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Towards a Basic Index Theory

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In this note, we give and explain the statement of the Hodge decomposition theorem for a transversely elliptic operator on a Riemannian foliation proved in [Ek]. Such an operator is Fredholm and therefore admits an index. This brings us to formulate the following problem: can this index be computed in terms of (topological) transverse invariants like in the Atiyah-Singer Index theorem? Some examples of transversely elliptic operators whose indices are topologically invariant are given, in particular the basic Euler-Poincaré number and the signature of a Riemannian foliation.

All the objects considered are assumed to be of class C^∞. Let M be a manifold of dimension $d+n$; for simplicity M will be connected and orientable. Differential forms (and in particular functions) will take their values in the field of complex numbers $\mathbf{C}$. If α is a form, $\overline{\alpha}$ will denote its complex conjugate; we say that α is *real* if $\overline{\alpha} = \alpha$.

1. Preliminaries

1.1. Definition. *A codimension n foliation $\mathcal{F}$ on M is given by an open cover $\{U_i\}_{i\in I}$ and submersions $f_i : U_i \longrightarrow T$ over an n-dimensional manifold T and, for each $U_i \cap U_j \neq \emptyset$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \longrightarrow f_j(U_i \cap U_j)$ such that*

$$f_j = \gamma_{ij} \circ f_i. \tag{1}$$

Every fibre of f_i is called a *plaque* of the foliation. Condition (1) says that, on the intersection $U_i \cap U_j$, the plaques defined respectively by f_i and f_j are the "same". The manifold M is decomposed into a family of disjoint immersed connected submanifolds of dimension d; each of these submanifolds is called a *leaf* of $\mathcal{F}$. We say that $\mathcal{F}$ is *transversely orientable* if T can be given an orientation which is preserved by all the γ_{ij}.

We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\Gamma(\mathcal{F})$ the space of its global sections *i.e.* vector fields tangent to $\mathcal{F}$. We say that a differential form α is *basic* if it satisfies $i_X\alpha = L_X\alpha = 0$ for every $X \in \Gamma(\mathcal{F})$. (This notion extends in obvious way to differential forms with values in a vector space.) Basic forms define a differential complex $(\Omega^*(M/\mathcal{F}), d)$; its homology $H^*(M/\mathcal{F})$ is called the *basic cohomology* of $\mathcal{F}$. A *basic function* is a function constant on the leaves; such functions form an algebra which we denote by A. The quotient $\nu\mathcal{F} = TM/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. A vector field $Y \in \chi(M)$ is said to be *foliated* if, for every $X \in \Gamma(\mathcal{F})$, the bracket $[X,Y] \in \Gamma(\mathcal{F})$. The space $\chi(M,\mathcal{F})$ of foliated vector fields is an algebra and a A-module for which $\Gamma(\mathcal{F})$ is an ideal. The quotient $\chi(M/\mathcal{F}) = \chi(M,\mathcal{F})/\Gamma(\mathcal{F})$ is called the algebra of *basic vector fields* on M; it is also a A-module.

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In all this paper a system of local coordinates means coordinates $(x_1, \cdots, x_d, y_1, \cdots, y_n)$ on an open set U on which the foliation is trivial and defined by the equations $dy_1 = \cdots = dy_n = 0$. If $\mathcal{F}$ is transversely holomorphic (see definition (1.2.2) below) $y_1, \cdots, y_n$ will be complex numbers.

1.2. Definition. *A transverse structure to $\mathcal{F}$ is a geometric structure on T invariant by all the local diffeomorphisms γ_{ij}.*

A transverse structure can be considered as a geometric structure on the leaf space $M/\mathcal{F}$ (which is not a manifold in general). Let us give some examples.

(1.2.1) – Suppose that T is a Riemannian manifold and all the γ_{ij} are isometries; then $\mathcal{F}$ is said to be *Riemannian.* This means that the normal bundle $\nu\mathcal{F}$ is equipped with a Riemannian metric which is "invariant along the leaves".

(1.2.2) – Suppose that T is a complex manifold and all the γ_{ij} are biholomorphic maps; then we say that $\mathcal{F}$ is *transversely holomorphic.* In that case, any transversal to $\mathcal{F}$ inherts a complex structure.

(1.2.3) – Suppose that T is a Hermitian manifold and all the γ_{ij} preserve the Hermitian structure; then we say that $\mathcal{F}$ is *Hermitian.* (The γ_{ij} are in particular biholomorphic maps and isometries.) The complexified normal bundle $\nu = \nu\mathcal{F} \otimes_{\mathbf{R}} \mathbf{C}$ is equipped with a Hermitian metric "invariant along the leaves".

(1.2.4) – If T is a Kählerian manifold and all the γ_{ij} preserve the Kähler structure we say that $\mathcal{F}$ is *transversely Kählerian.* In particular such a foliation is Hermitian. This is equivalent to the existence of a Hermitian metric h on the normal bundle $\nu\mathcal{F}$ which can be writen in a transverse local system of coordinates $(y_1, \cdots, y_n)$ in the form $h = \sum_{k,\ell=1}^{n} h_{k\ell}(y) dy_k \otimes d\overline{y}_\ell$ such that its skew-symmetric part $\omega = \frac{i}{2} \sum_{k,\ell=1}^{n} h_{k\ell}(y) dy_k \wedge d\overline{y}_\ell$ is closed (ω is a basic 2-form called the *Kähler basic form* of $\mathcal{F}$.) It is easy to see that every Riemannian transversely orientable foliation of codimension 2 can be given a structure of a complex codimension 1 transversely Kählerian foliation.

(1.2.5) – If T is a Spin manifold and all the γ_{ij} preserve the Spin structure, we say that $\mathcal{F}$ is a *Spin foliation.* Of course, a foliation by points on a Spin manifold is a Spin foliation.

(1.2.6) – If T is parallelizable and the parallelism is preserved by all the γ_{ij}, we say that $\mathcal{F}$ is *transversely parallelizable.* This is equivalent to say that the A-module $\chi(M/\mathcal{F})$ is free of rank n.

2. Foliated bundles

Let $\mathcal{P} : G \hookrightarrow P \stackrel{\iota}{\longrightarrow} M$ be a principal bundle with structural group $G \subset \mathrm{GL}(n, \mathbf{C})$. The group G acts on P on the right and on its Lie algebra $\mathcal{G}$ by the adjoint representation Ad *i.e.*, for $g \in G$ and $X \in \mathcal{G}$, $Ad_g(X) = gXg^{-1}$. Denote by $\mathcal{V}$ the vector bundle whose fibre V_z at a point $z \in P$ is the tangent space at z of the fibre of $\mathcal{P}$.

Let $\mathcal{E} : E \longrightarrow M$ be a complex vector bundle defined by a cocycle $\{U_i, g_{ij}, G\}$ where $\{U_i\}$ is an open cover of M and $g_{ij} : U_i \cap U_j \longrightarrow G \subset \mathrm{GL}(n, \mathbf{C})$ are the transition functions. To such a vector bundle we can always associate a principal bundle $G \longrightarrow P \longrightarrow M$ whose fibre is the group G and the transition functions are exactly the g_{ij} (viewed as translations on G).

There are different ways to define a connection on a vector bundle $\mathcal{E}$: on $\mathcal{E}$ directly or by using the associated principal bundle. We shall make use of all these possibilities.

<u>First definition</u>

A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ of TP such that

(a) - for every $z \in \mathcal{P} : T_zP = V_z \oplus H_z$ (H_z is the fibre of $\mathcal{H}$ at z),

(b) - for every $g \in G$ and every $z \in P : H_{zg} = (R_g)_* H_z$ where R_g is the right action of g on P.

<u>Second definition</u>

A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ given by the kernel of a G-invariant 1-form ξ on P with values in $\mathcal{G}$. The G-invariance of ξ means: $(R_g)^*(\xi) = Ad_{g^{-1}}(\xi)$ *i.e.* for $z \in P$, $X \in T_zP$ and $g \in G$, $\xi_{zg}((R_g)_*(X)) = g^{-1}\xi_z(X)g$ (where $(R_g)_*$ is the derivative of R_g).

Third definition

A *linear connection* on the vector bundle $\mathcal{E}$ is a map $\nabla : (X,\alpha) \in \chi(M) \times C^\infty(\mathcal{E}) \longmapsto \nabla_X\alpha \in C^\infty(\mathcal{E})$ satisfying the following properties

(c) - ∇ is $C^\infty(M)$-linear on the first factor, that is, for $\alpha \in C^\infty(\mathcal{E})$, $X, Y \in \chi(M)$ and functions $f, g \in C^\infty(M)$, we have $\nabla_{fX+gY}\alpha = f\nabla_X\alpha + g\nabla_Y\alpha$,

(d) - for $\alpha \in C^\infty(\mathcal{E})$, $X \in \chi(M)$ and $f \in C^\infty(M)$ we have $\nabla_X(f\alpha) = f\nabla_X\alpha + (Xf)\alpha$ where Xf is the derivative of the function f in the direction of the vector field X.

In fact, the map ∇ is the *covariant derivative* of the connection. The *curvature* of this connection is the 2-form $\mathcal{R}$ with values in $\mathrm{End}(\mathcal{E})$ (the space of endomorphisms of $\mathcal{E}$) defined by

$$\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Now, suppose that we are given a connection $\mathcal{H}$ (like in the first definition or the second one) on the principal bundle $\mathcal{P}$. It is easy to see that the restriction of ι_* (the derivative of ι) to H_z is an isomorphism onto $T_{\iota(z)}M$. Let $\tau = \iota_*^{-1}(T\mathcal{F})$. We say that $\mathcal{P}$ is *foliated* if τ is integrable. In this case, τ defines a foliation $\widetilde{\mathcal{F}}$ on P such that

(e) - $\dim(\widetilde{\mathcal{F}}) = \dim(\mathcal{F})$,

(f) - $\widetilde{\mathcal{F}}$ is invariant under the action of G.

2.1. Definition. *We say that the connection $\mathcal{H}$ is <u>basic</u> if ξ is basic; a foliated bundle $\mathcal{E}$ is said to be a <u>$\mathcal{F}$-bundle</u> if it admits a basic connection. We say that $\mathcal{E}$ is a $\mathcal{F}$-bundle if the associated principal bundle is a $\mathcal{F}$-bundle.*

A vector bundle $\mathcal{E}$ with a linear connection is foliated if, and only if, its curvature form $\mathcal{R}$ satisfies $\mathcal{R}(X,Y) = 0$ for $X, Y \in \Gamma(\mathcal{F})$; $\mathcal{E}$ is a $\mathcal{F}$-bundle if, and only if, $i_X\mathcal{R} = 0$ for $X \in \Gamma(\mathcal{F})$ (*cf.* [KT1]).

The foliation $\mathcal{F}_{\widehat{E}}$ on $\widehat{E} = P \times \mathbf{C}^n$ whose leaves are of the form (leaf of $\widetilde{\mathcal{F}}$) $\times$ (point of $\mathbf{C}^n$) is invariant by the diagonal action of G ; so it induces a foliation $\mathcal{F}_E$ on $E = P \times_G \mathbf{C}^n$.

A $\mathcal{F}$-*morphism* $\varphi : (\mathcal{E},\xi) \longrightarrow (\mathcal{E}',\xi')$ between two $\mathcal{F}$-bundles is a morphism of vector bundles such that $\xi = \varphi^*(\xi')$.

The collection of $\mathcal{F}$-bundles and $\mathcal{F}$-morphisms on M is a category. So we can define the group $K(M,\mathcal{F})$ of *foliated K-theory* as in the classical case.

2.2. Examples

(2.2.1) – Suppose that we are given a Riemann metric on M. Let $T\mathcal{F}^\perp$ be the subbundle of TM orthogonal to $\mathcal{F}$ and $\Gamma(T\mathcal{F}^\perp)$ the space of its sections. Every $X \in \chi(M)$ can be uniquely writen $X = X_\mathcal{F} + X_\nu$ where $X_\mathcal{F} \in \Gamma(\mathcal{F})$ and $X_\nu \in \Gamma(T\mathcal{F}^\perp)$. Let $\pi : TM \longrightarrow \nu\mathcal{F}$ be the canonical projection. For every section Y of the bundle $\nu\mathcal{F}$ we denote by $\widetilde{Y}$ a vector field on M which projects on Y. For every $X_\mathcal{F} \in \Gamma(\mathcal{F})$ and every $Y \in C^\infty(\nu\mathcal{F})$, $\pi([X_\mathcal{F},\widetilde{Y}])$ is independant of the choice of the $\widetilde{Y}$. Let $\widehat{\nabla}$ be any linear connection on $\nu\mathcal{F}$. So we can define a linear connection on the vector bundle $\nu\mathcal{F}$

$$\nabla : \chi(M) \times C^\infty(\nu\mathcal{F}) \longrightarrow C^\infty(\nu\mathcal{F}),$$

by

$$\nabla_X Y = \pi([X_\mathcal{F},\widetilde{Y}]) + \widehat{\nabla}_{X_\nu}Y.$$

It is called a *Bott connection* of $\mathcal{F}$. A simple calculation, using the integrability of the subbundle $T\mathcal{F}$ and the Jacobi identity, shows that the curvature form $\mathcal{R}$ satifies the equation $\mathcal{R}(X,Y) = 0$ for $X, Y \in \Gamma(\mathcal{F})$; this implies that the vector bundle $\nu\mathcal{F}$ is foliated.

(2.2.2) – Every flat vector bundle $\mathcal{E} : E \longrightarrow M$ (*i.e.* the transition functions of $\mathcal{E}$ are constant) is a $\mathcal{F}$-bundle.

(2.2.3) – Let $\mathcal{E} : E \longrightarrow M$ be a $\mathcal{F}$-bundle. Then the dual bundle $\mathcal{E}^*$ and all of its exterior and symmetric powers $\Lambda^*\mathcal{E}^*$ and $\mathcal{S}^*\mathcal{E}^*$ are $\mathcal{F}$-bundles; also $\mathcal{H}^2\mathcal{E} = \{\text{Hermitian forms on } \mathcal{E}\}$ is a $\mathcal{F}$-bundle.

3. Transversely Elliptic Operators

Let $\mathcal{E}$ be a $\mathcal{F}$-bundle and denote by $C^\infty(\mathcal{E})$ the space of its global sections; let ∇ denotes the covariant derivative $\chi(M) \times C^\infty(\mathcal{E}) \longrightarrow C^\infty(\mathcal{E})$ associated to the connection.

3.1. Definition. *We say that a section $\alpha \in C^\infty(\mathcal{E})$ is basic if it satisfies the condition $\nabla_X \alpha = 0$ for every $X \in \Gamma(\mathcal{F})$.*

The space $C^\infty(\mathcal{E}/\mathcal{F})$ of basic sections of $\mathcal{E}$ is an A-module. Let $\mathcal{E}$ and $\mathcal{E}'$ be two $\mathcal{F}$-bundles with ranks respectively N and N'.

3.2. Definition. *A basic differential operator of order $m \in \mathbf{N}$ from $\mathcal{E}$ to $\mathcal{E}'$ is a linear map $D : C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}'/\mathcal{F})$ such that, on local coordinates $(x_1, \ldots, x_d, y_1, \ldots y_n)$, D has the expression*

$$D = \sum_{|s| \leq m} a_s(y) \frac{\partial^{|s|}}{\partial y_1^{s_1} \ldots \partial y_n^{s_n}} \tag{2}$$

where $s = (s_1, \ldots, s_n) \in \mathbf{N}^n$, $|s| = s_1 + \cdots + s_n$ and a_s are $N \times N'$-matrices whose coefficients are basic functions.

The *principal symbol* of D at the point z and the covector $\zeta \in T_z^*M$ is the linear map $\sigma(D)(z, \zeta) : E_z \longrightarrow E'_z$ defined by

$$\sigma(D)(z, \zeta)(\eta) = \sum_{|s|=m} \zeta_1^{s_1} \cdot \cdots \cdot \zeta_n^{s_n} a_s(y)(\eta).$$

We say that D is *transversely elliptic* if $\sigma(D)(z, \zeta)$ is an isomorphism for every $z \in M$ and every transverse covector ζ different from 0. If $\mathcal{F}$ is Riemannian, its conormal bundle $\nu^*\mathcal{F}$ is a $\mathcal{F}$-bundle and is equipped with a foliation $\mathcal{F}^*$. If in addition M is compact, $\sigma(D)(z, \zeta)$ defines an element $[D]$ in the group $K(\nu^*\mathcal{F}, \mathcal{F}^*)$.

A *Hermitian metric* on $\mathcal{E}$ is a positive definite section h of $\mathcal{H}^2\mathcal{E}$. If h is basic we say that $\mathcal{F}$ is a *Hermitian $\mathcal{F}$-bundle*. If $(\mathcal{E}, h)$ is a Hermitian $\mathcal{F}$-bundle and $D : C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}/\mathcal{F})$ is a basic operator of order $m = 2\ell$ we can define a quadratic form A on $\mathcal{E}$ by

$$A_z(\eta) = (-1)^\ell h\big(\sigma(D)(z, \zeta)(\eta), \eta\big)$$

where $\eta \in E_z$. We say that D is *strongly transversely elliptic* if A is positive definite for every $z \in M$ and every transverse covector ζ different from zero. Of course, a strongly transversely elliptic operator is transversely elliptic.

Let $\{\mathcal{E}^r\}$ $(r \in \{0, \cdots, n\})$ be a sequence of Hermitian $\mathcal{F}$-bundles and basic operators $D_r : C^\infty(\mathcal{E}^r/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}^{r+1}/\mathcal{F})$ such that the sequence

$$0 \longrightarrow C^\infty(\mathcal{E}^0/\mathcal{F}) \xrightarrow{D_0} \cdots \xrightarrow{D_{r-1}} C^\infty(\mathcal{E}^r/\mathcal{F}) \xrightarrow{D_r} C^\infty(\mathcal{E}^{r+1}/\mathcal{F}) \xrightarrow{D_{r+1}} \cdots \xrightarrow{D_{n-1}} C^\infty(\mathcal{E}^n/\mathcal{F}) \longrightarrow 0 \tag{$\mathcal{C}$}$$

is a differential complex. Let $\sigma_r = \sigma(D_r)(z, \zeta) : E_z^r \longrightarrow E_z^{r+1}$ denotes the principal symbol of D_r at the point $z \in M$ and the transverse covector ζ. We say that the complex $(\mathcal{C})$ is *transversely elliptic* if the sequence

$$0 \longrightarrow E_z^0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{r-1}} E_z^r \xrightarrow{\sigma_r} E_z^{r+1} \xrightarrow{\sigma_{r+1}} \cdots \xrightarrow{\sigma_{n-1}} E_z^n \longrightarrow 0$$

is exact for every $z \in M$ and every non zero transverse covector ζ. On each $C^\infty(\mathcal{E}^r/\mathcal{F})$ we can define an inner product given by the formula (3) below. Let D^* be the formal of D which is a basic operator from $C^\infty(\mathcal{E}^{r+1}/\mathcal{F})$ to $C^\infty(\mathcal{E}^r/\mathcal{F})$. Then $L_r = DD^* + D^*D$ is a selfadjoint operator on $C^\infty(\mathcal{E}^r/\mathcal{F})$. We can easily show that the differential complex $(\mathcal{C})$ is transversely elliptic if, and only if, for every $r \in \{0, \cdots, n\}$, L_r is strongly transversely elliptic.

From now on we suppose that M is compact and connected. Assume that the foliation $\mathcal{F}$ is Riemannian transversely oriented. Let

$$G = \mathrm{SO}(n) \longrightarrow M^{\#} \xrightarrow{p} M$$

be the principal bundle of the orthonormal direct frames transverse to $\mathcal{F}$. Then, the foliation $\mathcal{F}$ lifts to a transversely parallelizable foliation $\mathcal{F}^{\#}$ on $M^{\#}$ of the same dimension and invariant under the action of the group G. Moreover, the leaf closures of $\mathcal{F}^{\#}$ are the fibres of a locally trivial fibration $F \longrightarrow M^{\#} \longrightarrow W$ where W is a compact manifold called the *basic manifold* of $\mathcal{F}$ (*cf.* [Mo]). Let $\mathcal{E}^{\#}$ be the pullback by p of the bundle $\mathcal{E}$; then $\mathcal{E}^{\#}$ is a G-bundle and a Hermitian $\mathcal{F}^{\#}$-bundle with respect to a Hermitian metric $h^{\#}$. The basic sections of $\mathcal{E}$ are canonically identified to basic sections of $\mathcal{E}^{\#}$ which are invariant under the action of G. In particular, if $f : M \longrightarrow \mathbf{C}$ is a basic function, $f \circ p$ is a basic function on $M^{\#}$ (with respect to $\mathcal{F}^{\#}$); moreover $f \circ p$ is invariant by the action of G. Because $f \circ p$ is continuous, it is constant on the leaf closures of $\mathcal{F}^{\#}$, so it induces a G-invariant C^{∞} function on the basic manifold W. We can prove, by the converse process, that any G-invariant C^{∞} function on the basic manifold W defines a C^{∞} basic function on M; so, the algebra A of basic functions on M is canonically isomorphic to the algebra $A_G(W)$ of functions on W invariant by G. The transverse metric on $M^{\#}$ (which makes $\mathcal{F}^{\#}$ Riemannian) induces a Riemannian metric on W for which G acts by isometries. Let μ be the measure on W associated to this metric (μ is a volume form if W is orientable, otherwise it is just a density).

On $C^{\infty}(\mathcal{E}/\mathcal{F})$ we define an inner product as follows. Let α and β be two elements of $C^{\infty}(\mathcal{E}/\mathcal{F})$. The function $z \in M \longmapsto h_z(\alpha(z), \beta(z)) \in \mathbf{C}$ is basic; so it defines a G-invariant function $\Theta(\alpha, \beta)$ on W. We set

$$\langle \alpha, \beta \rangle = \int_W \Theta(\alpha, \beta)(w) d\mu(w). \tag{3}$$

For any transversely elliptic operator D from a Hermitian $\mathcal{F}$-bundle $\mathcal{E}$ to a Hermitian $\mathcal{F}$-bundle $\mathcal{E}'$, denote by $N(D)$ its kernel D and $R(D)$ its range. Let D^* be the formal adjoint of D; D^* is a basic operator from $\mathcal{E}'$ to $\mathcal{E}$ and it is transversely elliptic.

3.3. Theorem. *The kernel $N(D)$ of D is finite dimensional, the range $R(D^*)$ of D^* is closed and finite codimensional and we have an orthogonal decomposition*

$$C^{\infty}(\mathcal{E}/\mathcal{F}) = N(D) \oplus R(D^*). \tag{4}$$

The proof of this theorem is very long; it can be found in [Ek]. This theorem says in particular that the operator D is Fredholm and then it has an index defined as usual by the formula

$$\mathrm{ind}_{\mathcal{F}}(D) = \dim \mathrm{Ker} D - \dim \mathrm{Ker} D^* \in \mathbf{Z}. \tag{5}$$

3.4. Problem. *Compute this integer in terms of invariants of the bundles $\mathcal{E}$ and $\mathcal{E}'$ and transverse topological invariants of $\mathcal{F}$. More precisely, is there an Atiyah-Singer Index Theorem for a transversely elliptic operator on a Riemannian foliation on a compact manifold?*

4. Examples

4.1. The basic de Rham complex

We suppose as in Theorem 3.3 that $\mathcal{F}$ is Riemannian of codimension n, transversely oriented and that M is compact. For every $r \in \{0, \cdots, n\}$, let $\mathcal{E}^r$ denotes the vector bundle $\Lambda^r(\nu^*\mathcal{F})$. Then $\mathcal{E}^r$ is a Hermitian $\mathcal{F}$-bundle. Its basic sections are exactly the basic forms of degree r which form a vector space denoted

$\Omega^r(M/\mathcal{F})$. The exterior differential $d : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^{r+1}(M/\mathcal{F})$ is a basic differential operator of order 1. The differential complex

$$0 \longrightarrow \Omega^0(M/\mathcal{F}) \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} \Omega^r(M/\mathcal{F}) \stackrel{d}{\longrightarrow} \Omega^{r+1}(M/\mathcal{F}) \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} \Omega^n(M/\mathcal{F}) \longrightarrow 0 \tag{6}$$

is called the *basic de Rham complex* of $\mathcal{F}$; its homology is the basic cohomology $H^*(M/\mathcal{F})$ we have already defined in section 1.

To make things more simple, we suppose that $\mathcal{F}$ is *homologically orientable*, that is, the vector space $H^n(M/\mathcal{F})$ is non trivial, so it is necessarily one dimensional (*cf.* [EH]). This condition is equivalent to the existence of a (real) volume form on the leaves χ which is $\mathcal{F}$-relatively closed, that is, $d\chi(X_1, \cdots, X_d, Y) = 0$ for $X_1, \cdots, X_d \in \Gamma(\mathcal{F})$ (*cf.* [Ma]). (In that case, we can complete the transverse metric by a Riemannian metric along the leaves to obtain a Riemann metric on the whole manifold for which the leaves are minimal and χ is associated to this metric.) This hypothesis will enable one to define an inner product on $\Omega^r(M/\mathcal{F})$ without using the basic manifold W. As in the classical case we define the Hodge star operator

$$* : \Omega^*(M/\mathcal{F}) \longrightarrow \Omega^*(M/\mathcal{F}) \tag{7}$$

in the following way. Let U be an open set on which the foliation is trivial. Let $\theta_1, \cdots, \theta_n$ be real 1-forms such that $(\theta_1, \cdots, \theta_n)$ is an orthonormal basis of the free module $\Omega^1(U/\mathcal{F})$ (over the algebra of basic functions on U). Then define $*$ by

$$*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}) = \varepsilon\theta_{j_1} \wedge \cdots \wedge \theta_{j_{n-r}}$$

where $\{j_1, \cdots, j_{n-r}\}$ is the increasing complementary sequence of $\{i_1, \cdots, i_r\}$ in $\{1, \cdots, n\}$ and ε is the signature of the permutation $\{i_1, \cdots, i_r, j_1, \cdots, j_{n-r}\}$. A straightforward calculation shows that $*$ satisfies the identity $** = (-1)^{r(n-r)}\mathrm{id}$. On $\Omega^r(M/\mathcal{F})$ we define a Hermitian product by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \overline{*\beta} \wedge \chi. \tag{8}$$

Then it is easy to see that the operator $\delta : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^{r-1}(M/\mathcal{F})$ defined by the formula $\delta = (-1)^{n(r-1)-1} * d*$ is the formal adjoint of $d : \Omega^{r-1}(M/\mathcal{F}) \longrightarrow \Omega^r(M/\mathcal{F})$ *i.e.* for every $\alpha \in \Omega^{r-1}(M/\mathcal{F})$ and every $\beta \in \Omega^r(M/\mathcal{F})$ we have $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$. Indeed

$$\begin{aligned} d(\alpha \wedge *\beta \wedge \chi) &= d\alpha \wedge *\beta \wedge \chi + (-1)^{r-1}\alpha \wedge d(*\beta) \wedge \chi + (-1)^{n-1}\alpha \wedge *\beta \wedge d\chi \\ &= d\alpha \wedge *\beta \wedge \chi + (-1)^{(2-r)(r-1)-1}\alpha \wedge *(\delta\beta) \wedge \chi + (-1)^{n-1}\alpha \wedge *\beta \wedge d\chi \\ &= d\alpha \wedge *\beta \wedge \chi - \alpha \wedge *(\delta\beta) \wedge \chi + (-1)^{n-1}\alpha \wedge *\beta \wedge d\chi. \end{aligned}$$

Integrating the two members and using the fact that χ is $\mathcal{F}$-relatively closed, we obtain the desired equality. In the more general case in which the leaves are not minimal, the formula for the adjoint has a correction term involving the mean curvature of the foliation (*cf.* [Al], [To] or [PR]). Let $\Delta_b : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^r(M/\mathcal{F})$ be the operator $\Delta_b = \delta d + d\delta$; Δ_b is selfadjoint; it is called the *basic Laplacian* (on the basic r-forms); a simple calculation, using local coordinates, proves that Δ_b is strongly transversely elliptic and therfore the complex (6) is transversely elliptic. Let

$$\mathcal{H}^r(M/\mathcal{F}) = \mathrm{Ker}\Delta_b = \{\alpha \in \Omega^r(M/\mathcal{F}) : d\alpha = 0 \text{ and } \delta\alpha = 0\}.$$

An element of $\mathcal{H}^r(M/\mathcal{F})$ is called a *basic harmonic form* (of degree r). Then, applying Theorem 3.3, we obtain

(i) $\dim \mathcal{H}^r(M/\mathcal{F}) < +\infty$;
(ii) *we have orthogonal decompositions*

$$\Omega^r(M/\mathcal{F}) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(\Delta_b) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(d) \oplus R(\delta). \tag{9}$$

As a consequence, the basic cohomology $H^r(M/\mathcal{F})$ is finite dimensional and is represented by $\mathcal{H}^r(M/\mathcal{F})$. Moreover the Hermitian map $(\alpha,\beta) \in \Omega^r(M/\mathcal{F}) \times \Omega^{n-r}(M/\mathcal{F}) \longmapsto \int_M \alpha \wedge \overline{\beta} \wedge \chi \in \mathbf{C}$ induces a non degenerate pairing

$$Q : H^r(M/\mathcal{F}) \times H^{n-r}(M/\mathcal{F}) \longrightarrow \mathbf{C} \tag{10}$$

i.e. the basic cohomology $H^*(M/\mathcal{F})$ satisfies the Poincaré duality.

These results were originally obtained by B. Reinhart in [Re] without the assumption that $H^n(M/\mathcal{F})$ is nonzero. But in 1981, Y. Carrière [Ca] constructed an example of a foliation whose basic cohomology does not satisfy Poincaré duality; this makes false a part of Reinhart's theorem. One year later F. Kamber and P. Tondeur [KT2] proved the same result as B. Reinhart for Riemannian foliations with minimal leaves (this is equivalent by [Ma] to $H^n(M/\mathcal{F}) \neq \{0\}$). We can easily observe that, with this hypothesis, Reinhart's proof is still valid. The general case (without any assumption) was completely established in [EH]. But as we have already pointed, these results are direct consequences of Theorem 3.3.

If $n = 2k = 4\ell$, Q defines a non degenerate quadratic form on $H^k(M/\mathcal{F})$; its signature is called the *signature* of $\mathcal{F}$ and is denoted $\mathrm{Sign}(\mathcal{F})$.

Now let $\mathcal{E}$ and $\mathcal{E}'$ be the vector bundles

$$\mathcal{E} = \bigoplus_{i\geq 0} \Lambda^{2i}(\nu^*\mathcal{F}) \quad \text{and} \quad \mathcal{E}' = \bigoplus_{i\geq 0} \Lambda^{2i+1}(\nu^*\mathcal{F}).$$

They are Hermitian $\mathcal{F}$-bundles and we have

$$C^\infty(\mathcal{E}/\mathcal{F}) = \bigoplus_{i\geq 0} \Omega^{2i}(M/\mathcal{F}) \quad \text{and} \quad C^\infty(\mathcal{E}'/\mathcal{F}) = \bigoplus_{i\geq 0} \Omega^{2i+1}(M/\mathcal{F}).$$

The operator $d+\delta : C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}'/\mathcal{F})$ is basic and transversely elliptic, then it is a Fredholm operator. Its index

$$\mathrm{ind}_{\mathcal{F}}(d+\delta) = \sum_{i=0}^{n} (-1)^i \dim H^i(M/\mathcal{F}). \tag{11}$$

is the *basic Euler-Poincaré number* $\chi(M/\mathcal{F})$ of the foliation $\mathcal{F}$. As in the classical case, it is an obstruction to the existence of a nonsingular foliated vector field transverse to $\mathcal{F}$ (*cf.* [BPR]).

4.2. The basic Dolbeault complex

We suppose that $\mathcal{F}$ is Hermitian and, for simplicity, homologically orientable. Let ν be the complexified normal bundle $\nu\mathcal{F} \otimes_{\mathbf{R}} \mathbf{C}$ of $\nu\mathcal{F}$. Let J be the automorphism of ν associated to the complex structure; J satistfies $J^2 = -\mathrm{id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles respectively denoted ν^{10} and ν^{01}. We have a splitting $\nu = \nu^{10} \oplus \nu^{01}$ which gives rise to a decomposition

$$\Lambda^r \nu^* = \bigoplus_{p+q=r} \Lambda^{p,q}$$

where $\Lambda^{p,q} = \Lambda^p \nu^{10^*} \otimes \Lambda^q \nu^{01^*}$. Basic sections of $\Lambda^{p,q}$ are called *basic forms of type* (p,q). They form a vector space denoted $\Omega^{p,q}(M/\mathcal{F})$. We have

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F}).$$

The exterior differential decomposes into a sum of two operators

$$\partial : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p+1,q}(M/\mathcal{F}) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p,q+1}(M/\mathcal{F})$$

as in the classical case of a complex manifold. We have $\overline{\partial}^2 = 0$; so we obtain a differential complex

$$0 \longrightarrow \Omega^{p,0}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,q}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n}(M/\mathcal{F}) \longrightarrow 0 \tag{12}$$

called the *basic Dolbeault complex* of $\mathcal{F}$; its homology $H^{p,q}(M/\mathcal{F})$ is the *basic Dolbeault cohomology* of the foliation $\mathcal{F}$: even though the leaf space is bad, it can be considered as a "complex manifold" whose Dolbeault cohomology is $H^{p,*}(M/\mathcal{F})$!

The star operator defined in (7) induces an isomorphism $* : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{n-q,n-p}(M/\mathcal{F})$. Moreover the restriction of the operator δ to the space $\Omega^{p,q}(M/\mathcal{F})$ decomposes into a sum of two operators $\delta' = - * \overline{\partial} *$ and $\delta'' = - * \partial *$ respectively of types $(-1,0)$ and $(0,-1)$. We can easily verify that δ'' is the formal adjoint of $\overline{\partial}$ for the inner product (8). Then the operator $\Delta_b'' = \delta''\overline{\partial} + \overline{\partial}\delta''$ is selfadjoint; a simple computation in local coordinates, like for the basic Laplacian, shows that Δ_b'' is strongly transversely elliptic and that the complex (12) is transversely elliptic. Let

$$\mathcal{H}^{p,q}(M/\mathcal{F}) = \mathrm{Ker}\Delta_b'' = \{\alpha \in \Omega^{p,q}(M/\mathcal{F}) : \overline{\partial}\alpha = 0 \text{ and } \delta''\alpha = 0\}.$$

Applying Theorem 3.3, we obtain

(i) dim $\mathcal{H}^{p,q}(M/\mathcal{F}) < +\infty$;

(ii) *we have orthogonal decompositions*

$$\Omega^{p,q}(M/\mathcal{F}) = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\Delta_b'') = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\overline{\partial}) \oplus R(\delta''). \tag{13}$$

Consequently, the basic Dolbeault cohomology $H^{p,q}(M/\mathcal{F})$ is finite dimensional and is represented by $\mathcal{H}^{p,q}(M/\mathcal{F})$. Moreover the star operator induces an unitary isomorphism (of real vector spaces) $\overline{*} : \alpha \in \mathcal{H}^{p,q}(M/\mathcal{F}) \longmapsto \overline{*\alpha} \in \mathcal{H}^{n-p,n-q}(M/\mathcal{F})$ and then an isomorphism

$$\overline{*} : H^{p,q}(M/\mathcal{F}) \longrightarrow H^{n-p,n-q}(M/\mathcal{F}) \tag{14}$$

i.e. the basic Dolbeault cohomology $H^{*,*}(M/\mathcal{F})$ satisfies the Serre duality.

Suppose now that $\mathcal{F}$ is transversely Kählerian with Kähler form ω (it is a basic differential form of degree 2; it is closed and non degenerate). In this case, we can prove that $\Delta_b = 2\Delta_b''$. Because of the decomposition

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F}),$$

every basic differential r-form can be uniquely writen $\alpha = \sum_{p+q=r} \alpha_{pq}$ where $\alpha_{pq} \in \Omega^{p,q}(M/\mathcal{F})$. Then we have the following assertions.

(iii) *α is Δ_b-harmonic if, and only if, each component α_{pq} is Δ_b''-harmonic. So we have a direct decomposition*

$$H^r(M/\mathcal{F}) = \bigoplus_{p+q=r} H^{p,q}(M/\mathcal{F}). \tag{15}$$

(iv) *The complex conjugacy induces an isomorphism (of real vector spaces)*

$$H^{p,q}(M/\mathcal{F}) \simeq H^{q,p}(M/\mathcal{F}).$$

(v) *For every odd* $r \in \{0, \cdots, 2n\}$, *the dimension of the space* $H^r(M/\mathcal{F})$ *is even. In particular if* $n = 1$ *we have* $b_1(M/\mathcal{F}) = 2\dim\, H^{01}(M/\mathcal{F})$.

The integer $\dim\, H^{01}(M/\mathcal{F})$ will be denoted $g(\mathcal{F})$ and called the *genus* of the foliation $\mathcal{F}$. It is similar to the genus of a compact Riemann surface; it counts the number of linearly independant basic holomorphic 1-forms.

(vi) *Por every* $p \in \{0, \cdots, n\}$ *the differential form* $\omega^p = \omega \wedge \cdots \wedge \omega$ *(wedge product p times) is harmonic. So, the space* $H^{p,p}(M/\mathcal{F})$ *is non zero.*

5. Signature and Index

The manifold M is supposed to be compact oriented and equipped with a Riemannian foliation $\mathcal{F}$ of even codimension $n = 2k$ and with minimal leaves. Let ν denotes the complexified of the normal bundle of $\mathcal{F}$. Let $\tau : \Omega^*(M/\mathcal{F}) \longrightarrow \Omega^*(M/\mathcal{F})$ be the isomorphism defined on $\alpha \in \Omega^r(M/\mathcal{F})$, by $\tau(\alpha) = i^{r(r-1)+k} * \alpha$. It satisfies $\tau^2 =$id *i.e.* τ is an involution; so it admits 1 and -1 as eingenvalues. Let $\Omega^*_+(M/\mathcal{F})$ and $\Omega^*_-(M/\mathcal{F})$ be the eigensubspaces associated respectively to these eigenvalues. Then we have a decomposition

$$\Omega^*(M/\mathcal{F}) = \Omega^*_+(M/\mathcal{F}) \oplus \Omega^*_-(M/\mathcal{F}).$$

Let $D = d + \delta$. It is clear that D is a transversely elliptic selfadjoint operator and its kernel $N(D)$ is identified, by Theorem 3.3, to $H^*(M/\mathcal{F})$. It is obvious to see that $D\tau = -\tau D$. Let

$$D_+ : \Omega^*_+(M/\mathcal{F}) \longrightarrow \Omega^*_-(M/\mathcal{F}) \text{ and } D_- : \Omega^*_-(M/\mathcal{F}) \longrightarrow \Omega^*_+(M/\mathcal{F})$$

be the restrictions of D respectively to the spaces $\Omega^*_+(M/\mathcal{F})$ and $\Omega^*_-(M/\mathcal{F})$. These operators are transversely elliptic and are adjoint to each other. By Theorem 3.3, the operators D_+ and D_- are Fredholm. Then each of them admits an index.

5.1. Definition. D_+ *is called the* <u>*signature operator*</u> *of the foliation* $\mathcal{F}$*; its index is the integer*

$$\text{ind}_{\mathcal{F}}(D_+) = \dim\, (\text{Ker } D_+) - \dim\, (\text{Ker } D_-). \tag{15}$$

Let $\Delta_b = D^2$ be the basic Laplacian; it is immediate to see that Δ_b and D have the same kernel which, by the basic Hodge Theorem (*cf.* subsection 4.1), is $\mathcal{H}^*(M/\mathcal{F})$ and which is identified to the space $H^*(M/\mathcal{F})$. Because $D\tau = -\tau D$, τ restricts to an involution $\tau : \text{Ker}D \longrightarrow \text{Ker}D$ which defines decompositions

$$H^*(M/\mathcal{F}) = H^*_+(M/\mathcal{F}) \oplus H^*_-(M/\mathcal{F}).$$

$$\mathcal{H}^*(M/\mathcal{F}) = \mathcal{H}^*_+(M/\mathcal{F}) \oplus \mathcal{H}^*_-(M/\mathcal{F}).$$

So

$$\text{ind}_{\mathcal{F}}(D_+) = \dim\, \mathcal{H}^*_+(M/\mathcal{F}) - \dim\, \mathcal{H}^*_-(M/\mathcal{F}) = \dim\, H^*_+(M/\mathcal{F}) - \dim\, H^*_-(M/\mathcal{F}).$$

Let V_r be the space $H^r(M/\mathcal{F}) \oplus H^{2k-r}(M/\mathcal{F})$ where $r \in \{0, \ldots, k-1\}$. Then the subspaces $H^r(M/\mathcal{F})$ and $H^{2k-r}H(M/\mathcal{F})$ are permuted by the action of τ on $H^*(M/\mathcal{F})$; so V_r is globally invariant by τ. This gives

$$\text{ind}_{\mathcal{F}}(D_+) = \dim\, \mathcal{H}^k_+(M/\mathcal{F}) - \dim\, \mathcal{H}^k_-(M/\mathcal{F}) = \dim\, H^k_+(M/\mathcal{F}) - \dim\, H^k_-(M/\mathcal{F}).$$

We will regard indifferently τ as an involution on the basic cohomology $H^*(M/\mathcal{F})$ or on the space $\mathcal{H}^*(M/\mathcal{F})$ of basic harmonic forms.

5.2. Remarks

i) If k is odd we have $\tau = i*$, then $\tau(\alpha) = -\overline{\tau(\alpha)}$ where $\alpha \in \Omega^k(M/\mathcal{F})$; this implies $** = -$id. The morphism $\tau : \mathcal{H}^k_+(M/\mathcal{F}) \longrightarrow \mathcal{H}^k_-(M/\mathcal{F})$ induced by τ is an isomorphism (of real vector spaces); consequently $\text{ind}_{\mathcal{F}}(D_+) = 0$.

ii) Suppose $n = 2k = 4\ell$. Recall that we have defined in the subsection 4.1 a non degenerate quadratic form Q on $H^k(M/\mathcal{F}) \simeq \mathcal{H}^k(M/\mathcal{F})$ whose signature is $\mathrm{Sign}(\mathcal{F})$. We verify easily that $\tau = *$ on $\mathcal{H}^k(M/\mathcal{F})$. Then, for non zero basic harmonic k-forms $\alpha \in \mathcal{H}^k_+(M/\mathcal{F})$ and $\beta \in \mathcal{H}^k_-(M/\mathcal{F})$, we have

$$Q(\alpha, \alpha) = \int_M \alpha \wedge \overline{\alpha} \wedge \chi = \int_M \alpha \wedge \tau\overline{\alpha} \wedge \chi = \int_M \alpha \wedge *\overline{\alpha} \wedge \chi = \langle \alpha, \alpha \rangle > 0$$

and

$$Q(\beta, \beta) = \int_M \beta \wedge \overline{\beta} \wedge \chi = -\int_M \beta \wedge \tau\overline{\beta} \wedge \chi = -\int_M \beta \wedge *\overline{\beta} \wedge \chi = -\langle \beta, \beta \rangle < 0$$

i.e. Q is positive definite on $\mathcal{H}^k_+(M/\mathcal{F})$ and negative definite on $\mathcal{H}^k_-(M/\mathcal{F})$. So, if we set, by convention, $\mathrm{Sign}(\mathcal{F}) = 0$ if the integer k is odd, we finally obtain the following theorem.

5.3. Theorem. *Let $\mathcal{F}$ be a homologically orientable Riemannian foliation of even codimension $n = 2k$ on a compact connected oriented manifold M. Then*

$$\mathrm{Sign}(\mathcal{F}) = \mathrm{ind}_{\mathcal{F}}(D_+).$$

As a conclusion to this paper : In [EN] it was shown that the basic cohomology $H^*(M/\mathcal{F})$ is invariant by homeomorphism in the category of Riemannian complete foliations. So the signature of $\mathcal{F}$ and the integers defined respectively in (11) and 4.2.(v) are topological invariants. This reinforces the idea that it is certainly interesting to attack Problem 3.4.

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