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GEOMETRY AND DIFFERENTIAL TOPOLOGY. ALGORITHMIC GEOMETRY

EXAMPLES OF TRANSVERSE STRUCTURES OF FOLIATIONS

by

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These are notes of lectures given at the CIMPA School "Geometry and Differential Topology. Algorithmic Geometry" in the University Cadi Ayyad, FST (Marrakech) in may 2004. Nothing is new; their aim was to give an elementary introduction, trough some simple and significant examples, to the rich and beautiful subject of transverse structures of foliations.

1. Generalities

One of the most important object in mathematics is the vector space $\mathbb{R}^d$ with its usual Euclidean product $\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i$. The fact that it admits a global system of coordinates makes easy the formulation of problems in analysis. But many problems may be formulated with some constraints like for instance:

$$\begin{cases} \text{Solve the problem } (P) \text{ in } \mathbb{R}^d \\ \text{with the constraints} \\ x_1^2 + \cdots + x_d^2 = 1 \end{cases}$$

The problem (P) is in fact formulated in a (closed) subset M of $\mathbb{R}^d$ (the $(d-1)$-sphere $\mathbb{S}^{d-1}$) and not in the hull space. Locally topological space M given by similar conditions looks like an Euclidean one: for every point $x \in M$, there exists an open neighborhood U and a homeomorphism $\varphi : \mathbb{R}^k \longrightarrow U$; we say that (U, φ) is a *local chart* or a *local system* of coordinates: $(x_1, \cdots, x_k) = \varphi^{-1}(x)$ are the coordinates of the point x in the chart (U, φ). If (V, ψ) is another chart around x for which $(x_1', \cdots, x_k')$ are the coordinates, the map $\psi^{-1} \circ \varphi : \varphi^{-1}(U \cap V) \longrightarrow \psi^{-1}(U \cap V)$ is a homeomorphism such that $\psi^{-1} \circ \varphi(x_1, \cdots, x_k) = (x_1', \cdots, x_k')$. We say that $\psi^{-1} \circ \varphi$ is the *coordinate change* from the chart (U, φ) to the chart (V, ψ). If such property is satisfied for every point $x \in M$, we say that M is a *topological manifold*. In that case, there exist an open cover $\{U_i\}$ of M and a family of homeomorphisms $\varphi_i : \mathbb{R}^n \longrightarrow U_i$; $\{(U_i, \varphi_i)\}$ is called an *atlas* defining M. Let M be a topological manifold of dimension k defined by an atlas (U_i, φ_i); we say that M is a *differentiable manifold* if, for every pair (i, j) such that $U_i \cap U_j \neq \emptyset$, the homeomorphism $\varphi_j^{-1} \circ \varphi_i$ is of class C^∞. We say that a differentiable manifold M is *connected, compact...* if the underlying topological space M is connected, compact... The notion of a “differentiable manifold” is central in differential geometry. Many geometrical structures can be defined on it; in this text we shall study a special type: *foliated structure*.

Let M be the Euclidean space $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ with canonical coordinates denoted $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and consider the family of affine subspaces F_y of M where $y \in \mathbb{R}^n$, defined by the differential system: $dy_1 = \ldots = dy_n = 0$. Then M, considered as a disjoint union of these spaces, is a non connected manifold of dimension m. Its topology is the product of the usual topology on $\mathbb{R}^m$ and the discrete one on $\mathbb{R}^n$. We say that M, with this structure, is a *foliated manifold* of *dimension* m and *codimension* n. It constitutes the *local model* of a *foliation* of codimension n on a manifold of dimension $m + n$. Let us denote this structure by $\mathfrak{F}$ on $\mathbb{R}^{m+n}$. Let $\mathcal{O}$ be an open set of $\mathbb{R}^{m+n}$; a $\mathfrak{F}$-*plaque* of $\mathcal{O}$ is any intersection of $\mathcal{O}$ with a horizontal space F_y. A diffeomorphism $\phi : \mathcal{O} \longrightarrow \mathcal{O}'$ is said to be $\mathfrak{F}$-*foliated* if it is of the form $\phi(x, y) = (\phi_1(x, y), \phi_2(y))$ that is, it sends any $\mathfrak{F}$-plaque of $\mathcal{O}$ into a $\mathfrak{F}$-plaque of $\mathcal{O}'$. This is summerazed in the following definition.

1.1. Definition *Let M be a manifold of dimension $m + n$. A codimension n foliation $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{m+n} \longrightarrow U_i$ such that, on each non empty intersection $U_i \cap U_j$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : (x, y) \in \varphi_i^{-1}(U_i \cap U_j) \longrightarrow (x', y') \in \varphi_j^{-1}(U_i \cap U_j)$ is a $\mathfrak{F}$-foliated diffeomorphism, that is it has the form $x' = \varphi_{ij}(x, y)$ and $y' = \gamma_{ij}(y)$.*

The manifold M is decomposed into connected submanifolds of dimension m. Each of them is called a *leaf* of $\mathcal{F}$. A subset U of M is *saturated* for $\mathcal{F}$ if it is a union of leaves: if $x \in U$ then the leaf passing through x is contained in U.

Coordinate patches (U_i, φ_i) satisfying conditions of definition 1.1 are said to be *distinguished* for the foliation $\mathcal{F}$.

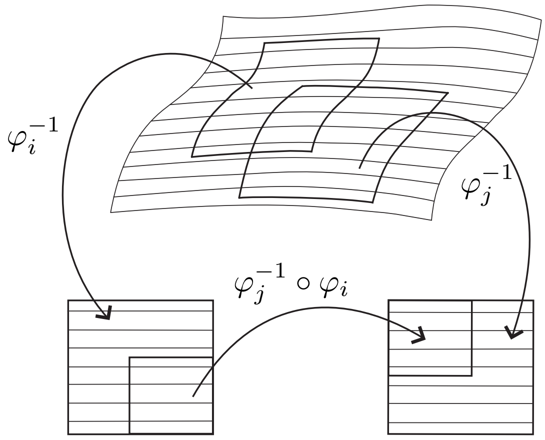

Fig.1

Let $\mathcal{F}$ be a codimension n foliation on M defined by a maximal atlas $\{(U_i, \varphi_i)\}_{i\in I}$ like in definition 1.1. Let $\pi : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the second projection. Then the map $f_i : U_i \overset{\pi\circ\varphi_i^{-1}}{\longrightarrow} \mathbb{R}^n$ is a submersion. On $U_i \cap U_j \neq \emptyset$ we have $f_j = \gamma_{ij} \circ f_i$. The fibres of the submersion f_i are the $\mathcal{F}$-plaques of U_i. The submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbb{R}^n$ give a complete characterization of $\mathcal{F}$.

1.2. Definition *A codimension n foliation on M is given by an open cover $(U_i)_{i\in I}$, submersions $f_i : U_i \longrightarrow T$ over a n dimensional transverse manifold T and, for $U_i \cap U_j \neq \emptyset$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$ satisfying $f_j(x) = \gamma_{ij} \circ f_i(x)$ for $x \in U_i \cap U_j$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a foliated cocycle defining $\mathcal{F}$.*

The foliation $\mathcal{F}$ is said to be *transversely orientable* if T can be given an orientation preserved by all the local diffeomorphisms γ_{ij}.

1.3. Induced foliations

Let N and M be two manifolds; suppose that we are given a codimension n foliation $\mathcal{F}$ on M. We say that map $f : N \longrightarrow M$ is *transverse* to $\mathcal{F}$ if, for each point $x \in N$, the tangent space T_yM of M at $y = f(x)$ is generated by $T_y\mathcal{F}$ and $(d_xf)(T_xN)$ (where d_xf is the tangent linear map of f at x) *i.e.*:

$$T_yM = T_y\mathcal{F} + (d_xf)(T_xN).$$

Equivalently, if we suppose that M is of dimension $m + n$, f is transverse to $\mathcal{F}$ if, for each $x \in N$, there exists a local system of coordinates $(x_1, \cdots, x_m, y_1, \cdots, y_n) : \mathbb{R}^{m+n} \longrightarrow V$ around y such that the map $g_U : (y_1^{-1}\circ f, \cdots, y_n^{-1}\circ f) : U = f^{-1}(V) \longrightarrow \mathbb{R}^n$ is a submersion. The collection of the local submersions (U, g_U) defines a codimension n foliation denoted $f^*(\mathcal{F})$ on N called the *pull-back foliation* of $\mathcal{F}$ by f.

If f is a submersion and $\mathcal{F}$ is the foliation by points, f is transverse to $\mathcal{F}$; in that case the leaves of $f^*(\mathcal{F})$ are exactly the fibres of f.

If $N = \widetilde{M}$ is the universal covering of M and f is the covering projection $f : \widetilde{M} \longrightarrow M$, then $f^*(\mathcal{F})$, denoted $\widetilde{\mathcal{F}}$, has the same dimension as $\mathcal{F}$; the two foliations $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ have the same local properties.

1.4. Morphisms of foliations

Let M and M' be two manifolds endowed respectively with two foliations $\mathcal{F}$ and $\mathcal{F}'$. A map $f : M \longrightarrow M'$ will be called *foliated* or a *morphism* between $\mathcal{F}$ and $\mathcal{F}'$ if, for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf of $\mathcal{F}'$; we say that f is an *isomorphism* if, in addition, f is a diffeomorphism; in this case the restriction of f to any leaf $L \in \mathcal{F}$ is a diffeomorphism on the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M. Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation $\mathcal{F}'$ on M; we say that $\mathcal{F}'$ is the *image* of $\mathcal{F}$ by the diffeomorphism f and we write $\mathcal{F} = f^*(\mathcal{F}')$. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are said to be C^r*-conjugated* (*topologically* if $r = 0$, *differentiably* if $r = \infty$ and *analytically* in the case $r = \omega$) if there exists a C^r-homeomorphism $f : M \longrightarrow M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

The set $\mathrm{Diff}^r(M, \mathcal{F})$ of C^r-diffeomorphisms of M which preseve the foliation $\mathcal{F}$ is a subgroup of the group $\mathrm{Diff}^r(M)$ of all the C^r-diffeomorphisms of M.

1.5. Frobenius Theorem

Let M be a manifold of dimension $m + n$. Denote by TM the tangent bundle of M and let E be a subbundle of rank m. Let U be an open set of M such that on U, TM is equivalent to the product $U \times \mathbb{R}^{m+n}$. At each point $x \in U$, the fibre E_x can be considered as the kernel of n differential 1-forms $\omega_1, \ldots, \omega_n$ linearly independent:

$$\text{(S)} \qquad E_x = \bigcap_{j=1}^{n} \ker\, \omega_j(x).$$

The subbundle E is called an *m-plane field* on M. We say that E is *involutive* if, for every vector fields X and Y tangent to E (*i.e.* sections of E), the bracket $[X, Y]$ is also tangent to E. We say that E is *completely integrable* if, through each point $x \in M$, there exists a submanifold P_x of dimension m which admits $E_{|P_x}$ (the restriction of E to P_x) as tangent bundle. The maximal connected submanifolds satisfying this property are called the *integral submanifolds* of the differential system (S). They define a partition of M *i.e.* a codimension n foliation. We have the following theorem:

Let E be a subbundle of rank m given locally by a differential system like in (S). *Then the following assertions are equivalent:*

– *E is involutive,*

– *E is completely integrable,*

– *there exist differential* 1*-forms (defined locally)* (β_{ij}) , $i, j = 1, \ldots, n$ *such that* $d\omega_i = \sum_{j=1}^{n} \beta_{ij} \wedge \omega_j \quad i = 1, \ldots, n$.

Let ω be a non ingular 1-form o M. Then ω defines a codimension 1 foliation if and only if there exists a 1-form β such that $d\omega = \beta \wedge \omega$; this is equivalent to $\omega \wedge d\omega = 0$. In particular this is the case if ω is closed.

On the other hand the non singular 1-form on $\mathbb{R}^3$ given by $\omega = dx - zdy$ satisfies the relation $\omega \wedge d\omega = dx \wedge dy \wedge dz$ and cannot define a foliation. The plane field $E \subset T\mathbb{R}^3$, kernel of the 1-form ω, has the remarkable following property: given two points a and b in $\mathbb{R}^3$, there exists a differentiable curve $\gamma : [0,1] \longrightarrow \mathbb{R}^3$ such that $\gamma(0) = a$ and $\gamma(1) = b$ and γ is tangent to E at every point. We say that ω defines a *contact structure*. Contact structures are the opposite of foliated structures.

1.6. Holonomy of a leaf

This is a very important notion in foliation theory. In many situations it determines completely the structure of the foliation. In this subsection, we will introduce this concept. We will give later some examples.

Let $\mathcal{F}$ be a codimension n foliation on M, let L be a leaf of $\mathcal{F}$ and $x \in L$. Let T be a small transversal to $\mathcal{F}$ passing through x. Let $\sigma : [0,1] \longrightarrow L$ be a continuous path such that $\sigma(0) = \sigma(1) = x$. Then there exist a finite open cover U_i, $i = 0, 1, \ldots, k$ of M with $U_0 = U_k$ and a subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1]$ such that:

– $\sigma([t_{i-1}, t_i]) \subset U_i$,

– if $U_i \cap U_j \neq \emptyset$ then $U_i \cup U_j$ is contained in a distinguished chart of $\mathcal{F}$.

We say that $\{U_i\}$ is a *subordinated chain* to σ. For $i = 0, 1, \ldots, k$ let T_i be a small transversal to $\mathcal{F}$ passing through $\sigma_i(t)$ with $T_0 = T_k = T$. For every point $z \in T_i$, sufficiently close to $\sigma(t_i)$, the plaque of $\mathcal{F}$ passing through z intersects T_{i+1} in a unique point $f_i(z)$. The domain of f_i contains a transversal T_i' passing through $\sigma(t_i)$ and homeomorphic to an open ball of $\mathbb{R}^n$. Then, it is clear that the map: $f_\sigma = f_{k-1} \circ f_{k-2} \circ \ldots \circ f_0$ is well defined on an open neighbourhood of x; it is called the *holonomy map* associated to σ. We can prove (see [CL] for instance) that the germ of f_σ:

– does not depend on the chain U_i, $i = 1, \ldots, k$ and in the choice of σ in its homotopy class in the group $\pi_1(L, x)$ of the homotopy classes of loops based at x,

– satisfies $f_\sigma(x) = x$.

So we get a homomorphism $h : [\sigma] \in \pi_1(L, x) \longrightarrow f_\sigma \in G(T, x)$ where $G(T, x)$ is the group of germs of diffeomorphisms of T fixing the point x. This representation h is called the *holonomy* of the leaf L at x; the image of $\pi_1(L)$ (L is path connected) is called the *holonomy group* of the leaf L. The foliation $\mathcal{F}$ is said to be *without holonomy* if every leaf L of $\mathcal{F}$ has a trivial holonomy group.

2. Transverse structures

Let us fix some notations. Let $\mathcal{F}$ be a codimension n foliation on M. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\nu\mathcal{F}$ the quotient $TM/T\mathcal{F}$ which is the *normal bundle* to $\mathcal{F}$; $\mathfrak{X}(\mathcal{F})$ will denote the space of sections of $T\mathcal{F}$ (elements of $\mathfrak{X}(\mathcal{F})$ are vector fields $X \in \mathfrak{X}(M)$ tangent to $\mathcal{F}$). A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. (Here i_X and L_X denote respectively the inner product and the Lie derivative with respect to the vector field X.) For a function $f : M \longrightarrow \mathbb{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$ *i.e.* f is constant on the leaves of $\mathcal{F}$; we denote by $\Omega^r(M/\mathcal{F})$ the space of basic forms of degree r on the foliated manifold $(M, \mathcal{F})$; this is a module over the algebra $\mathfrak{B}$ of basic functions. A vector field $Y \in \mathfrak{X}(M)$ is said to be *foliated*, if for every $X \in \mathfrak{X}(\mathcal{F})$, the bracket $[X, Y] \in \mathfrak{X}(\mathcal{F})$. We can easily see that the set $\mathfrak{X}(M, \mathcal{F})$ of foliated vector fields is a Lie algebra and a $\mathfrak{B}$-module;

by definition $\mathfrak{X}(\mathcal{F})$ is an ideal of $\mathfrak{X}(M,\mathcal{F})$ and the quotient $\mathfrak{X}(M/\mathcal{F}) = \mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F})$ is called the Lie algebra of *transverse* (or *basic*) vector fields on the foliated manifold $(M,\mathcal{F})$. Also, it has a module structure over the algebra $\mathfrak{B}$.

Let M be a manifold of dimension $m+n$ endowed with a codimension n foliation $\mathcal{F}$ defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$ like in definition 1.2.

2.1. Definition. *A transverse structure to $\mathcal{F}$ is a geometric structure on T invariant by the local diffeomorphisms γ_{ij}.*

This is a very important notion in foliation theory. To make it clear, let us give the main examples of such structures.

2.2. Lie foliations

We say that $\mathcal{F}$ is a *Lie G-foliation*, if T is a Lie group G and γ_{ij} are restrictions of left translations on G. Such foliation can also be defined by a 1-form ω on M with values in the Lie algebra $\mathcal{G}$ such that:

i) $\omega_x : T_xM \to \mathcal{G}$ is surjective for every $x \in M$,

ii) $d\omega + \frac{1}{2}[\omega,\omega] = 0$.

If $\mathcal{G}$ is Abelian, ω is given by n linearly independant closed scalar 1-forms $\omega_1, \dots, \omega_n$.

In the general case, the structure of a Lie foliation on a compact manifold, is given by the following theorem due to E. Fédida [Féd]:

Let $\mathcal{F}$ be a Lie G-foliation on a compact manifold M. Let $\widetilde{M}$ be the universal covering of M and $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\widetilde{M}$. Then there exist a homomorphism $h : \pi_1(M) \longrightarrow G$ and a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ whose fibres are the leaves of $\widetilde{\mathcal{F}}$ and such that, for every $\gamma \in \pi_1(M)$, the following diagram is commutative:

$$(*) \qquad \begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma} & \widetilde{M} \\ D\downarrow & & \downarrow D \\ G & \xrightarrow{h(\gamma)} & G \end{array}$$

where the first line denotes the deck transformation of $\gamma \in \pi_1(M)$ on $\widetilde{M}$.

The subgroup $\Gamma = h(\pi_1(M)) \subset G$ is called the *holonomy group* of $\mathcal{F}$ although the holonomy of each leaf is trivial. The fibration $D : \widetilde{M} \longrightarrow G$ is called the *developing map* of $\mathcal{F}$.

This theorem gives also a way to construct a Lie foliation. Let us see explicitly a particular example. Let M be the 2-torus $\mathbb{T}^2$; its universal covering $\widetilde{M}$ is $\mathbb{R}^2$ and his fundamental group is $\Gamma = \mathbb{Z}^2$. Denote by h the morphism from Γ to the Lie group $G = \mathbb{R}$ given by $h(m,n) = n + \alpha m$ where α is a real positive number. For convenience we will consider that the action of an element $(m,n) = \gamma \in \Gamma$ on $\mathbb{R}^2$ is given by:

$$(x+y) \in \mathbb{R}^2 \overset{\gamma}{\longmapsto} (x-m, y+n) \in \mathbb{R}^2.$$

Let $D : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the submersion defined by $D(x,y) = y - \alpha x$. It is not difficult to see that, for any $\gamma \in \Gamma$, the diagram:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\gamma} & \mathbb{R}^2 \\ D\downarrow & & \downarrow D \\ \mathbb{R} & \xrightarrow{h(\gamma)} & \mathbb{R} \end{array}$$

is commutative *i.e.* the fibration $D : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is equivariant under the action of Γ on $\mathbb{R}^2$ and then induces a Lie foliation on $\mathbb{T}^2$ transversely modeled on the Lie group $\mathbb{R}$.

2.3. Transversely parallelizable foliations

We say that $\mathcal{F}$ is *transversely parallelizable* if there exist on M foliated vector fields $Y_1, \ldots, Y_n$, transverse to $\mathcal{F}$ and everywhere linearly independent. This means that the manifold T admits a parallelism $(Y_1, \ldots, Y_n)$ invariant by all the local diffeomorphisms γ_{ij} or, equivalently, that the $\mathfrak{B}$-module $\mathfrak{X}(M/\mathcal{F})$ is free of rank n. The structure of a transversely parallelizable foliation on a compact manifold is given by the following theorem due to L. Conlon [Con] for $n = 2$ and in general to P. Molino [Mol].

Let $\mathcal{F}$ be a transversely parallelizable foliation of codimension n on a compact manifold M. Then:

(1) *the closures of the leaves are submanifolds which are fibres of a locally trivial fibration $\pi : M \longrightarrow W$ where W is a compact manifold,*

(2) *there exists a simply connected Lie group G_0 such that the restriction $\mathcal{F}_0$ of $\mathcal{F}$ to any leaf closure F is a Lie G_0-foliation,*

(3) *the cocycle of the fibration $\pi : M \longrightarrow W$ has values in the group of diffeomorphisms of F preserving $\mathcal{F}_0$.*

The fibration $\pi : M \longrightarrow W$ and the manifold W are called respectively the *basic fibration* and the *basic manifold* associated to $\mathcal{F}$. This theorem says that if, in particular, the leaves of $\mathcal{F}$ are closed, then the foliation is just a fibration over W. This is still true even if the leaves are not closed: the manifold M is a fibration over the leaf space $M/\mathcal{F}$ which is, in this case, a *Q-manifold* in the sense of [Bar]. The preceding theorem is still valid for *transversely complete foliations* on non compact manifolds (cf. [Mol]).

Any Lie foliation is transversely parallelizable. This is a consequence of the fact that a Lie group is parallelizable and that the parallelism can be chosen invariant by left translations.

2.4. Riemannian foliations

The foliation $\mathcal{F}$ is said to be *Riemannian* if there exists on T a Riemannian metric such that the local diffeomorphisms γ_{ij} are isometries. Using the submersions $f_i : U_i \longrightarrow T$ one can construct on M a Riemannian metric which can be written in local coordinates:

$$ds^2 = \sum_{i,j=1}^{m} \theta_i \otimes \theta_j + \sum_{k,\ell=1}^{n} g_{k\ell}(y) dy_k \otimes dy_\ell.$$

(This metric is said to be *bundle like.*) Equivalently, $\mathcal{F}$ is Riemannian, if any geodesic orthogonal to the leaves at a point is orthogonal to the leaves everywhere. See the paper [Rei] by B. Reinhart who introduced firstly the notion of Riemannian foliation.

Let $\mathcal{F}$ be Riemannian. Then there exists a Levi-Civita connection, transverse to the leaves which, by unicity argument, coincides on any distinguished open set, with the pull-back of the Levi-Civita connection on the Riemannian manifold T. This connection is said to be *projectable.* Let:

$$\mathrm{O}(n) \longrightarrow M^{\#} \stackrel{\tau}{\longrightarrow} M$$

be the principal bundle of orthonormal frames transverse to $\mathcal{F}$. The following theorem is due to P. Molino [Mol]:

Suppose M is compact. Then, the foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}^{\#}$ on $M^{\#}$ of the same dimension and such that:
(1) $\mathcal{F}^{\#}$ *is transversely parallelizable,*
(2) $\mathcal{F}^{\#}$ *is invariant under the action of $O(n)$ on $M^{\#}$ and projects, by τ, on $\mathcal{F}$.*

The basic manifold $W^{\#}$ and the basic fibration $F^{\#} \longrightarrow M^{\#} \stackrel{\pi^{\#}}{\longrightarrow} W^{\#}$ are called respectively the *basic manifold* and the *basic fibration* of $\mathcal{F}$.

We have the following properties:

– the restriction of τ to a leaf of $\mathcal{F}^{\#}$ is a covering over a leaf of $\mathcal{F}$. So all leaves of $\mathcal{F}$ have the same universal covering (*cf.* [Rei]),

– the closure of any leaf of $\mathcal{F}$ is a submanifold of M and the leaf closures define a *singular foliation* (the leaves have different dimensions) on M. (For more details about this notion see [Mol].)

Another interesting result for Riemannian foliations is the Global Reeb Stability Theorem which is valid even if the codimension is greater than 1.

Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. If there exists a compact leaf with finite fundamental group, then all leaves are compact with finite fundamental group.

The property $\mathcal{F}$ is Riemannian means that the leaf space $Q = M/\mathcal{F}$ is a Riemannian manifold even if Q does not support any differentiable structure !

2.5. $\mathcal{G}/\mathcal{H}$-foliations

This is a class of foliations which possess interesting transverse properties (see [EGN]). Let $\mathcal{G}$ be a Lie algebra of dimension d and $\mathcal{H}$ a Lie subalgebra of $\mathcal{G}$. We fix a basis $e_1, \ldots, e_d$ of $\mathcal{G}$ such that $e_{n+1}, \ldots, e_d$ span $\mathcal{H}$ and denote by $\theta^1, \ldots, \theta^d$ the corresponding dual basis. One has $[e_i, e_j] = \sum_k K_{ij}^k e_k$, where the *structure constants* K_{ij}^k fulfill the relations

(C1) $$K_{ij}^k = -K_{ji}^k$$

(C2) $$\sum_i (K_{ij}^k K_{rs}^i + K_{ir}^k K_{sj}^i + K_{is}^k K_{jr}^i) = 0 \;\; \text{(Jacobi identity)}$$

(C3) $$K_{ij}^k = 0 \text{ if } k \leq n \text{ and } n+1 \leq i, j$$

The set of constants K_{ij}^k satisfying (C1) and (C2) determines the Lie algebra structure of $\mathcal{G}$ while (C3) states that $\mathcal{H}$ is a Lie subalgebra of $\mathcal{G}$. We denote by G the simply connected

Lie group with Lie algebra $\mathcal{G}$ and by H the connected Lie subgroup of G corresponding to the Lie subalgebra $\mathcal{H}$.

We shall denote by θ the $\mathcal{G}$-valued 1-form on G which is the identity over the left invariant vector fields on G, i.e. $\theta = \sum_k \theta^k \otimes e_k$. Let $\omega = \sum_k \omega^k \otimes e_k$ be a $\mathcal{G}$-valued 1-form on a manifold M. An element $g \in G$ transforms ω into the $\mathcal{G}$-valued form $\mathrm{Ad}_g\omega$ where $\mathrm{Ad}_g\omega(X) = \mathrm{Ad}_g \cdot (\omega(X))$ for any vector field X on M. Once the basis $e_1, \ldots, e_d$ of $\mathcal{G}$ has been fixed, we shall identify ω with the n-tuple of scalar 1-forms $(\omega^1, \ldots, \omega^d)$. In particular $\theta = (\theta^1, \ldots, \theta^d)$.

Let a $\mathcal{G}$-valued 1-form $\omega = (\omega^1, \ldots, \omega^d)$ on a connected manifold M be given. Assume that ω fulfills the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$, i.e.

$$d\omega^k = -\frac{1}{2} \sum_{i,j=1}^{d} K_{ij}^k \omega^i \wedge \omega^j \tag{C4}$$

and that $\omega^1, \ldots, \omega^n$ are linearly independent. Then the differential system $\omega^1 = \ldots = \omega^n = 0$ is integrable and defines a codimension n foliation $\mathcal{F}$. We shall say that $\mathcal{F}$ is a $\mathcal{G}/\mathcal{H}$-foliation defined by the $\mathcal{G}$-valued form ω.

Main example. Let $M = G$. Then $\theta = (\theta^1, \ldots, \theta^d)$ defines a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ whose leaves are the left cosets of H.

Remark. The notion of $\mathcal{G}/\mathcal{H}$-foliation includes several classes of geometric structures:

(a) If $n = \dim M$ and H is closed then a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}$ defines a structure of locally homogeneous space on M; that is, the manifold M is locally modeled on the homogeneous space G/H with coordinate changes given by left translations by elements of G and $\mathcal{F}$ is the foliation by points. The homogeneous space G/H is endowed with a $\mathcal{G}/\mathcal{H}$-foliation when the projection $G \to G/H$ admits a global section.

(b) When $\mathcal{H} = 0$, $\mathcal{G}/\mathcal{H}$-foliations are just Lie foliations modeled over G. For instance a non-singular closed 1-form ω on M defines a Lie foliation modeled over $\mathbb{R}$.

(c) If H is closed then a $\mathcal{G}/\mathcal{H}$-foliation is a transversely homogeneous foliation modeled over the homogeneous space G/H. Every transversely homogeneous foliation is given locally by a collection of 1-forms $\omega^1, \ldots, \omega^d$ fulfilling (C4) (cf. [Blu]). If these forms are global then they define a $\mathcal{G}/\mathcal{H}$-foliation. This is the case if $H^1(M, H) = 0$ (cf. [Blu]).

(d) In general, when H is not necessarily closed, a $\mathcal{G}/\mathcal{H}$-foliation is a locally transversely homogeneous foliation as it is defined in [Mol].

Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω. A map $\varphi : N \to M$ transverse to $\mathcal{F}$ induces a $\mathcal{G}/\mathcal{H}$-foliation $\varphi^*\mathcal{F}$ on N which is defined by $\varphi^*\omega$. We say that $\varphi^*\mathcal{F}$ is the *pull-back* of $\mathcal{F}$ by φ. In particular, the universal covering space $\widetilde{M}$ of M is endowed with the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ defined by $\pi^*\omega$ where $\pi : \widetilde{M} \to M$ is the canonical projection. The following proposition states that the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ is a pull-back of the $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ on G which was considered as the main example.

Proposition [blu]. *Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω and let $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ be its pull-back to the universal covering space $\widetilde{M}$ of M. There are a map $\mathcal{D} : \widetilde{M} \to G$ and a group representation $\rho : \pi_1(M) \to G$ such that* *(i)* $\mathcal{D}$ *is* $\pi_1(M)$*-equivariant, i.e.* $\mathcal{D}(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot \mathcal{D}(\tilde{x})$ *for any* $\gamma \in \pi_1(M)$*, and*

(ii) $\widetilde{\omega} := \pi^*\omega = \mathcal{D}^*\theta$*, i.e.* $\widetilde{\mathcal{F}} = \mathcal{D}^*\mathcal{F}_{G,H}$.

The map $\mathcal{D}$ *is called the developing map of* $\mathcal{F}$ *and it is uniquely determined up to left translations by elements of* G*.*

2.6. Transversely holomorphic foliations

The foliation $\mathcal{F}$ is said to be *transversely holomorphic* if T is a complex manifold and the γ_{ij} are local biholomorphisms. Particular case is a *holomorphic foliation*: the manifolds M and T are complex, all the f_i are holomorphic and all γ_{ij} are local biholomorphisms.

If T is Kählerian and γ_{ij} are biholomorphisms preserving the Kähler form on T we say that $\mathcal{F}$ is *transversely Kählerian.* For example, any codimension 2 Riemannian foliation which is transversely orientable is transversely Kählerian.

Let us give concrete examples of such foliations. Let M be the unit sphere in the Hermitian space $\mathbb{C}^{n+1}$: $M = \mathbb{S}^{2n+1} = \left\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} |z_k|^2 = 1\right\}$. Let Z be the holomorphic vector field on $\mathbb{C}^{n+1}$ given by the formula: $Z = \displaystyle\sum_{k=1}^{n+1} a_k z_k \frac{\partial}{\partial z_k}$ where $a_k = \alpha_k + i\beta_k \in \mathbb{C}$. There exists a good choice of the numbers a_k such that the orbits of Z intersect transversely the sphere M; then Z induces on M a real vector field X which defines a foliation $\mathcal{F}$. It is not difficult to see that $\mathcal{F}$ is transversely holomorphic. It is transversely Kählerian if we choose in addition $\alpha_k = 0$ for any $k = 1, \ldots, n+1$.

3. More examples

3.1. Simple foliations.

Two trivial foliations can be defined on a manifold M: the first one is obtained by considering that all the leaves are the points; the second one has only one leaf, namely, M itself.

Every submersion $M \stackrel{\pi}{\longrightarrow} B$ with connected fibres defines a foliation. The leaves being the fibres $\pi^{-1}(b)$, $b \in B$. In particular, every product $F \times B$ is a foliation with leaves $F \times \{b\}$, $b \in B$. These foliations are transversely orientable if, and only if, the manifold B is orientable.

3.2. Linear foliation on the torus $\mathbb{T}^2$.

This example already differently described in the subsection 2.2. Let $\widetilde{M} = \mathbb{R}^2$ and consider the linear differential equation $dy - \alpha dx = 0$ where α is a real number. This equation has $y = \alpha x + c$, $c \in \mathbb{R}$ as general solution. When c varies, we obtain a family of parallel lines which defines a foliation $\widetilde{\mathcal{F}}$ in $\widetilde{M}$. The natural action of $\mathbb{Z}^2$ on $\widetilde{M}$ preserves the foliation $\widetilde{\mathcal{F}}$ (*i.e.* the image of any leaf of $\widetilde{\mathcal{F}}$ by an integer translation is a leaf of $\widetilde{\mathcal{F}}$). Then $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The leaves are all diffeomorphic to the circle $\mathbb{S}^1$ if α is rational and to the real line if α is not rational (*cf.* Fig. 2). In fact, if α is not rational, every leaf of $\mathcal{F}$ is dense; this shows that even if locally a foliation is simple, globally it can be complicated.

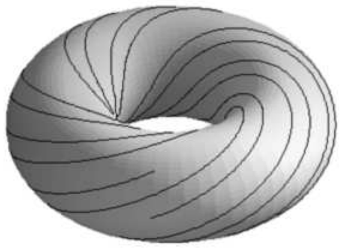

Fig. 2

3.3. One dimensional foliations.

Let M be a closed (i.e. without boundary) manifold of dimension n. Let X be a non singular vector field on M that is, for every $x \in M$, the vector T_xM is non zero. Then its integral curves are leaves of a 1-dimensional foliation. This is also the case for a line bundle on M (not necessarily a vector field). In fact there is a natural one-to-one correspondence between the set of C^∞-line bundles and the set of 1-dimensional C^∞-foliations.

The fact that M admits a one dimensional foliation depends on its topology. For each $r = 0, 1, \ldots, n$, let $H^r(M, \mathbb{R})$ denote the real r-th *cohomology space* of M which is finite dimensional. Then the number:

$$\chi(M) = \sum_{r=0}^{n} (-1)^r \dim\, H^r(M, \mathbb{R})$$

is a *topological invariant* called the *Euler-Poincaré number* of M. We have the following theorem:

The manifold M admits a one dimensional foliation if, and only if, $\chi(M) = 0$.

3.4. Reeb foliation on the 3-sphere $\mathbb{S}^3$.

Let M be the 3 dimensional sphere $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure which is the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. The two subsets:

$$M_+ = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 \leq \frac{1}{2}\right\} \text{ and } M_- = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_2|^2 \leq \frac{1}{2}\right\}$$

are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$. They have $\mathbb{T}^2$ as common boundary:

$$\mathbb{T}^2 = \partial M_+ = \partial M_- = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 = |z_2|^2 = \frac{1}{2}\right\}$$

and their union is equal to $\mathbb{S}^3$. Then $\mathbb{S}^3$ can be obtained by gluing M_+ and M_- along their boundaries by the diffeomorphism $(z_1, z_2) \in \partial M_+ \longrightarrow (z_2, z_1) \in \partial M_-$, i.e. we identify (z_1, z_2) with (z_2, z_1) in the disjoint union $M_+ \coprod M_-$. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the function defined by:

$$f(z) = \exp\left(\frac{1}{1 - |z|^2}\right).$$

Let t denote the second coordinate in $\mathbb{D} \times \mathbb{R}$. The family of surfaces $(S_t)_{t\in\mathbb{R}}$ obtained by translating the graph S of f along the t-axis defines a foliation on $\mathbb{D} \times \mathbb{R}$. If we add the cylinder $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is viewed as the boundary of $\overline{\mathbb{D}}$, we obtain a codimension one foliation $\widetilde{\mathcal{F}}$ on $\overline{\mathbb{D}} \times \mathbb{R}$. By construction, $\widetilde{\mathcal{F}}$ is invariant by the transformation $(z,t) \in \overline{\mathbb{D}} \times \mathbb{R} \longmapsto (z, t+1) \in \overline{\mathbb{D}} \times \mathbb{R}$; so it induces a foliation $\mathcal{F}_0$ on the quotient:

$$\overline{\mathbb{D}} \times \mathbb{R}/(z,t) \sim (z,t+1) \simeq \overline{\mathbb{D}} \times \mathbb{S}^1.$$

It has the boundary $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as a closed leaf. The others are diffeomorphic to $\mathbb{R}^2$ (see Fig. 3).

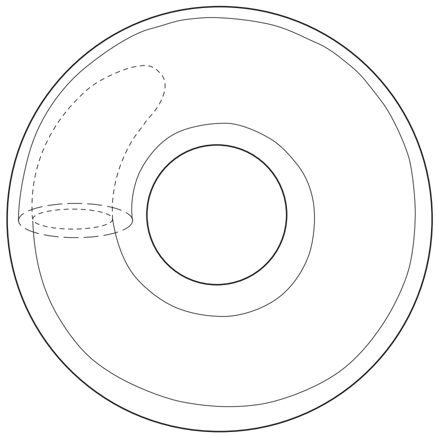

Fig. 3

Because M_+ and M_- are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$, $\mathcal{F}_0$ defines on M_+ and M_- respectively two foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ which give a codimension one foliation $\mathcal{F}$ on $\mathbb{S}^3$ called the *Reeb foliation*. All the leaves are diffeomorphic to the plane $\mathbb{R}^2$ except one which is the torus, the common boundary L of the two components M_+ and M_-. Its holonomy is given by the representation:

$$h : \pi_1(L) = \mathbb{Z}^2 \longrightarrow G(\mathbb{R}, 0)$$

($G(\mathbb{R}, 0)$ is the group of germs at 0 of diffeomorphisms of $\mathbb{R}$ fixing the point 0) which sends the generator $(1,0)$ (resp. the generator $(0,1)$) to the germ of the diffeomorphism h_1 (resp. h_2) equal to the identity on $[0,+\infty[$ (resp. on $]-\infty,0]$) different from the identity on $]-\infty,0]$ and infinitely tangent to it at 0 (resp. on $[0,+\infty[$).

3.5. Lie group actions.

Let M be a manifold of dimension $m+n$ and G a connected Lie group of dimension m. An *action* of G on M is a map $G \times M \xrightarrow{\Phi} M$ such that:

– $\Phi(e,x) = x$ for every $x \in M$ (where e is the unit element of G),

– $\Phi(g', \Phi(g,x)) = \Phi(g'g, x)$ for every $x \in M$ and every $g, g' \in G$.

Suppose that, for every point $x \in M$, the dimension of the *isotropy subgroup*:

$$G_x = \{g \in G : \Phi(g,x) = x\}$$

is independent of x. Then the action Φ defines a foliation $\mathcal{F}$ of dimension $= m - \dim G_x$; its leaves are the orbits $\{\Phi(g,x) : g \in G\}$. In particular this is the case if Φ is *locally* *free* i.e. if, for every $x \in M$, the isotropy subgroup G_x is discrete. An explicit example is given when M is the quotient H/Γ of a Lie group H by a discrete subgroup Γ and G is a connected Lie subgroup of H; the action of G on M being induced by the left action of G on H. We say that $\mathcal{F}$ is a *homogeneous foliation.*

Let us give an example. Let $A \in \mathrm{SL}(m+n-1, \mathbb{Z})$, where $m+n \geq 3$, be a matrix diagonalizable on the field $\mathbb{R}$ and having all its eigenvalues $\mu_1, \ldots, \mu_{m-1}, \lambda_1, \ldots, \lambda_n$ positive. Let $u_1, \cdots, u_{m-1}, v_1, \cdots, v_n$ be the corresponding eigenvectors in $\mathbb{R}^{m+n-1}$. As we can think of A as a diffeomorphism of the $(m+n-1)$-torus $\mathbb{T}^{m+n-1}$, the vectors $u_1, \ldots, u_{m-1}$, $v_1, \ldots, v_n$ can be considered as linear vector fields on $\mathbb{T}^{m+n-1}$ such that:

$$A_* u_j = \mu_j u_j, \quad A_* v_k = \lambda_k v_k \quad \text{for} \quad j = 1, \ldots, m-1 \quad \text{and} \quad k = 1, \ldots, n.$$

Let $(x_1, \cdots, x_{m-1}, y_1, \cdots, y_n, t)$ be the coordinates of a vector in $\mathbb{R}^{m+n} = \mathbb{R}^{m+n-1} \times \mathbb{R}$. Then the vector fields $u_1, \cdots, u_{m-1}, v_1, \cdots, v_n, u_m = \frac{\partial}{\partial t}$ generate the Lie algebra (over the ring of C^∞-functions) $\mathfrak{X}(\mathbb{R}^{m+n})$. The vector fields:

$$X_i = \mu_i^t u_i, \qquad Y_j = \lambda_j^t v_j \qquad \text{and} \qquad X_m = \frac{\partial}{\partial t} \qquad (\text{for } i = 1, \cdots, m-1,\ j = 1, \cdots, n)$$

satisfy the bracket relations:

$$[X_i, X_\ell] = [X_i, Y_j] = [Y_j, Y_k] = 0 \qquad \text{and} \qquad [X_m, X_i] = \ln(\mu_i) X_i, \;\; [X_m, Y_j] = \ln(\lambda_j) Y_j$$

(for $i, \ell = 1, \cdots, m-1$ and $j, k = 1, \cdots, n$,) and then generate over the field $\mathbb{R}$ a finite dimensional Lie algebra $\mathcal{H}$. It is the semi-direct product of the abelian algebra $\mathcal{H}_0$ generated by $X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_n$ and the one dimensional Lie algebra generated by X_m; $\mathcal{H}$ is solvable. The Lie subalgebra $\mathcal{G}$ defined by $X_1, \cdots, X_m$ is also solvable and it is an ideal of $\mathcal{H}$. The simply connected Lie groups H and G corresponding respectively to $\mathcal{H}$ and $\mathcal{G}$ can be constructed as follows. As the eigenvalues of the matrix A are real positive, the group $\mathbb{R}$ acts on $\mathbb{R}^{m+n-1}$: $(t, z) \in \mathbb{R} \times \mathbb{R}^{m+n-1} \longmapsto A^t z \in \mathbb{R}^{m+n-1}$ (where $z = (x_1, \cdots, x_{m-1}, y_1, \cdots y_n)$) leaving invariant the eigenspace E corresponding to $\mu_1, \cdots, \mu_{m-1}$; this action defines the groups H and G respectively as the semi-direct products $\mathbb{R} \ltimes \mathbb{R}^{m+n-1}$ and $\mathbb{R} \ltimes E$. Because the coefficients of A are in $\mathbb{Z}$, the preceding action restricted to $\mathbb{Z}$ preserves the subgroup $\mathbb{Z}^{m+n-1}$; this gives a subgroup $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^{m+n-1}$ which is a cocompact lattice of H. The quotient $\mathbb{T}_A^{m+n} = H/\Gamma$ is a compact manifold of dimension $m+n$. As we have already pointed, any subgroup of H induces a locally free action on H/Γ which defines a foliation. In our example we have two subgroups: G and the normal abelian subgroup K whose Lie algebra is the ideal generated by $Y_1, \cdots, Y_n$. Their actions on $\mathbb{T}_A^{m+n}$ give respectively two foliations $\mathcal{F}$ and $\mathcal{V}$; $\mathcal{V}$ is a Lie foliation transversely modeled on the Lie group G.

4. Suspension of diffeomorphism groups

One of the main class of foliations is obtained by the suspension procedure of groups of diffeomorphisms. This section will be devoted to the definition of this procedure and to give some examples of groups of diffeomorphisms which give interesting foliations.

4.1. General construction

Let B and F be two manifolds, respectively of dimensions m and n. Suppose that the fundamental group $\pi_1(B)$ of B is finitely generated. Let $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ be a representation, where $\mathrm{Diff}(F)$ is the diffeomorphism group of F. Denote by $\widetilde{B}$ the universal covering of B and $\widetilde{\mathcal{F}}$ the horizontal foliation on $\widetilde{M} = \widetilde{B} \times F$, i.e., the foliation whose leaves are the subsets $\widetilde{B} \times \{y\}$, $y \in F$. This foliation is invariant by all the transformations $T_\gamma : \widetilde{M} \longrightarrow \widetilde{M}$ defined by $T_\gamma(\widetilde{x}, y) = (\gamma \cdot \widetilde{x}, \rho(\gamma)(y))$ where $\gamma \cdot \widetilde{x}$ is the natural action of $\gamma \in \pi_1(B)$ on $\widetilde{B}$; then $\widetilde{\mathcal{F}}$ induces a codimension n foliation $\mathcal{F}_\rho$ on the quotient manifold:

$$M = \widetilde{M}/(\widetilde{x}, y) \sim (\gamma \cdot \widetilde{x}, \rho(\gamma)(y)).$$

We say that $\mathcal{F}_\rho$ is the *suspension* of the diffeomorphism group $\Gamma = \rho(\pi_1(B))$. The leaves of $\mathcal{F}_\rho$ are transverse to the fibres of the natural fibration induced by the projection on the first factor $\widetilde{B} \times F \longrightarrow \widetilde{B}$.

Conversely, suppose that $F \longrightarrow M \stackrel{\pi}{\longrightarrow} B$ is a fibration with compact fibre F and that $\mathcal{F}$ is a codimension n foliation (n = dimension of F) transverse to the fibres of π. Then there exists a representation $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ such that $\mathcal{F} = \mathcal{F}_\rho$.

The geometric transverse structures of the foliation $\mathcal{F}$ are exactly the geometric structures on the manifold F invariant by the action of Γ. So to give examples of foliations obtained by suspension, it is sufficient to construct examples of diffeomorphism groups. This is what we shall do now.

4.2. Let G be any compact Lie group of dimension n and Γ a finitely presented subgroup of G (in particular Γ is numerable). Then there exists a compact manifold B such that $\pi_1(B) = \Gamma$. Of course, the subgroup Γ can be viewed as a group of diffeomorphisms of G (Γ acts on G by left multiplication). The suspension of Γ gives a codimension n foliation; it is in fact a Lie foliation transversely modeled on G. Concrete examples are given for instance by the following representation.

- The compact Abelian Lie group $\mathbb{T}^n$ is the maximal torus of the compact Lie group $\mathrm{SO}(2n, \mathbb{R})$. Any element g can be represented by a matrix:

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}$$

where:

$$A_j = \begin{pmatrix} \cos 2\pi\theta_j & \sin 2\pi\theta_j \\ -\sin 2\pi\theta_j & \cos 2\pi\theta_j \end{pmatrix}$$

and all the other entries are equal to 0. Let $\Gamma = \mathbb{Z}^m$; this group has a system of generators $(\gamma_1, \cdots, \gamma_m)$ where $\gamma_1 = (1, 0, \cdots, 0)$, $\gamma_2 = (0, 1, \cdots, 0)$, $\cdots$, $\gamma_m = (0, \cdots, 0, 1)$. Let $\rho : \Gamma \longrightarrow \mathbb{T}^n$ be the representation which sends the generator γ_j (for $j = 1, \cdots, n$) to the matrix:

$$\begin{pmatrix} \ddots & & \\ & \begin{pmatrix} \cos 2\pi\alpha_j & \sin 2\pi\alpha_j \\ -\sin 2\pi\alpha_j & \cos 2\pi\alpha_j \end{pmatrix} & \\ & & \ddots \end{pmatrix}$$

where $\alpha_1, \cdots, \alpha_n \in]0,1[$. The remaining entries in the diagonal are 1 and the other are equal to 0. The foliation obtained by suspending this representation is the linear codimension n foliation on the torus $\mathbb{T}^{m+n}$.

4.3. Now let n an integer ≥ 2 and A a matrix of order n with coefficients in $\mathbb{Z}$ an determinant equal to 1 i.e. A is an element of $\mathrm{SL}(n,\mathbb{Z})$. Suppose that A admits n real positive eigenvalues $\lambda_1, \cdots, \lambda_n$ such that, for each $\lambda \in \{\lambda_1, \cdots, \lambda_n\}$, the components $(v^1, \cdots, v^n)$ in $\mathbb{R}^n$ of an eigenvector v associated to λ are linearly independent over $\mathbb{Q}$ i.e. , for $\mathbf{m} \in \mathbb{Z}^n$, every relation $\langle \mathbf{m}, v \rangle = 0$ implies $\mathbf{m} = 0$ (where $\langle \ , \ \rangle$ is the Euclidean product in $\mathbb{R}^n$). Such matrices exist ; take for instance(cf. [EN1]):

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & d_n \end{pmatrix}$$

with $d_1 = 1$ and $d_{i+1} = 1 + d_1 d_2 \cdots d_i$ for $i = 1, \cdots, n-1$. This fact is easy to verify for $n \leq 3$. Let G be the solvable Lie group semi-direct $\mathbb{R} \ltimes \mathbb{R}^n$ where $\mathbb{R}$ acts on $\mathbb{R}^n$ by the matrix A:

$$(t, u) \in \mathbb{R} \times \mathbb{R}^n \longmapsto A^t u \in \mathbb{R}^n.$$

Because the coefficients of A are integers, this action preserves the lattice $\mathbb{Z}^n$, so we can construct the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}^n$ exactly in the same way; then we obtain a cocompact discrete subgroup Γ of G. The quotient manifold $B = G/\Gamma$ is a flat fibre bundle with fibres the n-torus $\mathbb{T}^n$ over the circle $\mathbb{S}^1$.

Now let $\lambda \in \{\lambda_1, \cdots, \lambda_n\}$ and v an associated eigenvector. Since $\lambda\langle \mathbf{m}, v\rangle = \langle \mathbf{m}', v\rangle$ where $A'(\mathbf{m}) = \mathbf{m}' \in \mathbb{Z}^n$ (A' is the transpose matrix of A), Γ can be embedded in $\mathrm{SL}(n,\mathbb{C})$ as follows: choose integers $a_1, \cdots, a_{n-1}$, set $a = a_1 + \cdots + a_{n-1}$ and associate to $(\mathbf{m}, \ell) \in \mathbb{Z} \ltimes \mathbb{Z}^n$ the matrix $n \times n$

$$\lambda^{-\frac{a\ell}{n}} \begin{pmatrix} \lambda^{a_1 \ell} & \cdots & 0 & \langle \mathbf{m}, v \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \lambda^{a_{n-1}\ell} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(only the terms in the diagonal and the term in the first line and the n^{th} column are nonzero). So we obtain a representation

$$\rho : \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^{n-1}(\mathbb{C})).$$

The action of Γ on $P^{n-1}(\mathbb{C})$ extends to the point ∞ the affine action:

$$(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} \longmapsto (\lambda^{a_1\ell} z_1 + \langle \mathbf{m}, \ell \rangle, \lambda^{a_2\ell} z_2, \cdots, \lambda^{a_{n-1}\ell} z_{n-1}) \in \mathbb{C}^{n-1}$$

for every $(\mathbf{m}, \ell) \in \Gamma$. The suspension of this representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension $n-1$ on the compact differentiable manifold M, quotient of $\widetilde{M} = P^{n-1}(\mathbb{C}) \times G$ by the equivalence relation which identifies (z, x) to $(\rho(\gamma)(z), \gamma x)$ with $\gamma \in \Gamma$ (Γ acts on G by left translation). The leaves of $\mathcal{F}$ are homogeneous spaces of G by discrete subgroups. Note that $\mathcal{F}$ is not transversely Kählerian because the image of the representation ρ does not preserve the Kḧlerian metric on $P^{n-1}(\mathbb{C})$.

4.4. Let $\mathrm{SL}(n, \mathbb{R})$ be the group of real matrices of order n and determinant 1. This is a real form of the group $\mathrm{SL}(n, \mathbb{C})$ (complex matrices of order n and determinant 1); This group acts by projective tranformations on $P^{n-1}(\mathbb{C})$ (complex projectice space of dimension $n-1$). Then every subgroup of $\mathrm{SL}(n, \mathbb{C})$ acts by the same transformations on $P^{n-1}(\mathbb{C})$.

The construction of the following group Γ and the study of its properties can be found in [Mil]. In the upper half plane $\mathbb{H} = \{z = x+iy : y > 0\}$ with the Poincaré metric $\frac{dx^2+dy^2}{y^2}$ we consider a geodesic triangle $T(p, q, r)$ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ et $\frac{\pi}{r}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. We denote by σ_1, σ_2 et σ_3 the reflections associated respectively to the sides of this triangle; they generate an isometry group Σ^*; elements which preserve the orientation form a subgroup Σ of Σ^* of index 2 called the *triangle group* and denoted $T(p, q, r)$. It is a subgroup of $\mathrm{SL}(2, \mathbb{R})$ and its pull-back Γ by the projection $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \longrightarrow \mathrm{SL}(2, \mathbb{R})$ ($\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the universal covering of $\mathrm{SL}(2, \mathbb{R})$) is a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \Sigma \longrightarrow 1.$$

The group Γ has the presentation

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^p = \gamma_2^q = \gamma_3^r = \gamma_1\gamma_2\gamma_3 \rangle.$$

The quotient $B = \widetilde{\mathrm{SL}}(2, \mathbb{R})/\Gamma$ is a compact manifold of dimension 3. If the integers p, q et r are mutually prime the cohomology (with coefficients in $\mathbb{Z}$) of B is exactly the cohomology of the sphere $\mathbb{S}^3$. Since Γ is a subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, it acts on $P^1(\mathbb{C})$. So we obtain a (non injective) representation $\rho : \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^1(\mathbb{C}))$. The suspension of such representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension 1 on the differentiable manifold M of dimension 5, which is the quotient of $\widetilde{M} = P^1(\mathbb{C}) \times \widetilde{\mathrm{SL}}(2, \mathbb{R})$ by the equivalence relation which identifies (z, x) with $(\rho(\gamma)(z), \gamma x)$ (Γ acts on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ by left translation). The leaves of $\mathcal{F}$ are homogeneous spaces of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ by discrete subgroups.

4.5. The 1-dimensional real projective space $P^1(\mathbb{R})$ is obtained by adding the point ∞ to the real line $\mathbb{R}$; it is also isomorphic to the circle $\mathbb{S}^1$. The group $\mathrm{SL}(2, \mathbb{R})$ of 2-order real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ acts analytically on $\mathbb{S}^1$ by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x \right) \in \mathrm{SL}(2, \mathbb{R}) \times \mathbb{S}^1 \longmapsto \frac{ax+b}{cx+d} \in \mathbb{S}^1.$$

For any integer m such that $m \geq 2$, the two elements $\gamma_1 = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ generate a free non Abelian subgroup Γ (*cf.* [KM]) of the group $\mathrm{Diff}(\mathbb{S}^1)$ of diffeomorphisms of the circle $\mathbb{S}^1$.

Let B_1 and B_2 be two copies of $\mathbb{S}^2 \times \mathbb{S}^1$; each one of them has its fundamental group isomorphic to $\mathbb{Z}$. By Van Kampen theorem, the connected sum $B = B_1 \# B_2$ (which is a 3-dimensional manifold) has the non Abelian free group on two generators α_1 and α_2 as fundamental group. Let $\rho : \pi_1(B) \longrightarrow \Gamma$ be the representation defined by $\rho(\alpha_1) = \gamma_1$ and $\rho(\alpha_2) = \gamma_2$. As usual, the suspension of this representation gives rise to a codimension 1 foliation on the 4-manifold M which is a flat bundle $\mathbb{S}^1 \longrightarrow M \longrightarrow B$. This foliation is transversely homogeneous (in fact transversely *projective*).

5. A transverse invariant: the basic cohomology

Let M be a manifold equipped with a codimension n foliation $\mathcal{F}$. Recall that a differential form α is said to be *basic* if it satisfies $i_X\alpha = L_X\alpha = 0$ for every vector field X tangent to $\mathcal{F}$. If α is basic so is $d\alpha$; then basic forms constitute a differential complex $(\Omega^r(M/\mathcal{F}), d)$ called the *basic complex* of the foliation $\mathcal{F}$; its homology $H^*(M/\mathcal{F})$ is the *basic cohomology* of $\mathcal{F}$. It can be viewed as the de Rham cohomology of the leaf space $M/\mathcal{F}$ even it does not admit any differentiable structure. If $\mathcal{F}$ is a fibration $M \longrightarrow B$, $H^*(M/\mathcal{F})$ is exactly the de Rham cohomology of B. The following question is natural: *in which sense the basic cohomology looks like the de Rham cohomology of a manifold* ? For instance *is $H^*(M/\mathcal{F})$ finite dimensional when M is compact* ? This is immediate for $H^0(M/\mathcal{F})$ and $H^1(M/\mathcal{F})$ which injects in $H^1(M)$. But in general this is not the case for $H^p(M/\mathcal{F})$ with $p \geq 2$ as it was shown by G.W. Schwarz [Sch] and E. Ghys [Ghy1] who gave an example of a real analytic foliation whose $H^2(M\mathcal{F})$ is infinite dimensional. But the answer is positive when the foliation is Riemannian.

5.1. The case of a Lie foliation

Suppose that $\mathcal{F}$ is a Lie G-foliation. Let $h : \pi_1(M) \longrightarrow G$ be the holonomy representation and $D : \widetilde{M} \longrightarrow G$ the developing map associated to $\mathcal{F}$. Denote by $\widetilde{\mathcal{F}}$ the pull-back of $\mathcal{F}$ to the universal covering $\widetilde{M}$. For every $\gamma \in \pi_1(M)$ the following diagramm is commutative:

$$(*) \qquad \begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma} & \widetilde{M} \\ D\downarrow & & \downarrow D \\ G & \xrightarrow{h(\gamma)} & G \end{array}.$$

The basic forms for $\mathcal{F}$ on M are the basic forms for $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ invariant by the action of $\pi_1(M)$. By the diagram $(*)$ the space $\Omega^*(M/\mathcal{F})$ is canonically isomorphic to the space of differential forms on G invariant by $\Gamma = h(\pi_1(M))$ and then (by continuity) to the space $\Omega^*_K(G)$ of differential forms on G invariant by the closure K of Γ in G. The two differential complexes $(\Omega^*(M/\mathcal{F}), d)$ and $(\Omega^*_K(G), d)$ are canonically isomorphic; so their homologies $H^*(M/\mathcal{F})$ and $H^*_K(G)$ are the same. In particular if the leaves of $\mathcal{F}$ are dense, $K = G$ and $H^*(M/\mathcal{F})$ is the cohomology $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G.

Let us give an explicite example where all the leaves are dense i.e. $K = G$. Let $X = \frac{\partial}{\partial x_0} + a_1\frac{\partial}{\partial x_1} + \ldots + a_n\frac{\partial}{\partial x_n}$ be a linear vector field on the torus $\mathbb{T}^{n+1}$. The orbits of X

define a one dimensional Lie $\mathbb{R}^n$-foliation $\mathcal{F}$. If the real numbers $1, a_1, \dots, a_n$ are linearly independent over $\mathbb{Q}$, the leaves of $\mathcal{F}$ are all dense. Then $H^*(\mathbb{T}^{n+1}/\mathcal{F})$ is the exterior algebra $\Lambda^*\mathbb{R}^n$ of $\mathbb{R}^n$.

The result is the same if $\mathcal{F}$ is a linear foliation on $\mathbb{T}^{m+n}$ with all leaves dense and defined by m vector field $X_1, \cdots, X_n$ which are linearly independent.

5.2. The case of a transversely parallelizable foliation

Suppose $\mathcal{F}$ is transversely parallelizable and let $F \hookrightarrow M \xrightarrow{\pi} W$ be its basic fibration. We know that there exists a simply connected Lie group G_0 such that the induced foliation $\mathcal{F}_0$ on each fibre F is a Lie G_0-foliation and the cocycle of the fibration π is with values in the group Diff$(F, \mathcal{F}_0)$. Let $\mathcal{G}_0$ be the Lie algebra of G_0. Then there exists a spectral sequence (E_r) (cf. [ESH]) with term:

$$E_2^{pq} = H^p(W, \mathcal{H}^q)$$

converging to $H^*(M/\mathcal{F})$. Here $\mathcal{H}^q$ is the locally constant presheaf on W with fibre $H^q(\mathcal{G}_0)$. As W is compact and $H^q(\mathcal{G}_0)$ is finite dimensional, the basic cohomology $H^*(M/\mathcal{F})$ is also finite dimensional. The vector space $H^n(M/\mathcal{F}) = E_2^{st}$ (where $s =$dim(W) and $t =$dim(G_0)) is trivial or isomorphic to $\mathbb{R}$; in the last case we say that $\mathcal{F}$ is *homologically orientable.* This property has an interesting geometric interpretation: *there exists a bundle-metric on M for which the leaves are minimal submanifolds if and only if $\mathcal{F}$ is homologically orintable.* (cf. [Mas]).

Let us compute an explicite example. Let GA be the Lie group of affine transformations $x \in \mathbb{R} \longmapsto ax + b \in \mathbb{R}$ where $a \in]0, +\infty[$ and $b \in \mathbb{R}$. It can be embedded in the group $G =$SL$(2, \mathbb{R})$ as follows:

$$\begin{pmatrix} x \longrightarrow ax + b \\ a > 0 \end{pmatrix} \in GA \longmapsto \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

There exists (cf. [EN3]) a compact manifold M equipped with a Lie G-foliation $\mathcal{F}$ with GA as the closure of its holonomy group. Then, the basic cohomology of $\mathcal{F}$ is the cohomology of differential forms on G invriant by GA. The quotient SL$(2, \mathbb{R})/GA$ is diffeomorphic to the circle $\mathbb{S}^1$. Then we have a spectral sequence:

$$E_2^{pq} = H^p(\mathbb{S}^1, \mathcal{H}^q(\mathcal{GA}))$$

converging to $H^*(M/\mathcal{F})$. Because the fibre bundle $GA \hookrightarrow \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathbb{S}^1$ is principal and GA connected, the action of $\pi_1(\mathbb{S}^1)$ on $H^q(\mathcal{GA})$ is trivial. So:

$$E_2^{pq} = H^p(\mathbb{S}^1) \otimes H^q(\mathcal{GA}).$$

As $\dim(\mathbb{S}^1) = 1$, this spectral sequence converges at the E_2 term, that is:

$$H^r(M/\mathcal{F}) = \bigoplus_{p+q=r} H^p(\mathbb{S}^1) \otimes H^q(\mathcal{GA}).$$

But $H^q(\mathcal{GA}) = \mathbb{R}$ for $q = 0, 1$ and 0 otherwise. So we get:

$$H^*(M/\mathcal{F}) = \begin{cases} \mathbb{R} & \text{if } * = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

5.3. The case of a Riemannian foliation

We will use the same idea as in the case of a transversely parallelizable foliation even if the leaf closures does not have the same dimension. For details see [EN2].

The topological space X, quotient of M by the leaf closures is not a manifold in general and the projection $\pi : M \longrightarrow X$ is not a fibration. However, locally the foliation $\mathcal{F}$ has a good behaviour (cf. [Hae]). It was proved in [EN2] that X admits a *good cover* i.e. an open cover $\mathcal{U} = \{U_i\}$ (which can be chosen finite $\{U_1, \cdots, U_k\}$ because X is compact) such that each intersection $U_{i_1} \cap \cdots \cap U_{i_p}$ is contractible. Then there exists a spectral sequence:

$$E_2^{pq} = H^p(\mathcal{U}, \mathcal{H}_b^q)$$

converging to $H^*(M/\mathcal{F})$. Here $\mathcal{H}_b^q$ is the preasheaf on X which associates to any open set U the vector space $H^q(\pi^{-1}(U)/\mathcal{F})$. If L is a leaf closure with minimal dimension contained in $\pi^{-1}(U)$, the injection $L \hookrightarrow \pi^{-1}(U)$ induces an isomorphism $H^q(\pi^{-1}(U)/\mathcal{F}) \simeq H^q(L/\mathcal{F})$. Since the foliation in L is with dense leaves, its basic cohomology is finite dimensional (the space of basic forms itself is finite dimensional) and so is $H^*(M/\mathcal{F})$ (because the cover $\mathcal{U}$ is finite). It was also proved in [EN2] that the basic cohomology is a *topological invariant* in the category of complete Riemannian foliations. (Any Riemannian foliation on a compact manifold is complete.)

To illustrate concretely these methods, let us give an example. Let $\mathcal{F}$ be the transversely Kählerian flow defined on $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ by the restriction of the holomorphic vector field:

$$Z = i\beta_1 \frac{\partial}{\partial z_1} + i\beta_2 \frac{\partial}{\partial z_2}$$

(cf. subsection 2.6) where β_1 and β_2 are real numbers such that $\frac{\beta_1}{\beta_2} \notin \mathbb{R}^-$. If $\beta_1 = \beta_2 = 1$, $\mathcal{F}$ is just the Hopf fibration:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \longrightarrow P^1(\mathbb{C}).$$

We will compute the basic cohomology of $\mathcal{F}$. If all the leaves of $\mathcal{F}$ are compact, the quotient $X = M/\mathcal{F}$ is an orbifold homeomorphic to the 2-sphere $\mathbb{S}^2$; then $H^*(M/\mathcal{F}) = H^*(\mathbb{S}^2)$. Suppose that the leaves are not closed. Then the leaf closures are diffeomorphic to the torus $\mathbb{T}^2$ except two of them L_1 and L_2 corresponding to the points $z_1 = 0$ and $z_2 = 0$. The space X is homeomorphic to $[0, 1]$. Take the open sets $U_1 = [0, 2/3[$ and $]1/3, 1]$; U_1, U_2 and the intersection $U_{12} = U_1 \cap U_2 =]1/3, 2/3[$ are contractible so $\mathcal{U} = \{U_1, U_2\}$ is a good cover of $[0, 1]$. Let $V_1 = \pi^{-1}(U_1)$, $V_2 = \pi^{-1}(U_2)$ and $V_{12} = \pi^{-1}(U_1 \cap U_2)$. It is easy to see that:

$$H^*(V_1/\mathcal{F}) = H^*(V_2/\mathcal{F}) = H^*(\text{point}) = \begin{cases} \mathbb{R} & \text{for } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^*(V_{12}/\mathcal{F}) = H^*(\mathbb{S}^1) = \begin{cases} \mathbb{R} & \text{for } * = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the cover $\mathcal{U}$ contains only two open sets, the spectral sequence is reduced to an exact sequence:

$$0 \longrightarrow H^0(\mathbb{S}^3) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(\mathbb{S}^3/\mathcal{F}) \longrightarrow 0 \longrightarrow \mathbb{R} \longrightarrow H^2(\mathbb{S}^3/\mathcal{F}) \longrightarrow 0.$$

Since $H^1(\mathbb{S}^3/\mathcal{F})$ injects in $H^1(\mathbb{S}^3) = 0$, $H^1(\mathbb{S}^2/\mathcal{F}) = 0$ and this exact sequence gives:

$$H^*(\mathbb{S}^3/\mathcal{F}) = \begin{cases} \mathbb{R} & \text{for } * = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

which is exaclty the de Rham cohomology of $P^1(\mathbb{C})$; in particular $H^*(\mathbb{S}^3/\mathcal{F})$ does not depend on β_1 and β_2. The following example is a generalization of this situation.

5.4. Flow of isometries

Suppose that $\dim(M) = n + 1$ and that $\mathcal{F}$ is defined by a locally free isometric action of the Lie group $\mathbb{R}$ (the action preserves a Riemannian metric g on M). This is always the case if M is simply connected (cf. [Ghy2]: a Riemannian flow on a simply connected manifold is an isometry flow). The image of $\mathbb{R}$ by the representation $\mathbb{R} \longrightarrow \mathrm{Isom}(M)$ which defines the action is a connected abelian subgroup of the compact group $\mathrm{Isom}(M)$ of isometries of Riemannian manifold (M, g); its closure is a compact abelian subgroup K (K is a torus). Then the de Rham cohomology of M is isomorphic to the cohomology of the complex $(\Omega_K(M), d)$ of differential forms on M invariant by the action of K. Let X be the fundamental vector field defining $\mathcal{F}$. It is easy to see that:

$$\Omega^*(M/\mathcal{F}) = \{\omega \in \Omega^*_K(M) : i_X\omega = 0\}.$$

For $r = 1, \cdots, n+1$ let $\phi : \Omega^r_K(M) \longrightarrow \Omega^{r-1}_K(M)$ be defined by $\phi(\omega) = (-1)^r i_X\omega$. The image of ϕ is $\Omega^{r-1}(M/\mathcal{F})$ and the sequence:

$$0 \longrightarrow \Omega^r(M/\mathcal{F}) \hookrightarrow \Omega^r_K(M) \xrightarrow{\phi} \Omega^{r-1}(M/\mathcal{F}) \longrightarrow 0$$

is exact. Then it gives a long exact cohomology sequence:

$$0 \longrightarrow H^0(M/\mathcal{F}) \longrightarrow H^0(M) \longrightarrow H^{-1}(M/\mathcal{F}) = 0 \longrightarrow$$

$$H^1(M/\mathcal{F}) \longrightarrow H^1(M) \longrightarrow H^0(M/\mathcal{F}) = \mathbb{R} \longrightarrow \cdots$$

$$\cdots \longrightarrow H^r(M/\mathcal{F}) \longrightarrow H^r(M) H^{r-1}(M/\mathcal{F}) \longrightarrow \cdots$$

If M is a homology sphere (i.e. it has the homology of a sphere), the integer n is necessarily even and the long exact sequence gives:

$$H^*(M/\mathcal{F}) = \begin{cases} \mathbb{R} & \text{if } r \text{ is even and } 0 \leq r \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The basic cohomology of $\mathcal{F}$ is exactly the de Rham cohomology of the n-dimensional complex projective space $P^n(\mathbb{C})$. An explicite example is the transversely Kählerian flow on $\mathbb{S}^{2n+1}$ induced by the holomorphic vector field in $\mathbb{C}^{N+1}$:

$$Z = \sum_{k=1}^{n+1} i\beta_k z_k \frac{\partial}{\partial z_k}.$$

For a good choice of the real numbers $\beta_1, \cdots, \beta_{n+1}$. For $\beta_1 = \cdots = \beta_{n+1} = 1$, this flow is just the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \longrightarrow P^n(\mathbb{C})$.

The study of the basic cohomology by Hodge theory was done by many authors. See for instance [Elk] where it appears as a particular case of the general theory of transversely elliptic operators on Riemannian foliations.

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