GUICHARD'S THEOREM FOR A COMPLEX SIMPLE FOLIATION

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The general problem

Let E be a Fréchet space. For instance, E may be :

$$E = C^\infty(\mathcal{E})$$

the space of C^∞-sections of a vector bundle : $\mathcal{E} \longrightarrow M$ (over a compact manifold M) equipped with the C^∞-topology.

Let $\gamma : E \longrightarrow E$ an automorphism. Denote by E^γ the set of elements of E invariant by γ.

E^γ is the kernel of the (bounded) *coboundary operator* :

$$\delta : f \in E \longmapsto (f - \gamma \cdot f) \in E$$

thus it is a Fréchet subspace of E.

The vector $g = \delta f$ is a 'measure' of the *invariance defect* of f.

Suppose that we are given a bounded operator $T : E \longrightarrow E$ commuting with γ and we are interested to solve, in the subspace E^γ, the equation $Tf = g$ for given $g \in E^\gamma$.

A natural way to do it is to solve firstly the equation in E (forgetting that g is γ-invariant) and correct a solution $f_0 \in E$ by adding an element h of $N =$ kernel of T to make the new solution $f = f_0 + h$ invariant by γ that is, satisfying the relation $\gamma \cdot (f_0 + h) = f_0 + h$ *i.e.* $h - \gamma \cdot h = \gamma \cdot f_0 - f_0$. (The element $(\gamma \cdot f_0 - f_0)$ is in N.)

This brings us naturally to solve the following problem :

Given $g \in N$, does there exist $h \in N$ such that $h - \gamma \cdot h = g$?

This is the *cohomological equation* of the 'dynamical system' (N, γ).

The terminology comes from the fact that the first cohomology space ;$H^1(\mathbb{Z}, N)$ of the discrete group $\mathbb{Z}$ with coefficients in the $\mathbb{Z}$-module N is exactly the cokernel of the operator $\delta : N \longrightarrow N$.

Problem

Let N be a Fréchet space and γ an automorphism of N. Compute the vector space :

$$H^1(\mathbb{Z}, N)$$

Guichard's Theorem

Let $\tau : z \in \mathbb{C} \longmapsto z + 1 \in \mathbb{C}$ and γ the automorphism of the Fréchet space $N = \mathcal{H}(\mathbb{C})$ of holomorphic functions on $\mathbb{C}$ defind by $\gamma \cdot f = f \circ \gamma$. Then

$$H^1(\mathbb{Z}, \mathcal{H}(\mathbb{C})) = 0.$$

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Complex foliations

Let M be a differentiable manifold of dimension $2m+n$ endowed with a codimension n foliation $\mathcal{F}$ (then the dimension of $\mathcal{F}$ is $2m$).

Definition

The foliation $\mathcal{F}$ is said to be complex if it can be defined by an open cover $\{U_i\}$ of M and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \longrightarrow U_i$ (Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$) such that, for $U_i \cap U_j \neq \emptyset$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \longrightarrow \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', t') = \left(\phi_{ij}^1(z,t), \phi_{ij}^2(t)\right)$ with $\phi_{ij}^1(z,t)$ holomorphic in z for t fixed.

Any leaf of $\mathcal{F}$ is a complex manifold of dimension m.

The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold.

Example 1

Any complex manifold M of dimension m is a complex foliation of dimension m. Its automorphism group is exactly the automorphism group of the complex manifold M.

Example 2

Any holomorphic foliation (on a complex manifold M) is a complex foliation.

Example 3

Let B be a differentiable manifold and M an open set of $\mathbb{C}^m \times B$. For $t \in B$, $M^t = \{z \in \mathbb{C}^m : (z, t) \in M\}$ is an open set of $\mathbb{C}^m$ called the section of M along t. Sections of M are leaves of a complex foliation $\mathcal{F}$ of dimension m called the complex canonical foliation of M.

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Example 4

Let F be a complex manifold and B a differentiable one. Any locally trivial fibration $F \hookrightarrow M \longrightarrow B$ whose cocycle takes values in the automorphism group $\mathrm{Aut}(F)$ of (the complex manifold) F is a complex foliation, the fibres being the leaves. If the fibration is trivial i.e. $M = F \times B$, we say that $\mathcal{F}$ is a complex product foliation. In that case all the leaves are holomorphically equivalent. Suppose that $\mathcal{F}$ is a complex foliation on $M = F \times B$ whose leaves are the factors $F \times \{t\}$ but the complex structure may depend on t ; then we say that $\mathcal{F}$ is a differentiable product.

Example 5

Let M be a Riemannian manifold equipped with an orientable foliation $\mathcal{F}$ by surfaces. To any vector $u \in T_y\mathcal{F}$ ($T_y\mathcal{F}$ is the tangent space to $\mathcal{F}$ at y) we associate the unique vector $v \in T_y\mathcal{F}$ with the same length and such that (u, v) is an orthonormal direct frame of the tangent space $T_y\mathcal{F}$ to the foliation $\mathcal{F}$. We set $J_{\mathcal{F}}u = v$; then $J_{\mathcal{F}}$ is an almost complex structure on the leaves. Since the dimension of the leaves is 2, $J_{\mathcal{F}}$ is integrable and then defines a complex structure which makes $\mathcal{F}$ a 1-dimensional complex foliation.

Example 6

Let $(M, \mathcal{F})$ be a complex foliation. Suppose that $\mathcal{F}$ is Riemannian that is, the normal bundle $\nu\mathcal{F} = TM/T\mathcal{F}$ supports a Riemannian metric invariant along the leaves and that the leaves are closed. Then the leaf space $B = M/\mathcal{F}$ is an orbifold. In that case we say that $(M, \mathcal{F})$ is complex simple foliation. One can always equip the manifold M with a Riemannian metric which is transversely complete (geodesics which are orthogonal to $\mathcal{F}$ are complete). Let $\iota : M^{\#} \longrightarrow M$ be the G-principal bundle (where G is the orthogonal group $O(n)$) of orthonormal frames transverse to $\mathcal{F}$. Using Molino's Theorem, one easily obtain the following :

Proposition

Let $(M, \mathcal{F})$ be a simple complex foliation. Then $\mathcal{F}$ lifts to a complex foliation $\mathcal{F}^{\#}$ on $M^{\#}$ such that :

1. $dim_{\mathbb{C}}\mathcal{F}^{\#} = dim_{\mathbb{C}}\mathcal{F}$;
2. *G acts on $M^{\#}$ by automorphisms of $\mathcal{F}^{\#}$;*
3. *$\mathcal{F}^{\#}$ is a locally trivial fibration $M^{\#} \longrightarrow B$ over a differentiable manifold B ;*
4. *every leaf $F^{\#}$ of $\mathcal{F}^{\#}$ projects (by ι) on a leaf F of $\mathcal{F}$ and $\iota : F^{\#} \longrightarrow F$ is a holomorphic covering.*

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$\overline{\partial}_{\mathcal{F}}$-cohomology

Let $(M,\mathcal{F})$ be a complex foliation of dimension m. Let $\Omega^{pq}(\mathcal{F})$ be the space of foliated differential forms of type (p,q) that is, differential forms on M which can be written in local coordinates adapted to the foliation $(z,t)=(z_1,\ldots,z_m,t_1,\ldots,t_n)$:

$$\alpha=\sum \alpha_{JK}(z,t)dZ_J\wedge d\overline{Z}_K$$

where $dZ_J=dz_{j_1}\wedge\ldots\wedge dz_{j_p}$, $d\overline{Z}_K=d\overline{z}_{k_1}\wedge\ldots\wedge d\overline{z}_{k_q}$ and α_{JK} is a C^∞-function on (z,t).

Let $\overline{\partial}_{\mathcal{F}}:\Omega^{pq}(\mathcal{F})\longrightarrow\Omega^{p,q+1}(\mathcal{F})$ be the Cauchy-Riemann operator along the leaves defined by :

$$\overline{\partial}_{\mathcal{F}}\left(\sum \alpha_{JK}dZ_J\wedge d\overline{Z}_K\right)=\sum_{s=1}^{m}\frac{\partial\alpha_{JK}}{\partial\overline{z}_s}(z,t)d\overline{z}_s\wedge dZ_J\wedge d\overline{Z}_K$$

where $\frac{\partial}{\partial\overline{z}_s}=\frac{1}{2}\left\{\frac{\partial}{\partial x_s}+i\frac{\partial}{\partial y_s}\right\}$ with $z_s=x_s+iy_s$.

It satifies $\overline{\partial}_{\mathcal{F}}^2 = 0$, hence we have a differential complex :

$$\cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p,r}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p,r+1}(\mathcal{F}) \longrightarrow \cdots$$

called the $\overline{\partial}_{\mathcal{F}}$-complex of $(M, \mathcal{F})$; its homology $H^{pq}_{\mathcal{F}}(M)$ is called the foliated Dolbeault cohomology (or the $\overline{\partial}_{\mathcal{F}}$*-cohomology*) of the complex foliation $(M, \mathcal{F})$. It is locally trivial that is we have a Foliated Dolbeault-Grothendieck Lemma :

Lemma

Let $x \in M$. Then there exists an open neighborhood U of x adapted to the foliation such that, for every $p = 0, \ldots, m$, $H^{pq}_{\mathcal{F}}(U) = 0$ for $q \geq 1$.

One can describe the cohomology $H^{p*}_{\mathcal{F}}(M)$ by using a sheaf which is analogous to the sheaf of germs of holomorphic p-forms on a complex manifold. A p-form α is said to be $\mathcal{F}$-holomorphic, if it is foliated, of type $(p,0)$ and satisfies $\overline{\partial}_{\mathcal{F}}\alpha = 0$. Locally, a *$\mathcal{F}$-holomorphic* p-form can be written :

$$\alpha = \sum \alpha_{j_1\cdots j_p}(z,t)dz_{j_1} \wedge \cdots \wedge dz_{j_p}$$

with $\alpha_{j_1\cdots j_p}$ holomorphic on z.

Let $\mathcal{H}^p_{\mathcal{F}}$ be the sheaf of germs of $\mathcal{F}$-holomorphic p-forms on M and $\widetilde{\Omega}^{pq}(\mathcal{F})$ the sheaf of germs of differential forms of type (p,q) on $\mathcal{F}$; $\widetilde{\Omega}^{pq}(\mathcal{F})$ is a fine sheaf. The preceding Lemma implies the :

Proposition

The sequence $0 \longrightarrow \mathcal{H}^p_{\mathcal{F}} \hookrightarrow \widetilde{\Omega}^{p0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} \widetilde{\Omega}^{pm}(\mathcal{F}) \longrightarrow 0$ *is a fine resolution of* $\mathcal{H}^p_{\mathcal{F}}$. *So we have* $H^q(M,\mathcal{H}^p_{\mathcal{F}}) = H^{pq}_{\mathcal{F}}(M)$, *for* $p,q = 0,1,\ldots m$.

Let us start with a simple example. Let F be a complex manifold of dimension m and B a differentiable manifold. We denote by $C^\infty(B)$ the complex vector space of complex C^∞ functions on B.

Proposition

Suppose that $\mathcal{F}$ is defined by a locally trivial fibration $F \stackrel{\pi}{\longrightarrow} M \longrightarrow B$ (the cocycle is with values in the biholomorphism group of the complex manifold F). Then $H^{p*}_{\mathcal{F}}(M) = H^{p*}(F) \otimes C^\infty(B)$ where $H^{p*}(F)$ is the Dolbeault cohomology of the complex manifold F. In particular, $H^{p*}_{\mathcal{F}}(M) = 0$ for $* \geq 1$ if the the fibre F is a Stein manifold.

Foliated Guichard's theorem

An open set of $\mathbb{C}$ is said to be a crown if it is of type $C(r,R) = \{z \in \mathbb{C} : r < |z| < R\}$ where $r \in \mathbb{R}$ and $R \in]0,+\infty]$.
Open crowns of $\mathbb{C}$ are of six *types* :

1. $C(r,R) = \mathbb{C}$ if $r < 0$ and $R = +\infty$;
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An open set M of $\mathbb{C} \times B$ (B is any differentiable manifold) is called $\mathcal{F}$-crowned if each leaf M^t is an open crown of $\mathbb{C}$.

Recall that a simply connected Riemann surface F is isomorphic to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (F is elliptic), the complex plane $\mathbb{C}$ (F is parabolic) or the open unit disc $\mathbb{D}$ (F is hyperbolic).

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Foliated Guichard's theorem

An open set of $\mathbb{C}$ is said to be a crown if it is of type $C(r,R) = \{z \in \mathbb{C} : r < |z| < R\}$ where $r \in \mathbb{R}$ and $R \in]0,+\infty]$.
Open crowns of $\mathbb{C}$ are of six *types* :

1. $C(r,R) = \mathbb{C}$ if $r < 0$ and $R = +\infty$;
2. $C(r,R)$ is a disc if $r < 0$ and $R < +\infty$;
3. $C(r,R)$ is a punctured disc if $r = 0$ and $R < +\infty$;
4. $C(r,R) = \mathbb{C}^*$ if $r = 0$ and $R = +\infty$;
5. $C(r,R)$ = complement of a closed disc if $r > 0$ and $R = +\infty$;
6. $C(r,R)$ is an annulus if $0 < r < R < +\infty$. Two annulus $C(r,R)$ and $C(r',R')$ are equivalent $\Longleftrightarrow \frac{R}{r} = \frac{R'}{r'}$.

An open set M of $\mathbb{C} \times B$ (B is any differentiable manifold) is called $\mathcal{F}$-crowned if each leaf M^t is an open crown of $\mathbb{C}$.

Recall that a simply connected Riemann surface F is isomorphic to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (F is elliptic), the complex plane $\mathbb{C}$ (F is parabolic) or the open unit disc $\mathbb{D}$ (F is hyperbolic).

Theorem

Let $(M,\mathcal{F})$ be a codimension n complex simple foliation whose leaves are simply connected non compact Riemann surfaces all parabolic or all hyperbolic. Let $\gamma : M \longrightarrow M$ be an automorphism of the complex foliation which fixes each leaf F and acts freely and properly on F. Then :

1. *the space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to basic functions ;*
2. *for any $g \in \mathcal{H}_{\mathcal{F}}(M)$, the cohomological equation $h - h \circ \gamma = g$ admits a solution $h \in \mathcal{H}_{\mathcal{F}}(M)$ i.e. the vector space $H^1(\mathbb{Z}, \mathcal{H}_{\mathcal{F}}(M))$ is trivial (here, as usual, $\mathcal{H}_{\mathcal{F}}(M)$ is viewed as a $\mathbb{Z}$-module via the action $(k,f) \in \mathbb{Z} \times \mathcal{H}_{\mathcal{F}}(M) \longmapsto f \circ \gamma^k \in \mathcal{H}_{\mathcal{F}}(M)$).*

The case B is a point, $F = \mathbb{C}$ and $\gamma(z) = z + 1$ is exactly the classical Guichard's Theorem.

Lemma 1

Consider the G-principal bundle $\iota : M^\# \longrightarrow M$ of orthonormal frames transverse to $\mathcal{F}$ with the lifted complex foliation $\mathcal{F}^\#$. Then $H^{0*}_{\mathcal{F}}(M)$ injects in $H^{0*}_{\mathcal{F}^\#}(M^\#)$.

Since $G = \mathrm{O}(n)$ acts on $M^\#$ by automorphisms of $\mathcal{F}^\#$, the foliated differential forms of type $(0, q)$ on M are the foliated differential forms on $M^\#$ of type $(0, q)$ which are invariant by G (they form a vector space denoted $\Omega^{0q}_G(\mathcal{F}^\#)$) ; then $H^{0q}_{\mathcal{F}}(M)$ is canonically isomorphic to the cohomology $H^{0q}_{\mathcal{F}^\#,G}(M^\#)$ of the differential complex : $\cdots \xrightarrow{\overline{\partial}_{\mathcal{F}^\#}} \Omega^{0,r}_G(\mathcal{F}^\#) \xrightarrow{\overline{\partial}_{\mathcal{F}^\#}} \Omega^{0r}_G(\mathcal{F}^\#) \longrightarrow \cdots$.

Let μ be a normalized Haar measure on the compact Lie group G. Then the averaging map $\sigma : \Omega^{0q}(\mathcal{F}^\#) \longrightarrow \Omega^{0q}_G(\mathcal{F}^\#)$ defined by $\sigma(\alpha) = \int_G g^*(\alpha) d\mu(g)$ is linear and contiuous. The induced map $\sigma : H^{0*}_{\mathcal{F}}(M) = H^{0*}_{\mathcal{F}^\#,G}(M^\#) \hookrightarrow H^{0*}_{\mathcal{F}^\#}(M^\#)$ is injective. Then to prove $H^{0*}_{\mathcal{F}}(M) = 0$, it is sufficient to prove that $H^{0*}_{\mathcal{F}^\#}(M^\#) = 0$.

Lemma 2

The cohomology vector space $H^{01}_{\mathcal{F}}(M)$ is zero and the space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to functions constant on the leaves.

From Lemma 1 one can assume that $\mathcal{F}$ is defined by a (differentiable) locally trivial fibration $\pi : M \longrightarrow B$ where B is a C^∞-manifold.
For any $t \in B$, the fibre $F_t = \pi^{-1}(t)$ is a simply connected non compact Riemann surface ; then it is isomorphic to the complex plane $\mathbb{C}$ or the open unit disc $\mathbb{D}$. (From now on F will denote $\mathbb{C}$ or $\mathbb{D}$.)

Leaves are hyperbolic

Any point $b \in B$ has an open neighborhood $V \subset B$ diffeomorphic to an open ball in $\mathbb{R}^n$ and such that $U = \pi^{-1}(V)$ is diffeomorphic to the product $F \times V$. By the uniformization theorem with parameter (Ahlfors-Bers), the complex foliation $(U, \mathcal{F})$ is isomorphic to the complex product $F \times V$. Then we have $H^{01}_{\mathcal{F}}(U) = H^{01}(F) \otimes C^\infty(V) = 0$ (sinc F is Stein).

Leaves are parabolic

In that case uniformization theorem with parameter does not work. But one can solve the $\overline{\partial}_{\mathcal{F}}$ by the same procedure as in Lemma 3.

Now let $\mathcal{V} = \{V_j\}_{j\in J}$ be an open cover of B whose elements V_j are diffeomorphic to a ball of $\mathbb{R}^n$. For any $j \in J$, the foliation $(U_j, \mathcal{F})$, where U_j is the open set $U_j = \pi^{-1}(V_j)$ of M, is isomorphic to the complex product $F \times V_j$ hence $H^{01}_{\mathcal{F}}(U_j) = 0$; furthermore $\mathcal{U} = \{U_j\}_{j\in J}$ is an open cover of M. Let $\{\overline{\rho}_j\}$ be a C^∞-partition of 1 on B subordinated to the cover $\mathcal{V}$; then $\rho_j = \overline{\rho}_j \circ \pi$ is a C^∞-partition of 1 on M subordinated to the cover $\mathcal{U}$. Let $\omega \in \Omega^{01}_{\mathcal{F}}(M)$; clearly ω and its restriction ω_j to U_j are $\overline{\partial}_{\mathcal{F}}$-closed. Then there exists $h_j \in \Omega^{00}_{\mathcal{F}}(U_j) = C^\infty(U_j)$ such that $\omega_j = \overline{\partial}_{\mathcal{F}} h_j$. Let $h \in \Omega^{00}_{\mathcal{F}}(M) = C^\infty(M)$ defined by $h = \sum_{j\in J} \rho_j h_j$.

By the linearity and the continuity of the operator $\overline{\partial}_{\mathcal{F}}$, we have :

$$\overline{\partial}_{\mathcal{F}} h = \overline{\partial}_{\mathcal{F}} \left(\sum_{j \in J} \rho_j h_j \right) = \sum_{j \in J} \rho_j \overline{\partial}_{\mathcal{F}} h_j = \sum_{j \in J} \rho_j \omega_j = \omega.$$

(In the preceding sequence of equalities we used the fact $\overline{\partial}_{\mathcal{F}} \rho_j = 0$ because ρ_j is constant along the fibres of π.) This proves that $H^{01}_{\mathcal{F}}(M) = 0$.

Now, to prove that there are sufficiently $\mathcal{F}$-holomorphic functions (*i.e.* the vector space $\mathcal{H}_{\mathcal{F}}(M)$ is not reduced to functions constant on the leaves) it is sufficient to see that locally, for instance on the open sets U_j, they abound and to glue them by the partition of the unity $\{\rho_j\}$ considered above. (This is always possible even the leaf space $B = M/\mathcal{F}$ is only an orbifold.)

Lemma 3

Let M be a crowned open set of $\mathbb{C} \times B$ where B is any differentiable manifold. Suppose that all the leaves M^t (which are crowns) are of the same type. Then the cohomology vector space $H^{01}_{\mathcal{F}}(M)$ is zero.

We have to prove that for any foliated 1-form $\omega = f(z,t)d\overline{z}$ of type $(0,1)$ (which is always $\overline{\partial}_{\mathcal{F}}$-closed) there exists a function $h \in C^\infty(M)$ such that $\overline{\partial}_{\mathcal{F}} h = \omega$.

For any $j \in \mathbb{N}$, let K_j be a subset of M such that, for any $t \in B$, the section K_j^t of K_j is a compact crowned subset of M^t ; let Ω_j be a $\mathcal{F}$-crowned open neighborhood of K_j. We suppose that $\overline{\Omega}_j$ is contained in the interior $\mathrm{int}(K_{j+1})$ of K_{j+1}. The sequence (K_j) is such that $\overline{\Omega}_j \subset \mathrm{int}(K_{j+1})$, is increasing and converges to M. Let $\varphi_j : \mathbb{C} \times B \longrightarrow \mathbb{R}$ be a C^∞-function with compact support in Ω_{j+1} and equal to 1 on Ω_j.

We set $\psi_j = \varphi_j$ for $j = 0$ and $\psi_j = \varphi_j - \varphi_{j-1}$ for $j \geq 1$. For any $j \geq 1$, the function ψ_j is zero on Ω_{j-1} and the sequence (ψ_j) satisfies $\sum_{j=0}^{\infty} \psi_j = 1$. Let :

$$h_j(z,t) = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{\psi_j(\xi,t) f(\xi,t)}{\xi - z} d\xi \wedge d\overline{\xi}.$$

The function h_j is well defined and C^{∞} on M. By the Cauchy formula, we easily verify that $\overline{\partial}_{\mathcal{F}} h_j = \psi_j f$. Furthermore, for $j \geq 1$, ψ_j is zero on Ω_{j-1}, then h_j is $\mathcal{F}$-holomorphic on Ω_{j-1}. Since Ω_{j-1} is $\mathcal{F}$-crowned, h_j admits a Laurent expansion on Ω_{j-1} :

$$\text{(L)} \qquad h_j(z,t) = \sum_{k=0}^{\infty} a_{jk}(t) z^k + \sum_{m=1}^{\infty} \frac{b_{jm}(t)}{z^m}$$

where the coefficients a_{jk} and b_{jm} are given by the integral formulaes :

$$a_{jk}(t) = \frac{1}{2i\pi}\int_{\gamma_1^t} \frac{h_j(\xi,t)}{\xi^{k+1}}d\xi \quad \text{and} \quad b_{jm}(t) = \frac{1}{2i\pi}\int_{\gamma_2^t} \xi^{m-1}h_j(\xi,t)d\xi;$$

γ_1^t and γ_2^t are respectively the smallest circle and the biggest circle of the boundary of the compact set K_{j-1}^t. (In case K_{j-1}^t is a disc, γ_2^t is empty and $b_{jm}(t) = 0$ for $m \geq 1$.) The way the coefficients a_{jk} and b_{jm} are given shows that they are C^∞ in t ; on the other hand, the series (L) is given by the integral Cauchy formula, then it converges with respect to the C^∞-topology to the function h_j. Let $j \geq 1$; truncating conveniently the two sides (left and right) of the series (L), one gets a function :

$$v_j(z,t) = \sum_{k=0}^{k_j} a_{jk}(t)z^k + \sum_{m=1}^{m_j} \frac{b_{jm}(t)}{z^m}$$

which is $\mathcal{F}$-holomorphic on M and such that $\delta(h_j, v_j) < \frac{1}{2^j}$ (where δ is the distance on $C^\infty(M)$ defining the C^∞-topology). The series :

$h_0 + \sum_{j=1}^{\infty}(h_j - v_j)$ converges with respect to the C^∞-topology to a function $h \in C^\infty(M)$. This function satisfies the equation $\overline{\partial}_{\mathcal{F}} h = f$; indeed, by the linearity and the continuity of $\overline{\partial}_{\mathcal{F}}$, one has :

$$\overline{\partial}_{\mathcal{F}} h = \overline{\partial}_{\mathcal{F}} \left(h_0 + \sum_{j=1}^{\infty}(h_j - v_j) \right) = \overline{\partial}_{\mathcal{F}} h_0 + \sum_{j=1}^{\infty} \overline{\partial}_{\mathcal{F}}(h_j - v_j) = \sum_{j=0}^{\infty} \psi_j f = f.$$

This ends the proof of the lemma. ◊

Lemma 4

Let $M \xrightarrow{\pi} B$ be as in the statement of Theorem. Let $(\overline{M}, \overline{\mathcal{F}})$ be the complex foliation obtained as the quotient on $(M, \mathcal{F})$ by the action of the automorphism γ on M. Then the cohomology vector space $H^{01}_{\overline{\mathcal{F}}}(\overline{M})$ is zero.

From Lemma 1 one can also assume that $\mathcal{F}$ is defined by a (differentiable) locally trivial fibration $\pi : M \longrightarrow B$ where B is a C^∞-manifold.
The action of γ is free and proper and it leaves each fibre invariant. Then, the quotient manifold $\overline{M}$ is a differentiable locally trivial fibration $\overline{\pi} : \overline{M} \longrightarrow B$. Let V be an open set of B diffeomorphic to a ball of $\mathbb{R}^n$ and $U = \pi^{-1}(V)$; then U is diffeomorphic to $\overline{F} \times V$ where $\overline{F}$ is a crown of $\mathbb{C}$. The open set U with the complex foliation $\overline{\mathcal{F}}$ is isomorphic to a crowned open set of $\mathbb{C} \times B$ with its canonical complex foliation. By Lemma 2, we have $H^{01}_{\overline{\mathcal{F}}}(U) = 0$. Now, a partition of 1 argument like in the same Lemma allows us to conclude that $H^{01}_{\overline{\mathcal{F}}}(\overline{M}) = 0$.

End of the proof of the theorm

Let us start with a more general fact. Let $\mathcal{F}$ be a complex foliation of dimension m on a manifold M and Γ a countable group which acts freely and properly by automorphisms of $(M, \mathcal{F})$. Then the quotient manifold $\overline{M} = M/\Gamma$ is endowed with the induced foliation $\overline{\mathcal{F}}$ in such way that the canonical projection $\pi : M \longrightarrow \overline{M}$ is a foliated covering map of complex foliations. The pull-back $\pi^*(\mathcal{H}_{\overline{\mathcal{F}}})$ of the sheaf $\mathcal{H}_{\overline{\mathcal{F}}}$ by π is exactly the sheaf $\mathcal{H}_{\mathcal{F}}$. Then there exists a spectral sequence whose E_2 term is :

$$E_2^{k\ell} = H^k(\Gamma, H^\ell(M, \mathcal{H}_{\mathcal{F}}))$$

and converging to $H^*(\overline{M}, \mathcal{H}_{\overline{\mathcal{F}}})$. Here $H^k(\Gamma, H^\ell(M, \mathcal{H}_{\mathcal{F}}))$ is the k-cohomology of the discrete group Γ with values in the Γ-module $H^\ell(M, \mathcal{H}_{\mathcal{F}})$ (Γ acts on $(M, \mathcal{F})$ by automorphisms so it acts on $H^\ell(M, \mathcal{H}_{\mathcal{F}})$).

If M is acyclic *i.e.* $H^{\ell}(M,\mathcal{H}_{\mathcal{F}}) = \mathcal{H}_{\mathcal{F}}(M)$ and $H^{\ell}(M,\mathcal{H}_{\mathcal{F}}) = 0$ if $\ell \geq 1$, the sequence E_r converges at the E_2 term and $H^k(\overline{M},\mathcal{H}_{\overline{\mathcal{F}}}) = H^k(\Gamma,\mathcal{H}_{\mathcal{F}}(M))$ where the action of Γ on $\mathcal{H}_{\mathcal{F}}(M)$ is given by $(\gamma, f) \in \Gamma \times \mathcal{H}_{\mathcal{F}}(M) \longmapsto f \circ \gamma^{-1} \in \mathcal{H}_{\mathcal{F}}(M)$. But in our case $\Gamma = \mathbb{Z}$. So $\mathcal{H}_{\mathcal{F}}(M)/\mathcal{C} = H^1(\mathbb{Z},\mathcal{H}_{\mathcal{F}}(M)) = H^1(\overline{M},\mathcal{H}_{\overline{\mathcal{F}}}) = H^{01}_{\overline{\mathcal{F}}}(\overline{M}) = 0$ where $\mathcal{C}$ is the subspace of $\mathcal{H}_{\mathcal{F}}(M)$ generated by elements of the form $h - h \circ \gamma$. (The last equality is obtained by applying Lemma 4.) This shows that the coboundary operator $h \in \mathcal{H}_{\mathcal{F}}(M) \longmapsto (h - h \circ \gamma) \in \mathcal{H}_{\mathcal{F}}(M)$ is surjective that is, for any $g \in \mathcal{H}_{\mathcal{F}}(M)$ the cohomological equation $h - h \circ \gamma = g$ admits a solution $h \in \mathcal{H}_{\mathcal{F}}(M)$. $\diamondsuit$

Mittag-Leffler for complex foliations

Let $\mathcal{P}$ be a countable union of subsets P of M ; we say that $\mathcal{P}$ is an *acyclic family*, if it satisfies the following conditions :
- $\mathcal{P}$ is discrete *i.e.* if $P, P' \in \mathcal{P}$ with $P \neq P'$, there exist two disjoint open sets V and V' adapted to $\mathcal{F}$ containing P and P' respectively ;
- there exists an acyclic cover $\mathcal{U}$ by open sets adapted to the foliation such that any $P \in \mathcal{P}$ is contained in some $U \in \mathcal{U}$ and each $U \in \mathcal{U}$ contains at most one element $P \in \mathcal{P}$. We say that the cover $\mathcal{U}$ is *associated* to $\mathcal{P}$.

Foliated Mittag-Leffler Theorem

Suppose that the complex foliation $(M, \mathcal{F})$ satifies the hypotheses of Theorem or Lemma 3. Let Σ be an acyclic family of submanifolds Σ_i transverse to $\mathcal{F}$ and $\mathcal{U} = \{U_i\}$ an acyclic open cover associated to Σ. Suppose that we are given on each U_i a $\mathcal{F}$-meromorphic function $h_i : U_i \longrightarrow \mathbb{C}$ which is $\mathcal{F}$-holomorphic outside Σ_i and that $h_i - h_j$ is $\mathcal{F}$-holomorphic on $U_i \cap U_j$. Then there exists a $\mathcal{F}$-meromorphic function $h : M \longrightarrow \mathbb{C}$ such that $h - h_i$ is $\mathcal{F}$-holomorphic on U_i.