-open sets of M an that M'_0 is contained and relatively compact in M''_0. Then the image of the natural morphism $H^1(M'', \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M', \mathcal{H}^{b,s}_{\mathcal{F}})$ induced by the restriction is a finitely generated $\mathcal{B}^s$-module.*

Proof. Let $\mathcal{U}^*$, $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ four $\mathcal{F}$-open covers as in Lemma 4.10 and such that:

(i) $M' \subset \bigcup_{i=1}^{n} W_i =: M_1$ and is relatively compact in $M_2 := \bigcup_{i=1}^{n} U_i \subset M''$;

(ii) the U_i^*, U_i and W_i are isomorphic to $\mathbb{D} \times I$.

By Lemma 4.10, the image of the morphism $H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}})$ is a finitely generated $\mathcal{B}^s$-module. On the other hand, the $\mathcal{F}$-open covers $\mathcal{U}$ and $\mathcal{W}$ are acyclic; then by Leray's theorem, $H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) = H^1(M_2, \mathcal{H}^{b,s}_{\mathcal{F}})$ and $H^1(\mathcal{W}, \mathcal{H}^{b,s}_{\mathcal{F}}) = H^1(M_1, \mathcal{H}^{b,s}_{\mathcal{F}})$. Then the result follows from the canonical factorization:

$$H^1(M'', \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M_2, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M_1, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(M', \mathcal{H}^{b,s}_{\mathcal{F}}).$$

The theorem is then proved. ♢

4.12. Corollary. *Suppose that all leaves are compact (they are all diffeomorphic to a compact Riemann surface of genus $g \geq 2$). Then $H^1(M, \mathcal{H}^{b,s}_{\mathcal{F}})$ is a finitely generated $\mathcal{B}^s$-module.*

4.13. Question. *Suppose that all leaves are compact. Is the $\mathcal{B}^s$-module $H^1(M, \mathcal{H}^{b,s}_{\mathcal{F}})$ free? If this is the case, is its dimension equal g?*

Now we have amassed all that is necessary to prove the Main Theorem. It is immediate to see that it follows from the following one.

4.14. Theorem. *Let $M' = M'_0 \times I$ be a $\mathcal{F}$-open set of M with M'_0 relatively compact and strictly contained in M_0. Then for any $a \in M'_0$, there exists a $\mathcal{F}$-meromorphic function $f : M' \longrightarrow \mathbb{C}$ nonconstant on any leaf, with set of poles $\{a\} \times I$ and $\mathcal{F}$-holomorphic on $M' \setminus \{a\} \times I$.*

Let us first recall a result on modules of finite type illustrated in the following lemma. Its proof can be found for instance in [AM] (Proposition 2.4 p. 21).

4.15. Lemma. *Let E be a finitely generated module over a ring R, $\mathfrak{a}$ an ideal of R and θ an R-endomorphism of E such that $\theta(E) \subset \mathfrak{a}E$. Then there exists $a_1, \cdots, a_n \in \mathfrak{a}$ such that $\theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n = 0$.*

Proof of Theorem 4.14. Let U_1 be a $\mathcal{F}$-open neighborhood of $\{a\} \times I$ isomorphic to $\mathbb{D} \times I$ by a diffeomorphism (which is a biholomorphism on the leaves) $\varphi_1 : U_1 \longrightarrow \mathbb{D} \times I$ sending $\{a\} \times I$ on $\{0\} \times I$; by restricting the open set U_1 if necessary we can assume that the transverse derivatives (up to the order s) of φ_1 are bounded. Denote by U_2 the open set $M \setminus \{a\} \times I$. Then $\mathcal{U} = \{U_1, U_2\}$ is a $\mathcal{F}$-open cover of M. For each $j \in \mathbb{N}^*$, the function $\frac{1}{\varphi(\cdot,0)^j}$ is in $\mathcal{H}^{b,s}_{\mathcal{F}}(U_1 \cap U_2)$ and represents a cocycle $\zeta_j \in Z^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}})$. By Lemma 4.10 the image of the $\mathcal{B}^s$-morphism:

$$H^1(\mathcal{U}, \mathcal{H}^{b,s}_{\mathcal{F}}) \longrightarrow H^1(\mathcal{U} \cap M', \mathcal{H}^{b,s}_{\mathcal{F}})$$

is a finitely generated $\mathcal{B}^s$-module E. Applying Lemma 4.15 to the $\mathcal{B}^s$-module E, the ring $R = \mathcal{B}^s$, its ideal $\mathfrak{a} = \mathcal{B}^s$ and the morphism $\theta : g \in E \longmapsto \zeta_1 g \in E$ one can find elements $a_1, \cdots, a_n \in \mathcal{B}^s$ such that: $\theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n = 0$. The value of the morphism $\Theta = \theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n$ at the constant function $\chi = 1$ gives a cocyle:

$$\Theta(\chi) = \theta^n(\chi) + a_1\theta^{n-1}(\chi) + \cdots + a_{n-1}\theta(\chi) + a_n(\chi)$$

which is cohomologous to zero. This means that there exist elements $c_1, \cdots, c_{n+1} \in \mathcal{B}^s$ and a 0-cochain $\eta = \{f_1, f_2\} \in C^0(\mathcal{U} \cap M', \mathcal{H}_{\mathcal{F}}^{b,s})$ such that:

$$c_1\zeta_1 + \cdots + c_{n+1}\zeta_{n+1} = \delta\eta \qquad \text{on } \mathcal{U} \cap M'$$

that is:

$$c_1\zeta_1 + \cdots + c_{n+1}\zeta_{n+1} = f_2 - f_1 \qquad \text{on } U_1 \cap U_2 \cap M'.$$

The desired $\mathcal{F}$-meromorphic function on M' is defined by: $c_1\zeta_1 + \cdots + c_{n+1}\zeta_{n+1} + f_1$ on $U_1 \cap M'$ and f_2 on $U_2 \cap M'$. ♢

4.16. Corollary. *We take the same hypotheses as previously and suppose that M' is not the whole manifold M. Then there exists a $\mathcal{F}$-holomorphic function $f : M' \longrightarrow \mathbb{C}$ which is not constant on any leaf of any connected component of M'.*

Proof. Let $M'' = M_0'' \times I$ be a $\mathcal{F}$-open set of M where M_0'' is a relatively compact open set of M_0 containing M_0' in which the latter is relatively compact. We apply then the preceding theorem by taking $a \in M_0'' \setminus M_0'$. ♢

5. Examples

In this section we give examples of differentiably trivial fibrations but far to be even locally trivial in the complex sense.

5.1. Leaves are simply connected

Let $\pi : M \longrightarrow B$ be a differentiably trivial fibration whose fibers are holomorphically equivalent to the unit disc $\mathbb{D}$ (or the half plane $\mathbb{H}$). Then, by Ahlfors-Bers Theorem [AB] it is isomorphic to the product $\mathbb{D} \times B$ as a complex foliation. As this case is not interesting for us here we shall give an example with parabolic leaves that is, each leaf is individually isomorphic to $\mathbb{C}$ but the complex foliation we obtain is not equivalent to a complex product.

Denote by $P^1(\mathbb{C})$ the complex projective space of dimension one. Let $I =]0, 1[$ and $\phi : I \longrightarrow P^1(\mathbb{C})$ be a C^k-map which is not C^{k+1}. Let $M = P^1(\mathbb{C}) \times I \setminus \mathcal{G}$ where $\mathcal{G}$ is the graph of ϕ. This is a differentiable trivial fibration over I all of whose fibers are isomorphic to $\mathbb{C}$ and the complex structure on the fibers varies in a C^∞ way in the transverse direction. We obtain a complex foliation $\mathcal{F}$ whose leaves are the fibers of the trivial fibration $\pi : M \longrightarrow I$ where π is the restriction to M of the second projection $(z, t) \in P^1(\mathbb{C}) \times I \longmapsto t \in I$.

The complex foliation $\mathcal{F}$ on M constructed above is not C^{k+1}-equivalent to the complex product $\mathbb{C} \times I$.

Indeed, suppose that there exists a C^{k+1}-diffeomorphism $\Psi : \mathbb{C} \times I \longrightarrow M$ which is holomorphic between the fibers. In the coordinates on M given by its inclusion in $P^1(\mathbb{C}) \times I$, this map Ψ is necessarily of the form:

$$\Psi(z,t) = \left(t, \frac{a(t)z + b(t)}{c(t)z + d(t)}\right)$$

(where, for each fixed $t \in I$, $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ is a matrix in $\mathrm{SL}(2,\mathbb{C})$) because any holomorphic embedding of $\mathbb{C}$ in $P^1(\mathbb{C})$ is given by a Moëbius map. From the fact that Ψ is C^{k+1} we see that the functions a, b, c and d are C^{k+1} on t. Then ϕ is also C^{k+1} because:

$$\phi(t) = \Psi(\infty, t) = \frac{a(t)}{b(t)}.$$

5.2. Leaves are annulus

The following example is more or less the one given in subsection 1.2 for the open set N (Example v)). Let $I =]0, +\infty[$ and $\widetilde{M} = \mathbb{H} \times I$. We have an action Φ of $\mathbb{Z}$ on $\widetilde{M}$ given by :

$$\Phi(k, (z,t)) = (t^k z, t).$$

This action is free and proper; moreover it is holomorphic on each leaf $\mathbb{H} \times \{t\}$ of the product complex foliation $\widetilde{\mathcal{F}}$. Then it defines a complex foliation $\mathcal{F}$ on the quotient $M = \widetilde{M}/\Phi$ which is differentiably isomorphic to the product $(\mathbb{H}/\Phi_t) \times I$ where Φ_t is the loxodromy $\Phi_t(z) = tz$. The complex foliation $\mathcal{F}$ is not a locally trivial fibration. Indeed, because each leaf $(\mathbb{H}/\Phi_t) \times \{t\}$ is an annulus whose complex structure is coded by the ratio t, two different leaves $\mathbb{H}/\Phi_t \times \{t\}$ and $\mathbb{H}/\Phi_{t'} \times \{t'\}$ with $t \neq t'$) cannot be holomorphically equivalent.

5.3. Remark

In Example 5.1 the leaves are simply connected (all parabolic) and in Example 5.2 they have $\mathbb{Z}$ as common fundamental group. In these two cases it was proved in [ElK] that the first foliated Dolbeault cohomology group $H^{01}_{\mathcal{F}}(M)$ is trivial. This permits to give a more stronger foliated version of Mittag-Leffler Theorem.

5.4. Fundamental group of leaves is non Abelian

Let $\widetilde{M} = \mathbb{H} \times I$. Fon any $t \in I$ let ϕ_t be the Moëbius transformation of $\mathbb{H}$ defined by:

$$\phi_t(z) = \frac{z+1}{tz + (1+t)}.$$

The family of matrices Θ_t (indexed by t) in the group $\mathrm{SL}(2,\mathbb{R})$ corresponding to the family ϕ_t is:

$$\Theta_t = \begin{pmatrix} 1 & 1 \\ t & (1+t) \end{pmatrix}.$$

Easy calculations show that, on the interval $I =]-\frac{1}{2}, 0[$:

- The matrices Θ_t and $\Theta_{t'}$ have different eigenvalues for $t \neq t'$; then Θ_t and $\Theta_{t'}$ are not conjugated in $\mathrm{SL}(2, \mathbb{R})$.
- Each ϕ_t has a unique fixed point $z_0(t)$ in $\mathbb{H}$.
- The family $\{z_0(t)\}_{t \in I}$ is the graph $\mathcal{G}$ in $\mathbb{H} \times I$ of a C^∞-function $\alpha : I \longrightarrow \mathbb{H}$.

Let a be a point in $\mathbb{H}$ different from $z_0(t)$ for any $t \in I$. For each $t \in I$, let $\mathcal{O}_t(a)$ be the orbit of a under the action of ϕ_t. Let:

$$\widetilde{M} = \mathbb{H} \times I \setminus \left\{ \mathcal{G} \cup \left(\bigcup_{t \in I} \mathcal{O}_t(a) \right) \right\}.$$

For each $t \in I$, $\widetilde{M}$ is a ϕ_t-invariant open set of $\mathbb{H} \times I$ and then it supports the action Ψ of $\mathbb{Z}$ defined by:

$$\Psi(k, (z, t)) = (\phi_t^k(z), t).$$

This action is free and proper; so the quotient $M = \widetilde{M}/\Psi$ is a manifold diffeomorphic to a product $M_0 \times I$ (where M_0 is a noncompact Riemann surface) with a complex foliation $\mathcal{F}$. Each leaf L_t of $\mathcal{F}$ is the quotient of:

$$\mathbb{H} \setminus (\mathcal{O}_t(a) \cup \{z_0(t)\})$$

by the automorphism ϕ_t. Because the matrices Θ_t and $\Theta_{t'}$ are not conjugated for $t \neq t'$, the two leaves L_t and $L_{t'}$ are not holomorphically equivalent. Then the foliation is not a locally trivial fibration in the complex sense.

In this example all leaves are diffeomorphic. Then they have the same fundamental group: the free non Abelain group generated by a countable infinite set.

5.5. All leaves are compact

Let $\varphi : \omega \longmapsto \omega' = \frac{a\omega + b}{c\omega + d}$ be a non trivial biholomorphism of $\mathbb{H}$ (where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\mathrm{SL}(2, \mathbb{R})$). The map:

$$\Phi : (p, (z, \omega)) \in \mathbb{Z} \times \mathbb{C}^* \times \mathbb{H} \longmapsto (e^{-ip\varphi(\omega)} z, \omega) \in \mathbb{C}^* \times \mathbb{H}$$

is a free and proper holomorphic action of $\mathbb{Z}$ on $\widehat{M} = \mathbb{C}^* \times \mathbb{H}$. It preserves the foliation $\widehat{\mathcal{F}}$ whose leaves are the factors $\mathbb{C}^* \times \{\omega\}$ (in fact the action Φ preserves each leaf individually). The quotient space $M = \mathbb{C}^* \times \mathbb{H}/\Phi$ is a complex manifold of dimension 2. The induced complex foliation $\mathcal{F}$ on M has dimension 1 and all its leaves are elliptic curves $\mathbb{T}_\omega$; the complex structure of each $\mathbb{T}_\omega$ depends on $\omega \in \mathbb{H}$. Two leaves $\mathbb{T}_\omega$ and $\mathbb{T}_{\omega'}$ are isomorphic if, and only if, there exists a matrix $B \in \mathrm{SL}(2, \mathbb{Z})$ such that $\varphi(\omega') = B\varphi(\omega)$. The complex equivalence class of a leaf is then a countable set. Hence this foliation is not a locally trivial complex fibration even if it is a differentiable product.

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