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Advanced Course on

Foliations: Dynamics-Geometry-Topology

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FUNDAMENTS OF FOLIATION THEORY

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by

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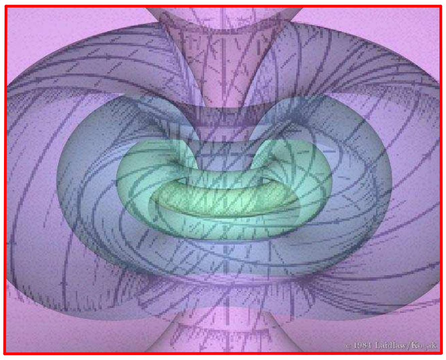


The sphere $\mathbb{S}^3$

FOREWORD

It is well known that there is no general methods to solve differential equations even in the case of the most very simple manifold namely the real line $\mathbb{R}$. Failing that, mathematicians rather try to study the geometrical and topological properties of their integral manifolds and their asymptotic behavior. This is exactly the purpose of *foliation theory*: qualitative study of differential equations. It was initiated by the works of H. Poincaré, I. Bendixon and developped later by C. Ehresmann, G. Reeb, A. Haefliger and many other people. Since then the subject has been a wide field in mathematical research. This motivates this course on *Fundaments of foliation theory* to take place in this session of *Advanced courses*. It is organized as follows.

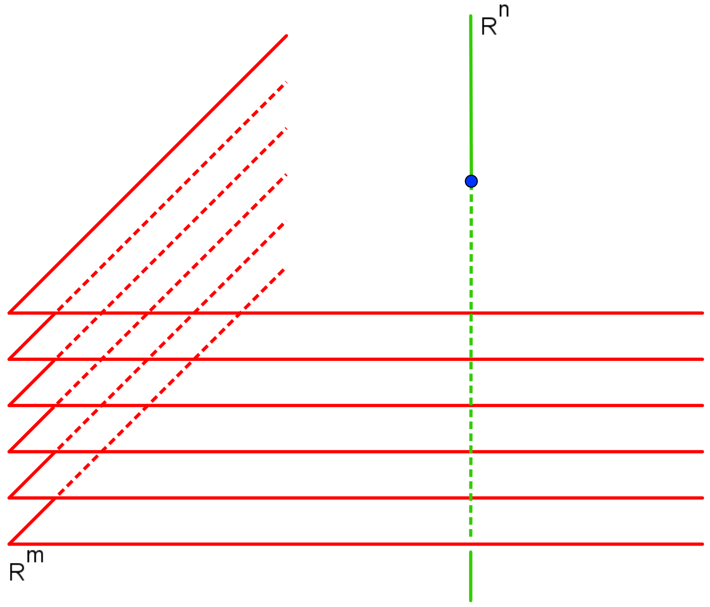
Part I is a very elementary introduction to foliation theory. We give the basic definitions and, through various simple examples, we introduce the notion of transverse structure which is capital in the study and classification of foliations. In Part II we give an elementary exposition of some results on transverse global analysis and thus lead to a basic index theory of transversely elliptic operators. Part III is devoted to open questions in some directions in the theory of foliations.

PART I

FOLIATIONS BY EXEMPLE

1. Generalities

A *foliation* is a geometric structure which a manifold can support. As we know a manifold is locally an Euclidean space. So to understand what is a foliation, it is more convenient to see its local model. Let us give a look at the Euclidean space $M = \mathbb{R}^{m+n}$ in the picture bellow:



Trivial foliation on $\mathbb{R}^{m+n}$ defined by the differential system $dy_1 = \cdots = dy_n = 0$

Fig. I.1

It can be seen as the product $\mathbb{R}^m \times \mathbb{R}^n$. Its usual topology is the product of the usual ones on the two factors for which it is a differentiable connected manifold of dimension $m+n$. But if we equip the second factor $\mathbb{R}^n$ by the discrete topology, M becomes a non connected manifold of dimension m; its connected components are the horizontal subspaces defined by the differential linear system $dy_1 = \cdots = dy_n = 0$ which can be seen as *leaves*. We then see that M is equipped with two topologies: the usual topology and a second one called the *leaf topology*.

Now let M be a (connected) manifold of dimension $m+n$. Intuitively one can define a *foliation* of *dimension* m or *codimension* n on M as a geometric structure such that around each point one can cut a small piece (that is, an open neighborhood) which looks like this picture. A first definition is the following.

1.1. Definition. *Let M be a manifold of dimension $m+n$. A codimension n* **foliation** *$\mathcal{F}$ on M is given by an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{m+n} \longrightarrow U_i$ such that, on each non empty intersection $U_i \cap U_j$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : (x, y) \in \varphi_i^{-1}(U_i \cap U_j) \longrightarrow (x', y') \in \varphi_j^{-1}(U_i \cap U_j)$ has the form $x' = \varphi_{ij}(x, y)$ and $y' = \gamma_{ij}(y)$.*

The manifold M is decomposed into connected submanifolds of dimension m. Each of them is called a *leaf* of $\mathcal{F}$. A subset U of M is *saturated* for $\mathcal{F}$ if it is a union of leaves: if $x \in U$ then the leaf passing through x is contained in U.

Coordinate patches (U_i, φ_i) satisfying conditions of definition 1.1 are said to be *distinguished* for the foliation $\mathcal{F}$.

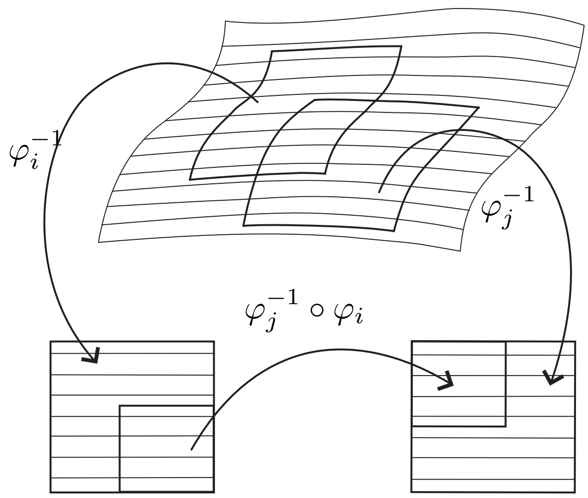


Fig. I.2

Let $\mathcal{F}$ be a codimension n foliation on M defined by a maximal atlas $\{(U_i, \varphi_i)\}_{i\in I}$ like in definition 1.1. Let $\pi : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the second projection. Then the map $f_i : U_i \overset{\pi\circ\varphi_i^{-1}}{\longrightarrow} \mathbb{R}^n$ is a submersion. On $U_i \cap U_j \neq \emptyset$ we have $f_j = \gamma_{ij} \circ f_i$. The fibres of the submersion f_i are the $\mathcal{F}$-plaques of U_i. The submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbb{R}^n$ give a complete characterization of $\mathcal{F}$.

1.2. Definition. *A codimension n* **foliation** *on M is given by an open cover $(U_i)_{i\in I}$, submersions $U_i \overset{f_i}{\longrightarrow} T$ over a n dimensional transverse manifold T and, for any non empty intersection $U_i \cap U_j$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$ satisfying $f_j(x) = \gamma_{ij} \circ f_i(x)$ for $x \in U_i \cap U_j$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a* **foliated cocycle** *defining $\mathcal{F}$.*

The foliation $\mathcal{F}$ is said to be *transversely orientable* if T can be given an orientation preserved by all the local diffeomorphisms γ_{ij}.

1.3. Induced foliations

Let N and M be two manifolds; suppose that we are given a codimension n foliation $\mathcal{F}$ on M. We say that a map $f : N \longrightarrow M$ is *transverse* to $\mathcal{F}$ if, for each point $x \in N$, the tangent space T_yM of M at $y = f(x)$ is generated by $T_y\mathcal{F}$ and $(d_xf)(T_xN)$ (where d_xf is the tangent linear map of f at x) *i.e.*:

$$T_yM = T_y\mathcal{F} + (d_xf)(T_xN). \tag{I.1}$$

Equivalently, if we suppose that M is of dimension $m+n$, f is transverse to $\mathcal{F}$ if, for each $x \in N$, there exists a local system of coordinates $(x_1, \cdots, x_m, y_1, \cdots, y_n) : \mathbb{R}^{m+n} \longrightarrow V$ around y such that the map $g_U : (y_1^{-1} \circ f, \cdots, y_n^{-1} \circ f) : U = f^{-1}(V) \longrightarrow \mathbb{R}^n$ is a submersion. The collection of the local submersions (U, g_U) defines a codimension n foliation denoted $f^*(\mathcal{F})$ on N called the *pull-back foliation* of $\mathcal{F}$ by f.

If f is a submersion and $\mathcal{F}$ is the foliation by points, then the vector space $T_y\mathcal{F}$ is reduced to $\{0\}$, d_xf is surjective and then the equality (I.1) is trivially satisfied; so f is transverse to $\mathcal{F}$; in that case the leaves of $f^*(\mathcal{F})$ are exactly the fibres of f.

If $N = \widehat{M}$ is a covering of M and f is the covering projection $f : \widehat{M} \longrightarrow M$, then $\widehat{\mathcal{F}} = f^*(\mathcal{F})$ has the same dimension as $\mathcal{F}$; the two foliations $\widehat{\mathcal{F}}$ and $\mathcal{F}$ have the same local properties.

1.4. Morphisms of foliations

Let M and M' be two manifolds endowed respectively with two foliations $\mathcal{F}$ and $\mathcal{F}'$. A map $f : M \longrightarrow M'$ will be called *foliated* or a *morphism* between $\mathcal{F}$ and $\mathcal{F}'$ if, for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf of $\mathcal{F}'$; we say that f is an *isomorphism* if, in addition, f is a diffeomorphism; in this case the restriction of f to any leaf $L \in \mathcal{F}$ is a diffeomorphism on the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M. Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation $\mathcal{F}'$ on M; we say that $\mathcal{F}'$ is the *image* of $\mathcal{F}$ by the diffeomorphism f and we write $\mathcal{F} = f^*(\mathcal{F}')$. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are said to be C^r-*conjugated* (*topologically* if $r = 0$, *differentiably* if $r = \infty$ and *analytically* in the case $r = \omega$) if there exists a C^r-homeomorphism $f : M \longrightarrow M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

The set of C^r-diffeomorphisms of M which preseve the foliation $\mathcal{F}$ is a group denoted $\mathrm{Diff}^r(M, \mathcal{F})$.

1.5. Frobenius Theorem

Let M be a manifold of dimension $m + n$. Denote by TM the tangent bundle of M and let E be a subbundle of rank m. Let U be an open set of M such that on U,

TM is equivalent to the product $U \times \mathbb{R}^{m+n}$. At each point $x \in U$, the fibre E_x can be considered as the kernel of n differential 1-forms $\omega_1, \ldots, \omega_n$ linearly independent:

$$E_x = \bigcap_{j=1}^{n} \ker\, \omega_j(x). \tag{I.2}$$

The subbundle E is called a *m-plane field* on M. We say that E is *involutive* if, for every vector fields X and Y tangent to E (*i.e.* sections of E), the bracket $[X, Y]$ is also tangent to E. We say that E is *completely integrable* if, through each point $x \in M$, there exists a submanifold P_x of dimension m which admits $E_{|P_x}$ (the restriction of E to P_x) as tangent bundle. The maximal connected submanifolds satisfying this property are called the *integral submanifolds* of the differential system $\omega_1 = \cdots = \omega_n = 0$. They define a partition of M *i.e.* a codimension n foliation. We have the following theorem:

Let E be a subbundle of rank m given locally by a differential system like in (I.2). *Then the following assertions are equivalent:*

– *E is involutive,*

– *E is completely integrable,*

– *there exist differential 1-forms (defined locally) (β_{ij}), $i, j = 1, \ldots, n$ such that $d\omega_i = \sum_{j=1}^{n} \beta_{ij} \wedge \omega_j \quad i = 1, \ldots, n$.*

<u>Trivial examples</u>

(i) Suppose that E is orientable and of rank 1. Then E has a global section ζ (vector field) such that at each point $x \in M$, $\zeta(x)$ is a basis of E_x. In that case two arbitrary sections $X = f\zeta$ and $Y = g\zeta$ satisfy $[X, Y] = \{f(\zeta \cdot g) - g(\zeta \cdot f)\}\zeta$; then the subbundle E is integrable and defines a one dimensional foliation. The leaves are exactly the integral curves of the vector field ζ. We will see in detail this particular situation.

(ii) Let ω be a non singular 1-form on M. Then ω defines a codimension 1 foliation if and only if there exists a 1-form β such that $d\omega = \beta \wedge \omega$; this is equivalent to $\omega \wedge d\omega = 0$. In particular this is the case if ω is closed.

(iii) On the other hand the non singular 1-form on $\mathbb{R}^3$ given by $\omega = dx - zdy$ satisfies the relation:

$$\omega \wedge d\omega = dx \wedge dy \wedge dz$$

and cannot define a foliation. The plane field $E \subset T\mathbb{R}^3$, kernel of the 1-form ω, has the remarkable following property: given two points a and b in $\mathbb{R}^3$, there exists a differentiable curve $\gamma : [0, 1] \longrightarrow \mathbb{R}^3$ such that $\gamma(0) = a$ and $\gamma(1) = b$ and γ is tangent to E at every point. We say that ω defines a *contact structure*. Contact structures are the opposite of foliated structures.

1.6. Holonomy of a leaf

This is a very important notion in foliation theory. In many situations it determines completely the structure of the foliation. In this subsection, we will introduce this concept. We will give later some examples.

Let $\mathcal{F}$ be a codimension n foliation on M, let L be a leaf of $\mathcal{F}$ and $x \in L$. Let T be a small transversal to $\mathcal{F}$ passing through x. Let $\sigma : [0,1] \longrightarrow L$ be a continuous path such that $\sigma(0) = \sigma(1) = x$. Then there exist a finite open cover U_i, $i = 0, 1, \ldots, k$ of M with $U_0 = U_k$ and a subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1]$ such that:

– $\sigma([t_{i-1}, t_i]) \subset U_i$,

– if $U_i \cap U_j \neq \emptyset$ then $U_i \cup U_j$ is contained in a distinguished chart of $\mathcal{F}$.

We say that $\{U_i\}$ is a *subordinated chain* to σ. For $i = 0, 1, \ldots, k$ let T_i be a small transversal to $\mathcal{F}$ passing through $\sigma_i(t)$ with $T_0 = T_k = T$. For every point $z \in T_i$, sufficiently close to $\sigma(t_i)$, the plaque of $\mathcal{F}$ passing through z intersects T_{i+1} in a unique point $f_i(z)$. The domain of f_i contains a transversal T'_i passing through $\sigma(t_i)$ and homeomorphic to an open ball of $\mathbb{R}^n$. Then, it is clear that the map: $f_\sigma = f_{k-1} \circ f_{k-2} \circ \ldots \circ f_0$ is well defined on an open neighbourhood of x; it is called the *holonomy map* associated to σ. We can prove (see [CL]) that the germ of f_σ:

– does not depend on the chain U_i, $i = 1, \ldots, k$ and on the choice of σ in its homotopy class in the group $\pi_1(L, x)$ of the homotopy classes of loops based at x,

– satisfies $f_\sigma(x) = x$.

So we get a homomorphism $h : [\sigma] \in \pi_1(L, x) \longrightarrow f_\sigma \in G(T, x)$ where $G(T, x)$ is the group of germs of diffeomorphisms of T fixing the point x. This representation h is called the *holonomy* of the leaf L at x; the image of $\pi_1(L)$ by h (L is path connected) is called the *holonomy group* of the leaf L. The foliation $\mathcal{F}$ is said to be *without holonomy* if every leaf L of $\mathcal{F}$ has a trivial holonomy group.

2. Transverse structures

Let us fix some notations. Let $\mathcal{F}$ be a codimension n foliation on M. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\nu\mathcal{F}$ the quotient $TM/T\mathcal{F}$ which is the *normal bundle* to $\mathcal{F}$.

– $\mathfrak{X}(\mathcal{F})$ will denote the space of sections of $T\mathcal{F}$ (elements of $\mathfrak{X}(\mathcal{F})$ are vector fields $X \in \mathfrak{X}(M)$ tangent to $\mathcal{F}$).

– A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. (Here i_X and L_X denote respectively the inner product and the Lie derivative with respect to the vector field X.) For a function $f : M \longrightarrow \mathbb{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$ *i.e.* f is constant on the leaves of $\mathcal{F}$; we denote by $\Omega^r(M/\mathcal{F})$ the space of basic forms of degree r on the foliated manifold $(M, \mathcal{F})$; this is a module over the algebra A of basic functions.

– A vector field $Y \in \mathfrak{X}(M)$ is said to be *foliated*, if for every $X \in \mathfrak{X}(\mathcal{F})$, the bracket $[X, Y] \in \mathfrak{X}(\mathcal{F})$. We can easily see that the set $\mathfrak{X}(M, \mathcal{F})$ of foliated vector fields is a Lie algebra and a A-module; by definition $\mathfrak{X}(\mathcal{F})$ is an ideal of $\mathfrak{X}(M, \mathcal{F})$ and the quotient $\mathfrak{X}(M/\mathcal{F}) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is called the Lie algebra of *basic* (or *transverse*) vector fields on the foliated manifold $(M, \mathcal{F})$. Also, it has a module structure over the algebra A.

Let M be a manifold of dimension $m + n$ endowed with a codimension n foliation $\mathcal{F}$ defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$ like in definition 1.2.

2.1. Definition. *A* ***transverse structure*** *to $\mathcal{F}$ is a geometric structure on T invariant by the local diffeomorphisms γ_{ij}.*

This is a very important notion in foliation theory. To make it clear, let us give the main examples of such structures.

2.2. Lie foliations

We say that $\mathcal{F}$ is a *Lie G-foliation*, if T is a Lie group G and γ_{ij} are restrictions of left translations on G. Such foliation can also be defined by a 1-form ω on M with values in the Lie algebra $\mathcal{G}$ such that:

i) $\omega_x : T_xM \to \mathcal{G}$ is surjective for every $x \in M$,

ii) $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

If $\mathcal{G}$ is Abelian, ω is given by n linearly independent closed real 1-forms $\omega_1, \ldots, \omega_n$.

In the general case, the structure of a Lie foliation on a compact manifold is given by the following theorem due to E. Fédida [Féd]:

Let $\mathcal{F}$ be a Lie G-foliation on a compact manifold M. Let $\widetilde{M}$ be the universal covering of M and $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\widetilde{M}$. Then there exist a homomorphism $h : \pi_1(M) \longrightarrow G$ and a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ whose fibres are the leaves of $\widetilde{\mathcal{F}}$ and such that, for every $\gamma \in \pi_1(M)$, the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{M} & \stackrel{\gamma}{\longrightarrow} & \widetilde{M} \\ D \downarrow & & \downarrow D \\ G & \stackrel{h(\gamma)}{\longrightarrow} & G \end{array} \tag{I.3}$$

where the first line denotes the deck transformation of $\gamma \in \pi_1(M)$ on $\widetilde{M}$ and $h(\gamma)$ is the left translation by γ.

The subgroup $\Gamma = h(\pi_1(M)) \subset G$ is called the *holonomy group* of $\mathcal{F}$ although the holonomy of each leaf is trivial. The fibration $D : \widetilde{M} \longrightarrow G$ is called the *developing map* of $\mathcal{F}$.

This theorem gives also a way to construct Lie foliations. Let us see explicitly a particular example. Let M be the 2-torus $\mathbb{T}^2$; its universal covering $\widetilde{M}$ is $\mathbb{R}^2$ and his fundamental group is $\Gamma = \mathbb{Z}^2$. Denote by h the morphism from Γ to the Lie group $G = \mathbb{R}$ given by $h(m,n) = n + \alpha m$ where α is a real positive number. For convenience we will consider that the action of an element $(m,n) = \gamma \in \Gamma$ on $\mathbb{R}^2$ is given by the map $(x,y) \in \mathbb{R}^2 \overset{\gamma}{\longmapsto} (x - m, y + n) \in \mathbb{R}^2$. Let $D : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the submersion defined by $D(x,y) = y - \alpha x$. It is not difficult to see that, for any $\gamma \in \Gamma$, the diagram:

$$\begin{array}{ccc} \mathbb{R}^2 & \overset{\gamma}{\longrightarrow} & \mathbb{R}^2 \\ D\downarrow & & \downarrow D \\ \mathbb{R} & \overset{h(\gamma)}{\longrightarrow} & \mathbb{R} \end{array}$$

is commutative *i.e.* the fibration $D : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is equivariant under the action of Γ on $\mathbb{R}^2$ and then induces a Lie foliation on $\mathbb{T}^2$ transversely modeled on the Lie group $\mathbb{R}$.

2.3. Transversely parallelizable foliations

We say that $\mathcal{F}$ is *transversely parallelizable* if there exist on M foliated vector fields $Y_1, \ldots, Y_n$, transverse to $\mathcal{F}$ and everywhere linearly independent. This means that the manifold T admits a parallelism $(Y_1, \ldots, Y_n)$ invariant by all the local diffeomorphisms γ_{ij} or, equivalently, that the A-module $\mathfrak{X}(M/\mathcal{F})$ is free of rank n. The structure of a transversely parallelizable foliation on a compact manifold is given by the following theorem due to L. Conlon [Con] for $n = 2$ and in general to P. Molino [Mol].

Let $\mathcal{F}$ be a transversely parallelizable foliation of codimension n on a compact manifold M. Then:

(1) *the closures of the leaves are submanifolds which are fibres of a locally trivial fibration $\pi : M \longrightarrow W$ where W is a compact manifold,*
(2) *there exists a simply connected Lie group G_0 such that the restriction $\mathcal{F}_0$ of $\mathcal{F}$ to any leaf closure F is a Lie G_0-foliation,*
(3) *the cocycle of the fibration $\pi : M \longrightarrow W$ has values in the group of diffeomorphisms of F preserving $\mathcal{F}_0$.*

The fibration $\pi : M \longrightarrow W$ and the manifold W are called respectively the *basic fibration* and the *basic manifold* associated to $\mathcal{F}$. This theorem says that if, in particular, the leaves of $\mathcal{F}$ are closed, then the foliation is just a fibration over W. This is still true even if the leaves are not closed: the manifold M is a fibration over the leaf space $M/\mathcal{F}$ which is, in this case, a *Q-manifold* in the sense of [Bar].

Any Lie foliation is transversely parallelizable. This is a consequence of the fact that a Lie group is parallelizable and that the parallelism can be chosen invariant by left translations.

2.4. Riemannian foliations

The foliation $\mathcal{F}$ is said to be *Riemannian* if there exists on T a Riemannian metric such that the local diffeomorphisms γ_{ij} are isometries. Using the submersions $f_i : U_i \longrightarrow T$ one can construct on M a Riemannian metric which can be written in local coordinates:

$$ds^2 = \sum_{i,j=1}^{m} \theta_i \otimes \theta_j + \sum_{k,\ell=1}^{n} g_{k\ell}(y) dy_k \otimes dy_\ell. \tag{I.4}$$

(This metric is said to be *bundle like.*) Equivalently, $\mathcal{F}$ is Riemannian, if any geodesic orthogonal to the leaves at a point is orthogonal to the leaves everywhere. See the paper [Rei1] by B. Reinhart who introduced firstly the notion of Riemannian foliation.

Let $\mathcal{F}$ be Riemannian. Then there exists a Levi-Civita connection, transverse to the leaves which, by unicity argument, coincides on any distinguished open set, with the pull-back of the Levi-Civita connection on the Riemannian manifold T. This connection is said to be *projectable.* Let $\mathrm{O}(n) \longrightarrow M^\# \xrightarrow{\tau} M$ be the principal bundle of orthonormal frames transverse to $\mathcal{F}$. The following theorem is due to P. Molino [Mol]:

Suppose M is compact. Then, the foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}^\#$ on $M^\#$ of the same dimension and such that:

(1) *$\mathcal{F}^\#$ is transversely parallelizable,*

(2) *$\mathcal{F}^\#$ is invariant under the action of $O(n)$ on $M^\#$ and projects, by τ, on $\mathcal{F}$.*

The basic manifold $W^\#$ and the basic fibration $F^\# \longrightarrow M^\# \xrightarrow{\pi^\#} W^\#$ are called respectively the *basic manifold* and the *basic fibration* of $\mathcal{F}$.

We have the following properties:

– the restriction of τ to a leaf of $\mathcal{F}^\#$ is a covering over a leaf of $\mathcal{F}$. So all leaves of $\mathcal{F}$ have the same universal covering (*cf.* [Rei1]),

– the closure of any leaf of $\mathcal{F}$ is a submanifold of M and the leaf closures define a *singular foliation* (the leaves have different dimensions) on M. (For more details about this notion see [Mol].)

Another interesting result for Riemannian foliations is the Global Reeb Stability Theorem which is valid even if the codimension is greater than 1.

Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. If there exists a compact leaf with finite fundamental group, then all leaves are compact with finite fundamental group.

The property '$\mathcal{F}$ is Riemannian' means that the leaf space $B = M/\mathcal{F}$ is a Riemannian manifold even if B does not support any differentiable structure!

2.5. $\mathcal{G}/\mathcal{H}$-foliations

This is a class of foliations which possess interesting transverse properties (see [EGN]). Let $\mathcal{G}$ be a Lie algebra of dimension d and $\mathcal{H}$ a Lie subalgebra of $\mathcal{G}$. We fix a basis $e_1, \ldots, e_d$ of $\mathcal{G}$ such that $e_{n+1}, \ldots, e_d$ span $\mathcal{H}$ and denote by $\theta^1, \ldots, \theta^d$ the corresponding dual basis. One has $[e_i, e_j] = \sum_k K_{ij}^k e_k$, where the *structure constants* K_{ij}^k fulfill the relations:

$$K_{ij}^k = -K_{ji}^k \tag{C1}$$

$$\sum_i (K_{ij}^k K_{rs}^i + K_{ir}^k K_{sj}^i + K_{is}^k K_{jr}^i) = 0 \quad \text{(Jacobi identity)} \tag{C2}$$

$$K_{ij}^k = 0 \text{ if } k \leq n \text{ and } n+1 \leq i, j \tag{C3}$$

The set of constants K_{ij}^k satisfying (C1) and (C2) determines the Lie algebra structure of $\mathcal{G}$ while (C3) states that $\mathcal{H}$ is a Lie subalgebra of $\mathcal{G}$. We denote by G the simply connected Lie group with Lie algebra $\mathcal{G}$ and by H the connected Lie subgroup of G corresponding to the Lie subalgebra $\mathcal{H}$.

We shall denote by θ the $\mathcal{G}$-valued 1-form on G which is the identity over the left invariant vector fields on G, *i.e.* $\theta = \sum_k \theta^k \otimes e_k$. Let $\omega = \sum_k \omega^k \otimes e_k$ be a $\mathcal{G}$-valued 1-form on a manifold M. An element $g \in G$ transforms ω into the $\mathcal{G}$-valued form $\mathrm{Ad}_g\omega$ where $\mathrm{Ad}_g\omega(X) = \mathrm{Ad}_g \cdot (\omega(X))$ for any vector field X on M. Once the basis $e_1, \ldots, e_d$ of $\mathcal{G}$ has been fixed, we shall identify ω with the n-tuple of scalar 1-forms $(\omega^1, \ldots, \omega^d)$. In particular $\theta = (\theta^1, \ldots, \theta^d)$.

Let a $\mathcal{G}$-valued 1-form $\omega = (\omega^1, \ldots, \omega^d)$ on a connected manifold M be given. Assume that ω fulfills the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$, i.e.

$$d\omega^k = -\frac{1}{2} \sum_{i,j=1}^{d} K_{ij}^k \omega^i \wedge \omega^j \tag{C4}$$

and that $\omega^1, \ldots, \omega^n$ are linearly independent. Then the differential system of equations $\omega^1 = \ldots = \omega^n = 0$ is integrable and defines a codimension n foliation $\mathcal{F}$. We shall say that $\mathcal{F}$ is a $\mathcal{G}/\mathcal{H}$-foliation defined by the $\mathcal{G}$-valued form ω.

Main example. Let $M = G$. Then $\theta = (\theta^1, \dots, \theta^d)$ defines a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ whose leaves are the left cosets of H.

Remark. The notion of $\mathcal{G}/\mathcal{H}$-foliation includes several classes of geometric structures:

(a) If $n = \dim M$ and H is closed then a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}$ defines a structure of locally homogeneous space on M; that is, the manifold M is locally modeled on the homogeneous space G/H with coordinate changes given by left translations by elements of G and $\mathcal{F}$ is the foliation by points. The homogeneous space G/H is endowed with a $\mathcal{G}/\mathcal{H}$-foliation when the projection $G \to G/H$ admits a global section.

(b) When $\mathcal{H} = 0$, $\mathcal{G}/\mathcal{H}$-foliations are just Lie foliations modeled over G. For instance a non-singular closed 1-form ω on M defines a Lie foliation modeled over $\mathbb{R}$.

(c) If H is closed then a $\mathcal{G}/\mathcal{H}$-foliation is a transversely homogeneous foliation modeled over the homogeneous space G/H. Every transversely homogeneous foliation is given locally by a collection of 1-forms $\omega^1, \dots, \omega^d$ fulfilling (C4) (cf. [Blu]). If these forms are global then they define a $\mathcal{G}/\mathcal{H}$-foliation. This is the case if $H^1(M, H) = 0$ (cf. [Blu]).

Let us give a concrete example constructed by R. Roussarie. It was the first for which the Godbillon-Vey class is non trivial. Let $G = \mathrm{SL}(2, \mathbb{R})$ be the linear group of real 2×2 matrices of determinant 1 and Γ a cocompact lattice (it well known that these subgroups abound in G); the homogeneous space $M = G/\Gamma$ is a 3-dimensional compact orientable manifold. The Lie algebra $\mathcal{G}$ of G has a basis consisting of:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the relations:

$$\begin{cases} [X, Y] = Z \\ [Z, X] = 2X \\ [Z, Y] = -2Y. \end{cases}$$

These elements are associated to left invariant vector fields X, Y and Z on G, then also on M. Let α, β and η be the dual basis of (X, Y, Z); we have the following relations which can be derived easily from (????):

$$\begin{cases} d\alpha = -\beta \wedge \eta \\ d\beta = -2\alpha \wedge \beta \\ d\eta = 2\alpha \wedge \eta. \end{cases}$$

Because $d\eta = \theta \wedge \eta$ (where $\theta = 2\alpha$), we have $\eta \wedge d\eta = 0$. Then by Frobenius theorem the 1-form η defines a codimension one foliation $\mathcal{F}$ on M.

(d) In general, when H is not necessarily closed, a $\mathcal{G}/\mathcal{H}$-foliation is a locally transversely homogeneous foliation as it is defined in [Mol].

Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω. A map $\varphi : N \to M$ transverse to $\mathcal{F}$ induces a $\mathcal{G}/\mathcal{H}$-foliation $\varphi^*\mathcal{F}$ on N which is defined by $\varphi^*\omega$. We say that $\varphi^*\mathcal{F}$ is the *pull-back* of $\mathcal{F}$ by φ. In particular, the universal covering space $\widetilde{M}$ of M is endowed with the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ defined by $\pi^*\omega$ where $\pi : \widetilde{M} \to M$ is the canonical projection. The following proposition states that the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ is a pull-back of the $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ on G which was considered as the main example.

Proposition [Blu]. *Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω and let $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ be its pull-back to the universal covering space $\widetilde{M}$ of M. There are a map $\mathcal{D} : \widetilde{M} \to G$ and a group representation $\rho : \pi_1(M) \to G$ such that:*

(i) $\mathcal{D}$ is $\pi_1(M)$-*equivariant, i.e.* $\mathcal{D}(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot \mathcal{D}(\tilde{x})$ *for any* $\gamma \in \pi_1(M)$, *and*

(ii) $\widetilde{\omega} := \pi^*\omega = \mathcal{D}^*\theta$, *i.e.* $\widetilde{\mathcal{F}} = \mathcal{D}^*\mathcal{F}_{G,H}$.

The map $\mathcal{D}$ is called the developing map of $\mathcal{F}$ and it is uniquely determined up to left translations by elements of G.

2.6. Transversely holomorphic foliations

The foliation $\mathcal{F}$ is said to be *transversely holomorphic* if T is a complex manifold and the γ_{ij} are local biholomorphisms. Particular case is a *holomorphic foliation*: the manifolds M and T are complex, all the f_i are holomorphic and all γ_{ij} are local biholomorphisms.

If T is Kählerian and γ_{ij} are biholomorphisms preserving the Kähler form on T we say that $\mathcal{F}$ is *transversely Kählerian.* For example, any codimension 2 Riemannian foliation which is transversely orientable is transversely Kählerian.

Some concrete examples

(i) Let $M = \mathbb{S}^{2n+1} = \left\{ (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} |z_k|^2 = 1 \right\}$ be the unit sphere in the Hermitian space $\mathbb{C}^{n+1}$. Let Z be the holomorphic vector field on $\mathbb{C}^{n+1}$ given by the formula: $Z = \sum_{k=1}^{n+1} a_k z_k \dfrac{\partial}{\partial z_k}$ where $a_k = \alpha_k + i\beta_k \in \mathbb{C}$. It is not difficult to see that there exists a good choice of the numbers a_k such that the orbits of Z intersect transversely the sphere M; then Z induces on M a real vector field X which defines a foliation $\mathcal{F}$. One can verify easily that $\mathcal{F}$ is transversely holomorphic. It is transversely Kählerian if we choose in addition $\alpha_k = 0$ for any $k = 1, \ldots, n+1$.

(ii) Let $\widetilde{M} = \mathbb{C}^m \times \mathbb{C}^n \setminus \{(0,0)\}$; the coordinates of a point (z, w) will be denoted $(z_1, \cdots, z_m, w_1, \cdots, w_n)$. Define the foliation $\widetilde{\mathcal{F}}$ by the system of differential equations $dw_1 = \cdots = dw_n = 0$; then $\widetilde{\mathcal{F}}$ is a holomorphic (and then transversely holomorphic) foliation on the complex manifold $\widetilde{M}$. The leaf of $\widetilde{\mathcal{F}}$ passing through a point (z, w) is the complex vector space $\mathbb{C}^m$ for $w \neq 0$ and $\widetilde{L}_0 = \mathbb{C}^m \setminus \{0\}$ for $w = 0$.

Let $\lambda \in \mathbb{C}^*$ such that $|\lambda| \neq 1$. The action of the group $\Gamma = \mathbb{Z}$ on $\widetilde{M}$ generated by the transformation $\gamma : (z, w) \in \widetilde{M} \longrightarrow (\lambda z, \lambda w) \in \widetilde{M}$ is free, proper and preserves the foliation $\widetilde{\mathcal{F}}$. Then $\widetilde{\mathcal{F}}$ induces a holomorphic foliation $\mathcal{F}$ of complex dimension m on the quotient manifold $M = \widetilde{M}/\Gamma$; M is analytically equivalent to $\mathbb{S}^{2(m+n)-1} \times \mathbb{S}^1$ and leaves of $\mathcal{F}$ are bihomolophically equivalent to $\mathbb{C}^m$ except the one L_0 coming from $\widetilde{L}_0$ which is isomorphic to the complex Hopf manifold $\mathbb{S}^{2m-1} \times \mathbb{S}^1$.

The foliation $\mathcal{F}$ is not Riemannian. But the complex normal bundle of $\mathcal{F}$ is equipped with the Hermitian metric $h = \displaystyle\sum_{k=1}^{n} dw_k \wedge d\overline{w}_k$ which makes $\mathcal{F}$ a transversely conformal foliation.

3. More examples

3.1. Simple foliations

Two trivial foliations can be defined on a manifold M: the first one is obtained by considering that all the leaves are the points; the second one has only one leaf, namely, M itself.

Every submersion $M \stackrel{\pi}{\longrightarrow} B$ with connected fibres defines a foliation. The leaves being the fibres $\pi^{-1}(b)$, $b \in B$. In particular, every product $F \times B$ is a foliation with leaves $F \times \{b\}$, $b \in B$. These foliations are transversely orientable if, and only if, the manifold B is orientable.

3.2. Linear foliation on the torus $\mathbb{T}^2$

This example was already differently described in the subsection 2.2. Let $\widetilde{M} = \mathbb{R}^2$ and consider the linear differential equation $dy - \alpha dx = 0$ where α is a real number. This equation has $y = \alpha x + c$, $c \in \mathbb{R}$ as general solution. When c varies, we obtain a family of parallel lines which defines a foliation $\widetilde{\mathcal{F}}$ in $\widetilde{M}$. The natural action of $\mathbb{Z}^2$ on $\widetilde{M}$ preserves the foliation $\widetilde{\mathcal{F}}$ (*i.e.* the image of any leaf of $\widetilde{\mathcal{F}}$ by an integer translation is a leaf of $\widetilde{\mathcal{F}}$). Then $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The leaves are all diffeomorphic to the circle $\mathbb{S}^1$ if α is rational and to the real line if α is not rational (*cf.* Fig. I.3). In fact, if α is not rational, every leaf of $\mathcal{F}$ is dense; this shows that even if locally a foliation is simple, globally it can be complicated.

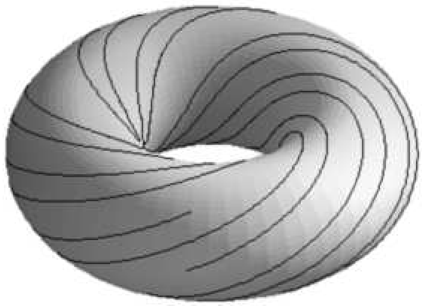


Fig. I.3

3.3. One dimensional foliations

Let M be a compact manifold (without boundary) of dimension n. Let X be a non singular vector field on M that is, for every $x \in M$, the vector X_x is non zero. Then its integral curves are leaves of a 1-dimensional foliation. This is also the case for a line bundle on M (not necessarily a vector field). In fact there is a natural one-to-one correspondence between the set of C^∞-line bundles and the set of 1-dimensional C^∞-foliations.

The fact that M admits a one dimensional foliation depends on its topology. For each $r = 0, 1, \ldots, n$, let $H^r(M, \mathbb{R})$ denote the real r-th *cohomology space* of M which is finite dimensional. Then the number:

$$\chi(M) = \sum_{r=0}^{n} (-1)^r \dim H^r(M, \mathbb{R}) \tag{I.5}$$

is a *topological invariant* called the *Euler-Poincaré number* of M. We have the following theorem:

The manifold M admits a one dimensional foliation if, and only if, $\chi(M) = 0$.

3.4. Reeb foliation on the 3-sphere $\mathbb{S}^3$

Let M be the 3-dimensional sphere $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure. The two subsets:

$$M_+ = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 \leq \frac{1}{2}\right\} \quad \text{and} \quad M_- = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_2|^2 \leq \frac{1}{2}\right\}$$

are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$. They have $\mathbb{T}^2$ as common boundary:

$$\mathbb{T}^2 = \partial M_+ = \partial M_- = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 = |z_2|^2 = \frac{1}{2}\right\}$$

and their union is equal to $\mathbb{S}^3$. Then $\mathbb{S}^3$ can be obtained by gluing M_+ and M_- along their boundaries by the diffeomorphism $(z_1, z_2) \in \partial M_+ \longrightarrow (z_2, z_1) \in \partial M_-$, i.e. we identify (z_1, z_2) with (z_2, z_1) in the disjoint union $M_+ \coprod M_-$. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the function defined by:

$$f(z) = \exp\left(\frac{1}{1 - |z|^2}\right).$$

Let t denote the second coordinate in $\mathbb{D} \times \mathbb{R}$. The family of surfaces $(S_t)_{t \in \mathbb{R}}$ obtained by translating the graph S of f along the t-axis defines a foliation on $\mathbb{D} \times \mathbb{R}$. If we add the cylinder $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is viewed as the boundary of $\overline{\mathbb{D}}$, we obtain a codimension one foliation $\widetilde{\mathcal{F}}$ on $\overline{\mathbb{D}} \times \mathbb{R}$. By construction, $\widetilde{\mathcal{F}}$ is invariant by the transformation $(z, t) \in \overline{\mathbb{D}} \times \mathbb{R} \longmapsto (z, t + 1) \in \overline{\mathbb{D}} \times \mathbb{R}$; so it induces a foliation $\mathcal{F}_0$ on the quotient:

$$\overline{\mathbb{D}} \times \mathbb{R}/(z, t) \sim (z, t + 1) \simeq \overline{\mathbb{D}} \times \mathbb{S}^1.$$

It has the boundary $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as a closed leaf. All other leaves are diffeomorphic to $\mathbb{R}^2$ (see Fig. I.4).

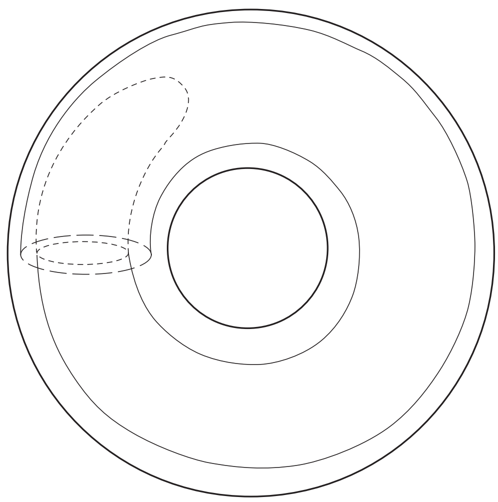


Fig. I.4

Because M_+ and M_- are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$, $\mathcal{F}_0$ defines on M_+ and M_- respectively two foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ which give a codimension one foliation $\mathcal{F}$ on $\mathbb{S}^3$ called the *Reeb foliation*. All the leaves are diffeomorphic to the plane $\mathbb{R}^2$ except one which is the torus, the common boundary L of the two components M_+ and M_-.

3.5. Lie group actions.

Let M be a manifold of dimension $m+n$ and G a connected Lie group of dimension m. An *action* of G on M is a map $G\times M \stackrel{\Phi}{\longrightarrow} M$ such that:

– $\Phi(e,x)=x$ for every $x\in M$ (where e is the unit element of G),

– $\Phi(g',\Phi(g,x))=\Phi(g'g,x)$ for every $x\in M$ and every $g,g'\in G$.

Suppose that, for every point $x\in M$, the dimension of the *isotropy subgroup*:

$$G_x=\{g\in G:\Phi(g,x)=x\}$$

is independent of x. Then the action Φ defines a foliation $\mathcal{F}$ of dimension $= m-\dim G_x$; its leaves are the orbits $\{\Phi(g,x):g\in G\}$. In particular this is the case if Φ is *locally free* i.e. if, for every $x\in M$, the isotropy subgroup G_x is discrete. An explicit example is given when M is the quotient H/Γ of a Lie group H by a discrete subgroup Γ and G is a connected Lie subgroup of H; the action of G on M being induced by the left action of G on H. We say that $\mathcal{F}$ is a *homogeneous foliation.* Let us give two examples.

The first one

Let $A\in \mathrm{SL}(m+n-1,\mathbb{Z})$, where $m+n\geq 3$, be a matrix diagonalizable on $\mathbb{R}$ and having all its eigenvalues $\mu_1,\ldots,\mu_{m-1},\lambda_1,\ldots,\lambda_n$ positive. Let $u_1,\cdots,u_{m-1},v_1,\cdots,v_n$ be the corresponding eigenvectors in $\mathbb{R}^{m+n-1}$. As we can think of A as a diffeomorphism of the $(m+n-1)$-torus $\mathbb{T}^{m+n-1}$, the vectors $u_1,\ldots,u_{m-1}$, $v_1,\ldots,v_n$ can be considered as linear vector fields on $\mathbb{T}^{m+n-1}$ such that:

$$A_*u_j=\mu_ju_j,\quad A_*v_k=\lambda_kv_k\quad \text{for}\quad j=1,\ldots,m-1\quad\text{and}\quad k=1,\ldots,n.$$

Let $(x_1,\cdots,x_{m-1},y_1,\cdots,y_n,t)$ be the coordinates of a vector in $\mathbb{R}^{m+n-1}\times\mathbb{R}$. Then the vector fields $u_1,\cdots,u_{m-1},v_1,\cdots,v_n,u_m=\frac{\partial}{\partial t}$ generate the Lie algebra (over the ring of C^∞-functions) $\mathfrak{X}(\mathbb{R}^{m+n})$. The vector fields:

$$X_i=\mu_i^tu_i,\qquad Y_j=\lambda_j^tv_j\qquad\text{and}\qquad X_m=\frac{\partial}{\partial t}\qquad(\text{for } i=1,\cdots,m-1,\ j=1,\cdots,n)$$

satisfy the bracket relations:

$$[X_i,X_\ell]=[X_i,Y_j]=[Y_j,Y_k]=0\qquad\text{and}\qquad[X_m,X_i]=\ln(\mu_i)X_i,\quad[X_m,Y_j]=\ln(\lambda_j)Y_j$$

(for $i,\ell=1,\cdots,m-1$ and $j,k=1,\cdots,n$,) and then generate over the field $\mathbb{R}$ a finite dimensional Lie algebra $\mathcal{H}$. It is the semi-direct product of the abelian algebra $\mathcal{H}_0$ generated by $X_1,\cdots,X_{m-1},Y_1,\cdots,Y_n$ and the one dimensional Lie algebra generated by X_m; $\mathcal{H}$ is solvable. The Lie subalgebra $\mathcal{G}$ defined by $X_1,\cdots,X_m$ is also solvable and it is an ideal of $\mathcal{H}$. The simply connected Lie groups H and G corresponding respectively to $\mathcal{H}$ and $\mathcal{G}$ can be constructed as follows. As the eigenvalues of the matrix A are real positive, the group $\mathbb{R}$ acts on $\mathbb{R}^{m+n-1}$:

$$(t,z) \in \mathbb{R} \times \mathbb{R}^{m+n-1} \longmapsto A^t z \in \mathbb{R}^{m+n-1}$$

(where $z = (x_1, \cdots, x_{m-1}, y_1, \cdots y_n)$) leaving invariant the eigenspace E corresponding to $\mu_1, \cdots, \mu_{m-1}$; this action defines the groups H and G respectively as the semi-direct products $\mathbb{R}^{m+n-1} \rtimes \mathbb{R}$ and $E \rtimes \mathbb{R}$. Because the coefficients of A are in $\mathbb{Z}$, the preceding action restricted to $\mathbb{Z}$ preserves the subgroup $\mathbb{Z}^{m+n-1}$; this gives a subgroup $\Gamma = \mathbb{Z}^{m+n-1} \rtimes \mathbb{Z}$ which is a cocompact lattice of H. The quotient $\mathbb{T}_A^{m+n} = H/\Gamma$ is a compact manifold of dimension $m+n$. As we have already pointed, any subgroup of H induces a locally free action on H/Γ which defines a foliation. In our example we have two subgroups: G and the normal abelian subgroup K whose Lie algebra is the ideal generated by $Y_1, \cdots, Y_n$. Their actions on $\mathbb{T}_A^{m+n}$ give respectively two foliations $\mathcal{F}$ and $\mathcal{V}$; $\mathcal{V}$ is a Lie foliation transversely modeled on the Lie group G.

The second one

Let Q be the quadratic form on $\mathbb{R}^{n+1}$ defined by $Q(x) = -x_0 + \sum_{i=1}^n x_i^2$ and let $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : Q(x) = -1 \text{ and } x_0 > 0\}$. The group of orientation preserving linear transformations of $\mathbb{R}^{n+1}$ which preserve Q is the group $H = \mathrm{SO}(1,n)$. It acts on $\mathbb{R}^{n+1}$; the isotropy subgroup of the point $(1, 0, \cdots, 0)$ is $K = \mathrm{SO}(n)$ and the quotient H/K is analytically equivalent to $\mathbb{H}^n$. Let Γ be a torsion-free subgroup of H (cf. [Bor]) such that the quotient $B = \Gamma \setminus \mathbb{H}^n = \Gamma \setminus H/K$ is a n-dimensional compact manifold.

Since H is a linear group (it is a subgroup of $\mathrm{GL}(n,\mathbb{R})$) the elements of its Lie algebra $\mathcal{H}$ can be represented by matrices; they are of the form $\begin{pmatrix} 0 & A \\ A^* & B \end{pmatrix}$ where $A = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$, A^* its transpose and B is an $n \times n$ skew-symmetric matrix. A basis of $\mathcal{H}$ is given by the $(n+1)\times(n+1)$-matrices A_i with $i = 1, \cdots, n$ and $B_{k\ell}$ with $k, \ell = 1, \cdots, n$ where A_i is symmetric and has 1 at line 0 and column i and 0 elsewhere, $B_{k\ell}$ is skew symmetric and has -1 at line k and column ℓ and 0 elsewhere. Easy computations show that the commutators of these elements are given by the following formulaes:

$$[A_i, A_j] = -B_{ij} \qquad [A_i, B_{k\ell}] = \begin{cases} -A_\ell & \text{if } i = k \\ -A_k & \text{if } i = \ell \\ 0 & \text{otherwise} \end{cases}$$

and:

$$[B_{k\ell}, B_{k'\ell'}] = \begin{cases} -B_{k\ell'} & \text{if } \ell = k' \\ 0 & \text{otherwise.} \end{cases}$$

The group H acts on the bundle $F(\mathbb{H}^n)$ of oriented orthonormal tangent frames of $\mathbb{H}^n$ in such way that for given two frames ε and ε' there exists only one element $h \in H$ such that $h \cdot \varepsilon = \varepsilon'$; then H is diffeomorphic to $F(\mathbb{H}^n)$. The subgroup $\widehat{K}$ corresponding to the subalgebra $\widehat{\mathcal{K}}$ generated by $\{B_{k\ell} : 2 \leq k < \ell \leq n\}$ fixes a point of $\mathbb{H}^n$ and a unit tangent vector at that point; hence the quotient $F(\mathbb{H}^n)/\widehat{K}$ is diffeomorphic to the bundle $S(\mathbb{H}^n)$ of unit tangent vectors to $\mathbb{H}^n$ which is of dimension $2n-1$.

The Lie algebra $\mathcal{H}$ has two n-dimensional subalgebras $\mathcal{G}^+$ and $\mathcal{G}^-$ whose bases are respectively given by the two families:

$$\left\{A_1, \frac{\sqrt{2}}{2}(-A_2 + B_{12}), \cdots, \frac{\sqrt{2}}{2}(-A_2 + B_{1n})\right\}$$

and:

$$\left\{A_1, \frac{\sqrt{2}}{2}(-A_2 - B_{12}), \cdots, \frac{\sqrt{2}}{2}(-A_2 - B_{1n})\right\}$$

These subalgebras define two foliations $\widetilde{\mathcal{F}}^+$ and $\widetilde{\mathcal{F}}^-$ both of dimension n. They are also the foliations defined by the left actions on H of the subgroups G^+ and G^- whose Lie subalgebras are respectively $\mathcal{G}^+$ and $\mathcal{G}^-$. The adjoint action of $\widehat{K}$ on H leaves the above two foliations invariant then they pass to the right quotient $F(\mathbb{H}^n)/\widehat{K}$ giving rise to two foliations $\widehat{\mathcal{F}}^+$ and $\widehat{\mathcal{F}}^-$.

Now the fundamental group of B is isomorphic to Γ and may be considered as a subgroup of H. The quotient of $S(\mathbb{H}^n)$ by the left action of Γ is the tangent sphere bundle M of the manifold B; it is a compact manifold of dimension $2n-1$. The two foliations $\widehat{\mathcal{F}}^+$ and $\widehat{\mathcal{F}}^-$ are left Γ-invariant and induce two foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on M both of dimension n and codimension $n-1$. Their intersection is the one dimensional foliation generated by the vector field A_1.

4. Suspension of diffeomorphism groups

One of the main class of foliations is obtained by the suspension procedure of groups of diffeomorphisms. This section will be devoted to the definition of this procedure and to some examples of groups of diffeomorphisms which give interesting foliations.

4.1. General construction

Let B and F be two manifolds, respectively of dimensions m and n. Suppose that the fundamental group $\pi_1(B)$ of B is finitely generated. Let $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ be a representation, where $\mathrm{Diff}(F)$ is the diffeomorphism group of F. Denote by $\widetilde{B}$ the universal covering of B and $\widetilde{\mathcal{F}}$ the horizontal foliation on $\widetilde{M} = \widetilde{B} \times F$, i.e., the foliation whose leaves are the subsets $\widetilde{B} \times \{y\}$, $y \in F$. This foliation is invariant by all the transformations $T_\gamma : \widetilde{M} \longrightarrow \widetilde{M}$ defined by $T_\gamma(\widetilde{x}, y) = (\gamma \cdot \widetilde{x}, \rho(\gamma)(y))$ where $\gamma \cdot \widetilde{x}$ is the natural action of $\gamma \in \pi_1(B)$ on $\widetilde{B}$; then $\widetilde{\mathcal{F}}$ induces a codimension n foliation $\mathcal{F}_\rho$ on the quotient manifold:

$$M = \widetilde{M}/(\widetilde{x}, y) \sim (\gamma \cdot \widetilde{x}, \rho(\gamma)(y)).$$

We say that $\mathcal{F}_\rho$ is the *suspension* of the diffeomorphism group $\Gamma = \rho(\pi_1(B))$. The leaves of $\mathcal{F}_\rho$ are transverse to the fibres of the natural fibration induced by the projection on the first factor $\widetilde{B} \times F \longrightarrow \widetilde{B}$.

Conversely, suppose that $F \longrightarrow M \stackrel{\pi}{\longrightarrow} B$ is a fibration with compact fibre F and that $\mathcal{F}$ is a codimension n foliation ($n =$ dimension of F) transverse to the fibres of π. Then there exists a representation $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ such that $\mathcal{F} = \mathcal{F}_\rho$.

The geometric transverse structures of the foliation $\mathcal{F}$ are exactly the geometric structures on the manifold F invariant by the action of Γ. So to give examples of foliations obtained by suspension, it is sufficient to construct examples of diffeomorphism groups. This is what we shall do now.

4.2. Let B be the circle $\mathbb{S}^1$ and $F = \mathbb{R}_+ = [0, +\infty[$. Let ρ be the representation of $\mathbb{Z} = \pi_1(\mathbb{S}^1)$ in $\mathrm{Diff}([0, +\infty[)$ defined by $\rho(1) = \varphi$ where $\varphi(y) = \lambda y$ with $\lambda \in]0, 1[$. Because φ is isotopic to the identity map of F, the manifold M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}_+$ and the foliation $\mathcal{F}_\rho$ has one closed leaf diffeomorphic to the circle $\mathbb{S}^1$, corresponding to the fixed point $\varphi(0) = 0$ (see Fig. I.5).

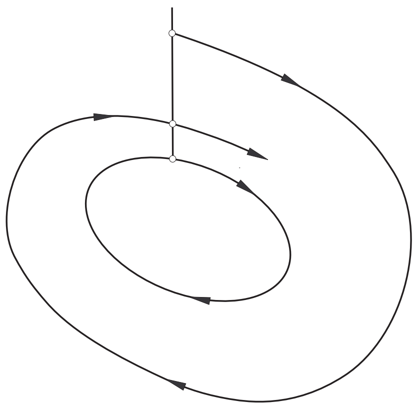


Fig. I.5

4.3. Let n be an integer ≥ 2 and A a matrix of order n with coefficients in $\mathbb{Z}$ and determinant equal to 1 *i.e.* A is an element of $\mathrm{SL}(n, \mathbb{Z})$. Suppose that A admits n real positive eigenvalues $\lambda_1, \cdots, \lambda_n$ such that, for each $\lambda \in \{\lambda_1, \cdots, \lambda_n\}$, the components $(v^1, \cdots, v^n)$ in $\mathbb{R}^n$ of an eigenvector v associated to λ are linearly independent over $\mathbb{Q}$

i.e. , for $\mathbf{m} \in \mathbb{Z}^n$, every relation $\langle \mathbf{m}, v\rangle = 0$ implies $\mathbf{m} = 0$ (where $\langle\ ,\ \rangle$ is the Euclidean product in $\mathbb{R}^n$). Such matrices exist ; take for instance(cf. [EN1]):

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & d_n \end{pmatrix}$$

with $d_1 = 1$ and $d_{i+1} = 1 + d_1 d_2 \cdots d_i$ for $i = 1, \cdots, n-1$. This fact is easy to verify for $n \leq 3$. Let G be the solvable Lie group and Γ its lattice like in the example one subsection 3.5. The quotient manifold $B = G/\Gamma$ is a flat fibre bundle with fibres the n-torus $\mathbb{T}^n$ over the circle $\mathbb{S}^1$.

Now let $\lambda \in \{\lambda_1, \cdots, \lambda_n\}$ and v an associated eigenvector. Since $\lambda\langle \mathbf{m}, v\rangle = \langle \mathbf{m}', v\rangle$ where $A'(\mathbf{m}) = \mathbf{m}' \in \mathbb{Z}^n$ (A' is the transpose matrix of A), Γ can be embedded in $\mathrm{SL}(n, \mathbb{C})$ as follows: choose integers $a_1, \cdots, a_{n-1}$, set $a = a_1 + \cdots + a_{n-1}$ and associate to $(\mathbf{m}, \ell) \in \Gamma = \mathbb{Z}^n \rtimes \mathbb{Z}$ the matrix $n \times n$:

$$\lambda^{-\frac{a\ell}{n}} \begin{pmatrix} \lambda^{a_1 \ell} & \cdots & 0 & \langle \mathbf{m}, v\rangle \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & & \lambda^{a_{n-1}\ell} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(only the terms in the diagonal and the term in the first line and the n^{th} column are nonzero). So we obtain an injective representation:

$$\rho : \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^{n-1}(\mathbb{C})).$$

The action of Γ on $P^{n-1}(\mathbb{C})$ extends to the point ∞ the affine action:

$$(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} \longmapsto (\lambda^{a_1\ell} z_1 + \langle \mathbf{m}, \ell\rangle, \lambda^{a_2\ell} z_2, \cdots, \lambda^{a_{n-1}\ell} z_{n-1}) \in \mathbb{C}^{n-1}$$

for every $(\mathbf{m}, \ell) \in \Gamma$. The suspension of this representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension $n-1$ on the compact differentiable manifold M, quotient of $\widetilde{M} = P^{n-1}(\mathbb{C}) \times G$ by the equivalence relation which identifies (z, x) to $(\rho(\gamma)(z), \gamma x)$ with $\gamma \in \Gamma$ (Γ acts on G by left translations). The leaves of $\mathcal{F}$ are homogeneous spaces of G by discrete subgroups. Note that $\mathcal{F}$ is not transversely Kählerian because the image of the representation ρ does not preserve the Kählerian metric on $P^{n-1}(\mathbb{C})$.

4.4. Let $\mathrm{SL}(n,\mathbb{R})$ be the group of real matrices of order n and determinant 1. This is a real form of the group $\mathrm{SL}(n,\mathbb{C})$ (complex matrices of order n and determinant 1). This group acts by projective tranformations on $P^{n-1}(\mathbb{C})$ (complex projectice space of dimension $n-1$). Then every subgroup of $\mathrm{SL}(n,\mathbb{C})$ acts by the same transformations on $P^{n-1}(\mathbb{C})$.

The construction of the following group Γ and the study of its properties can be found in [Mil]. In the upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$ with the Poincaré metric $\frac{dx^2+dy^2}{y^2}$ we consider a geodesic triangle $T(p,q,r)$ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$. We denote by σ_1, σ_2 and σ_3 the reflections associated respectively to the sides of this triangle; they generate an isometry group Σ^*; elements which preserve the orientation form a subgroup Σ of Σ^* of index 2 called the *triangle group* and denoted $T(p,q,r)$. It is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and its pull-back Γ by the projection $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \longrightarrow \mathrm{SL}(2,\mathbb{R})$ ($\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal covering of $\mathrm{SL}(2,\mathbb{R})$) is a central extension: $0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \Sigma \longrightarrow 1$. The group Γ has the presentation:

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^p = \gamma_2^q = \gamma_3^r = \gamma_1\gamma_2\gamma_3 \rangle.$$

The quotient $B = \widetilde{\mathrm{SL}}(2,\mathbb{R})/\Gamma$ is a compact manifold of dimension 3. If the integers p, q et r are mutually prime the cohomology (with coefficients in $\mathbb{Z}$) of B is exactly the cohomology of the sphere $\mathbb{S}^3$. Since Γ is a subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, it acts on $P^1(\mathbb{C})$. So we obtain a (non injective) representation $\rho : \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^1(\mathbb{C}))$. The suspension of such representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension 1 on the differentiable manifold M of dimension 5, which is the quotient of $\widetilde{M} = P^1(\mathbb{C}) \times \widetilde{\mathrm{SL}}(2,\mathbb{R})$ by the equivalence relation which identifies (z,x) with $(\rho(\gamma)(z), \gamma x)$ (Γ acts on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ by left translation). The leaves of $\mathcal{F}$ are homogeneous spaces of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ by discrete subgroups.

4.5. The 1-dimensional real projective space $P^1(\mathbb{R})$ is obtained by adding the point ∞ to the real line $\mathbb{R}$; it is also isomorphic to the circle $\mathbb{S}^1$. The group $\mathrm{SL}(2,\mathbb{R})$ of 2-order real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ acts analytically on $\mathbb{S}^1$ by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x \right) \in \mathrm{SL}(2,\mathbb{R}) \times \mathbb{S}^1 \longmapsto \frac{ax+b}{cx+d} \in \mathbb{S}^1.$$

For any integer m such that $m \geq 2$, the two elements $\gamma_1 = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ generate a free non Abelian subgroup Γ (*cf.* [KM]) of the group $\mathrm{Diff}(\mathbb{S}^1)$ of diffeomorphisms of the circle $\mathbb{S}^1$.

Let B_1 and B_2 be two copies of $\mathbb{S}^2 \times \mathbb{S}^1$; each one of them has its fundamental group isomorphic to $\mathbb{Z}$. By Van Kampen theorem, the connected sum $B = B_1 \# B_2$ (which is a 3-dimensional manifold) has the non Abelian free group on two generators α_1 and α_2 as fundamental group. Let $\rho : \pi_1(B) \longrightarrow \Gamma$ be the representation defined by $\rho(\alpha_1) = \gamma_1$ and $\rho(\alpha_2) = \gamma_2$. As usual, the suspension of this representation gives rise to a codimension 1 foliation on the 4-manifold M which is a flat bundle $\mathbb{S}^1 \longrightarrow M \longrightarrow B$. This foliation is transversely homogeneous (in fact transversely *projective*).

5. Codimension one foliations

The richness of this category of foliations comes from the existence of non singular transverse vector fields which give a way to pass from a leaf to an other one. Most of the results in Foliation Theory were first obtained in the codimension one case; we will summarize very few of them.

Existence

Let $\mathcal{F}$ be a codimension one foliation on a compact manifold M and let ν be a transverse vector bundle to $\mathcal{F}$. Because ν is of rank one, it is integrable and defines a foliation $\mathcal{V}$ transverse to $\mathcal{F}$. So we have clearly $\chi(M) = 0$. It is natural to ask if this condition is sufficient for the existence of a codimension one foliation on M. This question was answered by W. Thurston [Thu]:

5.1. Theorem. *Let M be a compact manifold. Then M admits a codimension one foliation if, and only if, the Euler-Poincaré number $\chi(M)$ of M is zero.*

The regularity property seems to be very important in the existence of foliations on compact manifolds. In particular there is a big difference in the treatment between the C^∞ case and the real analytic one. In this direction A. Haefliger proved in [Hae1] the following important theorem.

5.2. Theorem. *Let M be a compact manifold with a finite fundamental group. Then M has no real analytic codimension one foliation.*

Topological behavior of leaves

Let $\mathcal{F}$ be a codimension one foliation on a connected manifold M. A subset $A \subset M$ is called *invariant* (for $\mathcal{F}$) if it is saturated: if it contains x, then it contains the leaf passing through x. A leaf L can be of three types:

(i) L is *proper*, that is, its topology coincides with the topology induced by M. (For instance any closed leaf is proper.)
(ii) *Locally dense*: there exists an invariant open set $\mathcal{O}$ such that $\overline{L} \cap \mathcal{O} = \mathcal{O}$.
(iii) *exceptional*: it is neither proper nor locally dense.

A subset K of M is called *minimal* if it is nonempty, closed, invariant and minimal for these properties: if $K' \subset K$ has the same properties then $K' = K$. It can be of three types:

(i') K is a proper leaf (compact if M is compact).

(ii') K is equal to the whole manifold M; in this case every leaf of $\mathcal{F}$ is dense. We say that the foliation is *minimal*.

(iii') K is union of exceptional leaves. We say that K is an *exceptional minimal set*.

The construction of a foliation with prescribed type of minimal set is a problem which is far to be trivial. But many results and examples were obtained in this direction. One of them was by S. Novikov [Nov] on the existence of compact leaves on 3-manifolds.

5.3. Theorem. *Let M be a compact 3-manifold with a finite fundamental group. Then any codimension one foliation on M has a compact leaf diffeomorphic to the torus $\mathbb{T}^2$.*

The topology of a compact leaf may determine the nature of the foliation on its neighborhood. This is described for instance by the following theorem due to G. Reeb [Ree].

5.4. Theorem. (Local stability) *Suppose that $\mathcal{F}$ admits a compact leaf L with finite fundamental group. Then L admits a saturated neighborhood V such that every leaf contained in V is compact with finite fundamental group.*

This theorem is in fact valid even if the codimension is greater than 1.

The existence of an exceptional minimal set K for a codimension one foliation $\mathcal{F}$ forces the holonomy of some leaf $L \subset K$ to have a special behavior as it is stated in the following theorem due to R. Sacksteder [Sac].

5.5. Theorem. *Let $\mathcal{F}$ be a codimension one foliation on a compact manifold M with an exceptional minimal set K. Then there exist a leaf $L \subset K$ and a closed curve $\sigma : [0,1] \longrightarrow L$ such that if $h :]-\varepsilon, \varepsilon[\longrightarrow]-\varepsilon, \varepsilon[$ is a representative of the germ of holonomy of σ (the segment $]-\varepsilon, \varepsilon[$ is viewed as a small transversal to $\mathcal{F}$ at the point $x_0 = \sigma(0) = \sigma(1)$) then $h'(0) < 1$. In particular, the holonomy of the leaf L is non trivial.*

In the same order of ideas, R. Sacksteder has constructed, by the suspension procedure on the 3-manifold $\Sigma_2 \times \mathbb{S}^1$ a codimension one foliation with an exceptional minimal set. (Here Σ_2 denotes the compact orientable surface of genus 2.)

Of course, the consistence of a mathematical theme is measured by the quantity of interesting examples it can produce. For instance, one can ask: *does there exists a simply connected manifold M which supports a codimension one minimal foliation*? The first example was given by G. Hector:

5.6. Theorem. *The Euclidean space* $\mathbb{R}^3$ *supports a codimension one foliation whose leaves are all dense.*

The construction of this foliation is very laborious. The reader who is more interested in this example can see the original article [Hec] or the reference [CL] where it is also treated elementarily and in detail.

PART II

A DIGRESSION: BASIC GLOBAL ANALYSIS

A foliation $\mathcal{F}$ on a manifold M is the geometric realization of a completely differential system S: the leaves of $\mathcal{F}$ are exactly the integral manifolds of S. One passes from a leaf to another by changing the initial condition of S; so the leaf space $B = M/\mathcal{F}$ can be interpreted as a parameter space of the solutions of S. Even if, in general, B has no differentiable structure, one can define on it many geometric objects: they correspond to their analogous in the classical sense 'invariant along the leaves'. Then one can ask: *in which sense the space B looks like a good manifold?* The goal of this Part II is to show that, if $\mathcal{F}$ is Riemannian and M is compact, then B behaves like a compact Riemannian manifold in the sense of global analysis: for instance, on B, one can consider elliptic differential equations and solve them in the same conditions as in a compact manifold. This enables one to show that the cohomological properties of a compact Riemannian manifold or a compact Kählerian one can be transposed to the space B.

1. Foliated bundles

Let $\mathcal{P} : G \hookrightarrow P \xrightarrow{\iota} M$ be a principal bundle with structural group $G \subset \mathrm{GL}(n, \mathbb{C})$. The group G acts on P on the right and on its Lie algebra $\mathcal{G}$ by the adjoint representation Ad *i.e.*, for $g \in G$ and $X \in \mathcal{G}$, $Ad_g(X) = gXg^{-1}$. Denote by $\mathcal{V}$ the vector bundle whose fibre V_z at a point $z \in P$ is the tangent space at z of the fibre of $\mathcal{P}$.

Let $\mathcal{E} : E \longrightarrow M$ be a complex vector bundle defined by a cocycle $\{U_i, g_{ij}, G\}$ where $\{U_i\}$ is an open cover of M and $g_{ij} : U_i \cap U_j \longrightarrow G \subset \mathrm{GL}(n, \mathbb{C})$ are the transition functions. To such a vector bundle we can always associate a principal bundle $G \longrightarrow P \longrightarrow M$ whose fibre is the group G and the transition functions are exactly the g_{ij} (viewed as translations on G).

There are different ways to define a connection on a vector bundle $\mathcal{E}$: on $\mathcal{E}$ directly or by using the associated principal bundle. We shall make use of all these possibilities.

<u>First definition</u>

A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ of TP such that:

(a) - for every $z \in \mathcal{P} : T_zP = V_z \oplus H_z$ (H_z is the fibre of $\mathcal{H}$ at z),
(b) - for every $g \in G$ and every $z \in P$: $H_{zg} = (R_g)_* H_z$ where R_g is the right action of g on P and $(R_g)_*$ its derivative.

Second definition

A *connection* on $\mathcal{P}$ is a subbundle $\mathcal{H}$ given by the kernel of a G-invariant 1-form ξ on P with values in $\mathcal{G}$. The G-invariance of ξ means: $(R_g)^*(\xi) = Ad_{g^{-1}}(\xi)$ *i.e.* for $z \in P$, $X \in T_zP$ and $g \in G$, $\xi_{zg}((R_g)_*(X)) = g^{-1}\xi_z(X)g$.

Third definition

A *linear connection* on the vector bundle $\mathcal{E}$ is a map $\nabla : \chi(M) \times C^\infty(\mathcal{E}) \longmapsto C^\infty(\mathcal{E})$ which associates to each (X, α) a section $\nabla_X \alpha$ satisfying the following properties:

(c) - ∇ is $C^\infty(M)$-linear on the first factor, that is, for $\alpha \in C^\infty(\mathcal{E})$, $X, Y \in \chi(M)$ and functions $f, g \in C^\infty(M)$, we have $\nabla_{fX+gY}\alpha = f\nabla_X\alpha + g\nabla_Y\alpha$,

(d) - for $\alpha \in C^\infty(\mathcal{E})$, $X \in \chi(M)$ and $f \in C^\infty(M)$ we have $\nabla_X(f\alpha) = f\nabla_X\alpha + (Xf)\alpha$ where Xf is the derivative of the function f in the direction of the vector field X.

In fact, the map ∇ is the *covariant derivative* of the connection. The *curvature* of this connection is the 2-form $\mathcal{R}$ with values in $\mathrm{End}(\mathcal{E})$ (the space of endomorphisms of $\mathcal{E}$) defined by:

$$\mathcal{R}(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

Now, suppose that we are given a connection $\mathcal{H}$ (like in the first definition or the second one) on the principal bundle $\mathcal{P}$. It is easy to see that the restriction of ι_* (the derivative of ι) to H_z is an isomorphism onto $T_{\iota(z)}M$. Let $\tau = \iota_*^{-1}(T\mathcal{F})$. We say that $\mathcal{P}$ is *foliated* if τ is integrable. In this case, τ defines a foliation $\widetilde{\mathcal{F}}$ on P such that:

(e) - $\dim(\widetilde{\mathcal{F}}) = \dim(\mathcal{F})$,

(f) - $\widetilde{\mathcal{F}}$ is invariant under the action of G.

1.1. Definition. *We say that the connection $\mathcal{H}$ is* **basic** *if ξ is basic; a foliated bundle $\mathcal{E}$ is said to be a $\mathcal{F}$-***bundle** *if it admits a basic connection. We say that $\mathcal{E}$ is a $\mathcal{F}$-bundle if the associated principal bundle is a $\mathcal{F}$-bundle.*

A vector bundle $\mathcal{E}$ with a linear connection is foliated if, and only if, its curvature form $\mathcal{R}$ satisfies $\mathcal{R}(X, Y) = 0$ for $X, Y \in \Gamma(\mathcal{F})$; $\mathcal{E}$ is a $\mathcal{F}$-bundle if, and only if, $i_X\mathcal{R} = 0$ for $X \in \Gamma(\mathcal{F})$ (*cf.* [KT1]).

The foliation $\mathcal{F}_{\widehat{E}}$ on $\widehat{E} = P \times \mathbb{C}^n$ whose leaves are (leaf of $\widetilde{\mathcal{F}}$) $\times$ (point of $\mathbb{C}^n$) is invariant by the diagonal action of G ; so it induces a foliation $\mathcal{F}_E$ on $E = P \times_G \mathbb{C}^n$.

A *$\mathcal{F}$-morphism* $\varphi : (\mathcal{E}, \xi) \longrightarrow (\mathcal{E}', \xi')$ between two $\mathcal{F}$-bundles is a morphism of vector bundles such that $\xi = \varphi^*(\xi')$.

The collection of $\mathcal{F}$-bundles and $\mathcal{F}$-morphisms on M is a category. So we can define the group $K(M, \mathcal{F})$ of *foliated K-theory* as in the classical case.

1.2. Examples

(1.2.1) – Suppose that we are given a Riemannian metric on M. Let $T\mathcal{F}^{\perp}$ be the subbundle of TM orthogonal to $\mathcal{F}$ and $\Gamma(T\mathcal{F}^{\perp})$ the space of its sections. Every $X \in \chi(M)$ can be uniquely writen $X = X_{\mathcal{F}} + X_{\nu}$ where $X_{\mathcal{F}} \in \Gamma(\mathcal{F})$ and $X_{\nu} \in \Gamma(T\mathcal{F}^{\perp})$. Let $\pi : TM \longrightarrow \nu\mathcal{F}$ be the canonical projection. For every section Y of the bundle $\nu\mathcal{F}$ we denote by $\widetilde{Y}$ a vector field on M which projects on Y. For every $X_{\mathcal{F}} \in \Gamma(\mathcal{F})$ and every $Y \in C^{\infty}(\nu\mathcal{F})$, $\pi([X_{\mathcal{F}}, \widetilde{Y}])$ is independent of the choice of the $\widetilde{Y}$. Let $\widehat{\nabla}$ be any linear connection on $\nu\mathcal{F}$. So we can define a linear connection on the vector bundle $\nu\mathcal{F}$: $\nabla : \chi(M) \times C^{\infty}(\nu\mathcal{F}) \longrightarrow C^{\infty}(\nu\mathcal{F})$, by:

$$\nabla_X Y = \pi([X_{\mathcal{F}}, \widetilde{Y}]) + \widehat{\nabla}_{X_{\nu}} Y. \tag{II.1}$$

It is called a *Bott connection* of $\mathcal{F}$. A simple calculation, using the integrability of the subbundle $T\mathcal{F}$ and the Jacobi identity, shows that the curvature form $\mathcal{R}$ satisfies the equation $\mathcal{R}(X, Y) = 0$ for $X, Y \in \Gamma(\mathcal{F})$; this implies that the vector bundle $\nu\mathcal{F}$ is foliated.

(1.2.2) – Every flat vector bundle $\mathcal{E} : E \longrightarrow M$ (*i.e.* the transition functions of $\mathcal{E}$ are constant) is a $\mathcal{F}$-bundle.

(1.2.3) – Let $\mathcal{E} : E \longrightarrow M$ be a $\mathcal{F}$-bundle. Then the dual bundle $\mathcal{E}^*$ and all of its exterior and symmetric powers $\Lambda^*\mathcal{E}^*$ and $\mathcal{S}^*\mathcal{E}^*$ are $\mathcal{F}$-bundles; also $\mathcal{H}^2\mathcal{E} = \{\text{Hermitian forms on } \mathcal{E}\}$ is a $\mathcal{F}$-bundle.

2. Transversely Elliptic Operators

Let $\mathcal{E}$ be a $\mathcal{F}$-bundle and denote by $C^{\infty}(\mathcal{E})$ the space of its global sections; let ∇ denotes the covariant derivative $\chi(M) \times C^{\infty}(\mathcal{E}) \longrightarrow C^{\infty}(\mathcal{E})$ associated to the connection.

2.1. Definition. *We say that a section $\alpha \in C^{\infty}(\mathcal{E})$ is* ***basic*** *if it satisfies the condition $\nabla_X \alpha = 0$ for every $X \in \Gamma(\mathcal{F})$.*

For any $\mathcal{F}$-bundle $\mathcal{E}$, denote by $\widetilde{\mathcal{E}}$ the sheaf of germs of its basic sections. The space of its global sections $C^{\infty}(\mathcal{E}/\mathcal{F})$ is an A-module (A is the algebra of basic functions). Let $\mathcal{E}$ and $\mathcal{E}'$ be two $\mathcal{F}$-bundles with ranks respectively N and N'.

2.2. Definition. *A* ***basic differential operator*** *of order $m \in \mathbb{N}$ from $\mathcal{E}$ to $\mathcal{E}'$ is a morphism of sheaves $D : \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{E}}'$ such that, on a local system of coordinates $(x_1, \ldots, x_d, y_1, \ldots y_n)$, D has the expression:*

$$D = \sum_{|s| \leq m} a_s(y) \frac{\partial^{|s|}}{\partial y_1^{s_1} \ldots \partial y_n^{s_n}}$$

where $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$*,* $|s| = s_1 + \cdots + s_n$ *and* a_s *are* $N \times N'$*-matrices whose coefficients are basic functions.*

The *principal symbol* of D at the point z and the covector $\zeta \in T_z^*M$ is the linear map $\sigma(D)(z,\zeta) : E_z \longrightarrow E'_z$ defined by:

$$\sigma(D)(z,\zeta)(\eta) = \sum_{|s|=m} \zeta_1^{s_1} \cdot \cdots \cdot \zeta_n^{s_n} a_s(y)(\eta).$$

We say that D is *transversely elliptic* if $\sigma(D)(z,\zeta)$ is an isomorphism for every $z \in M$ and every transverse covector ζ different from 0. If $\mathcal{F}$ is Riemannian, its conormal bundle $\nu^*\mathcal{F}$ is a $\mathcal{F}$-bundle and is equipped with a foliation $\mathcal{F}^*$. If in addition M is compact, $\sigma(D)(z,\zeta)$ defines an element $[D]$ in the group $K(\nu^*\mathcal{F}, \mathcal{F}^*)$.

A *Hermitian metric* on $\mathcal{E}$ is a positive definite section h of $\mathcal{H}^2\mathcal{E}$. If h is basic we say that $\mathcal{F}$ is a *Hermitian $\mathcal{F}$-bundle.* If $(\mathcal{E}, h)$ is a Hermitian $\mathcal{F}$-bundle and $D : C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}/\mathcal{F})$ is a basic operator of order $m = 2\ell$ we can define a quadratic form A on $\mathcal{E}$ by $A_z(\eta) = (-1)^\ell h\big(\sigma(D)(z,\zeta)(\eta), \eta\big)$ where $\eta \in E_z$. We say that D is *strongly transversely elliptic* if A is positive definite for every $z \in M$ and every transverse covector ζ different from zero. Of course, a strongly transversely elliptic operator is transversely elliptic.

Let $\{\mathcal{E}^r\}$ $(r \in \{0, \cdots, n\})$ be a sequence of Hermitian $\mathcal{F}$-bundles and basic operators $D_r : E^r = C^\infty(\mathcal{E}^r/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}^{r+1}/\mathcal{F}) = E^{r+1}$ such that the sequence:

$$(\mathcal{C}) \qquad 0 \longrightarrow E^0 \xrightarrow{D_0} \cdots \xrightarrow{D_{r-1}} E^r \xrightarrow{D_r} E^{r+1} \xrightarrow{D_{r+1}} \cdots \xrightarrow{D_{n-1}} E^n \longrightarrow 0$$

is a differential complex. Let $\sigma_r = \sigma(D_r)(z,\zeta) : E_z^r \longrightarrow E_z^{r+1}$ denotes the principal symbol of D_r at the point $z \in M$ and the transverse covector ζ. We say that the complex $(\mathcal{C})$ is *transversely elliptic* if the sequence:

$$0 \longrightarrow E_z^0 \xrightarrow{\sigma_0} \cdots \xrightarrow{\sigma_{r-1}} E_z^r \xrightarrow{\sigma_r} E_z^{r+1} \xrightarrow{\sigma_{r+1}} \cdots \xrightarrow{\sigma_{n-1}} E_z^n \longrightarrow 0$$

is exact for every $z \in M$ and every non zero transverse covector ζ. On each $C^\infty(\mathcal{E}^r/\mathcal{F})$ we can define an inner product given by the formula (II.2). Let D^* be the formal adjoint of D which is a basic operator from $C^\infty(\mathcal{E}^{r+1}/\mathcal{F})$ to $C^\infty(\mathcal{E}^r/\mathcal{F})$. Then $L_r = DD^* + D^*D$ is a selfadjoint operator on $C^\infty(\mathcal{E}^r/\mathcal{F})$. We can easily show that the differential complex $(\mathcal{C})$ is transversely elliptic if, and only if, for every $r \in \{0, \cdots, n\}$, L_r is strongly transversely elliptic.

From now on we suppose that M is compact and connected. Assume that the foliation $\mathcal{F}$ is Riemannian transversely oriented. Let $G = \mathrm{SO}(n) \longrightarrow M^\# \xrightarrow{p} M$ be the principal bundle of the orthonormal direct frames transverse to $\mathcal{F}$. Then, the foliation $\mathcal{F}$ lifts to a transversely parallelizable foliation $\mathcal{F}^{\#}$ on $M^{\#}$ of the same dimension and invariant under the action of the group G. Moreover, the leaf closures of $\mathcal{F}^{\#}$ are the fibres of a locally trivial fibration $F \longrightarrow M^{\#} \longrightarrow W$ where W is a compact manifold called the *basic manifold* of $\mathcal{F}$ (*cf.* I. 2.4). Let $\mathcal{E}^{\#}$ be the pullback by p of the bundle $\mathcal{E}$; then $\mathcal{E}^{\#}$ is a G-bundle and a Hermitian $\mathcal{F}^{\#}$-bundle with respect to a Hermitian metric $h^{\#}$. The basic sections of $\mathcal{E}$ are canonically identified to basic sections of $\mathcal{E}^{\#}$ which are invariant under the action of G. In particular, if $f : M \longrightarrow \mathbb{C}$ is a basic function, $f \circ p$ is a basic function on $M^{\#}$ (with respect to $\mathcal{F}^{\#}$); moreover $f \circ p$ is invariant by the action of G. Because $f \circ p$ is continuous, it is constant on the leaf closures of $\mathcal{F}^{\#}$, so it induces a G-invariant C^∞ function on the basic manifold W. We can prove, by the converse process, that any G-invariant C^∞ function on the basic manifold W defines a C^∞ basic function on M; so, the algebra A of basic functions on M is canonically isomorphic to the algebra $A_G(W)$ of functions on W invariant by G. The transverse metric on $M^{\#}$ (which makes $\mathcal{F}^{\#}$ Riemannian) induces a Riemannian metric on W for which G acts by isometries. Let μ be the measure on W associated to this metric (μ is a volume form if W is orientable, otherwise it is just a density).

On $C^\infty(\mathcal{E}/\mathcal{F})$ we define an inner product as follows. Let α and β be two elements of $C^\infty(\mathcal{E}/\mathcal{F})$. The function $z \in M \longmapsto h_z(\alpha(z), \beta(z)) \in \mathbb{C}$ is basic; so it defines a G-invariant function $\Theta(\alpha, \beta)$ on W. We set:

$$\langle \alpha, \beta \rangle = \int_W \Theta(\alpha, \beta)(w) d\mu(w). \tag{II.2}$$

For any transversely elliptic operator D from a Hermitian $\mathcal{F}$-bundle $\mathcal{E}$ to a Hermitian $\mathcal{F}$-bundle $\mathcal{E}'$, denote by $N(D)$ its kernel D and $R(D)$ its range. Let D^* be the formal adjoint of D; D^* is a basic operator from $\mathcal{E}'$ to $\mathcal{E}$ and it is transversely elliptic.

2.3. Theorem. *The kernel $N(D)$ of D is finite dimensional, the range $R(D^*)$ of D^* is closed and finite codimensional and we have an orthogonal decomposition*:

$$C^\infty(\mathcal{E}/\mathcal{F}) = N(D) \oplus R(D^*). \tag{II.3}$$

The proof of this theorem is long; it can be found in [E1]. We will just sketch the three principal steeps. It is not difficult to see that one can restrict our attention to the case where $E = F$, D is of even order $m = 2\ell$ and transversely strongly elliptic.

Step one : $\mathcal{F}$ is a Lie foliation with dense leaves

This step will be very important even if it is almot immediate.

• The vector space $C^\infty(\mathcal{E}/\mathcal{F})$ is finite dimensional. Indeed, a basic section which is zero at a point is zero everywhere by the density of leaves.

• Let $\overline{E}_0 = C^\infty(\mathcal{E}/\mathcal{F})$ and $N_0' = \dim\overline{E}_0$. The Hermitian metric on $\mathcal{E}$ induces a Hermtian metric on $\overline{E}_0$.

• The Hodge decomposition for the operator D is just the decomposition of a linear operator on a finite dimensional Hermitian space.

<u>Step two : $\mathcal{F}$ is a TP foliation</u>

• Consider the basic fibration $F \hookrightarrow M \longrightarrow W$ of $\mathcal{F}$. For $u \in W$, let F_u be the fibre of π over u and $\overline{\mathcal{E}}_u = C^\infty(\mathcal{E}_u/\mathcal{F}_u)$ where $\mathcal{E}_u$ and $\mathcal{F}_u$ are respectively the restrictions of $\mathcal{E}$ and $\mathcal{F}$ to F_u. Then, by step one, $\overline{\mathcal{E}}_u$ is finite dimensional complex vector space and one can prove (*cf.* [E1]) that :

• The dimension of $\overline{\mathcal{E}}_u$ is independent of $u \in W$;

• The set $\overline{\mathcal{E}} = \displaystyle\bigcup_{u\in W} \overline{\mathcal{E}}_u$ is a Hermitian vector bundle over the manifold W;

The vector bundle $\overline{\mathcal{E}} \longrightarrow W$ is called the *useful bundle* associated to $\mathcal{E}$. It is a key ingredient in the proof of the Hodge decomposition for transversely elliptic operators on Riemannian foliations.

• The linear map $\psi : C^\infty(\mathcal{E}/\mathcal{F}) \longmapsto C^\infty(\overline{\mathcal{E}})$ defined by $\psi(\alpha)(u) = \alpha_{|F_u}$ is an isomorphism of Hermitian vector bundles.

• The operator $D : \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{E}}$ induces a strongly elliptic operator $\overline{D} : \widetilde{\overline{\mathcal{E}}} \longrightarrow \widetilde{\overline{\mathcal{E}}}$ of the same order and such that, for any open set $U \subset W$ trivializing the vector bundle $\overline{\mathcal{E}}$, the diagram:

$$\begin{array}{ccc} C^\infty_V(\mathcal{E}/\mathcal{F}) & \xrightarrow{D} & C^\infty_V(\mathcal{E}/\mathcal{F}) \\ \psi \downarrow & & \downarrow \psi \\ C^\infty_U(\overline{\mathcal{E}}) & \xrightarrow{\overline{D}} & C^\infty_U(\overline{\mathcal{E}}) \end{array}$$

is commutative where $V = \pi^{-1}(U)$. Then the classical Hodge decomposition for $\overline{D}$ gives the Hodge decompostion for the transversely elliptic operator D.

<u>Step three : the general case</u>

We suppose that the foliation $\mathcal{F}$ is transversely orientable. We denote by G the group SO(n) and let $G \longrightarrow M^\# \xrightarrow{\rho} M$ be the principal bundle of direct orthonormal frames transverse to $\mathcal{F}$.

• Denote by $\mathcal{E}^\#$ the pullback of $\mathcal{E}$ to $M^\#$; $\mathcal{E}^\#$ is also a $\mathcal{F}^\#$-Hermitian vector bundle of the same rank as $\mathcal{E}$.

• Let $C_G^\infty(\mathcal{E}^\#/\mathcal{F}^\#)$ be the subspace of $C^\infty(\mathcal{E}^\#/\mathcal{F}^\#)$ whose elements are $\mathcal{F}^\#$-basic sections of $\mathcal{E}^\#$ which are invariant by the action of G. Then one has a canonical isomorphism:

$$\theta : C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C_G^\infty(\mathcal{E}^\#/\mathcal{F}^\#).$$

• By using a basic connection on the principal bundle $\rho : M^\# \longrightarrow M$, one can lift the operator D to a basic differential operator $\overline{D}^\# : \widetilde{\widetilde{\mathcal{E}}}^\# \longrightarrow \widetilde{\widetilde{\mathcal{E}}}^\#$ which commutes with the action of G.

• Let $Q_1, \cdots, Q_N$ (where $N = \frac{n(n+1)}{2}$) be the fundamental vector fields of the action of G on $M^\#$. They can be considered as first order basic differential operators acting on the space $C^\infty(\mathcal{E}^\#/\mathcal{F}^\#)$. For each Q_j ($j \in \{1, \cdots, N\}$) let $\overline{Q}_j$ denote its complex conjugate; let:

$$Q' = \left(\sum_{j=1}^{N} Q_j \overline{Q}_j\right)^{\ell}, \qquad Q = (-1)^\ell Q' \quad \text{and} \quad D' = D^\# + Q.$$

• Then D' is a strongly transversely elliptic operator acting on $C^\infty(\mathcal{E}^\#/\mathcal{F}^\#)$; since the restriction of Q to the subspace $C_G^\infty(\mathcal{E}^\#/\mathcal{F}^\#)$ is zero, one has a commutative diagram:

$$(\mathcal{D}) \qquad \begin{array}{ccc} C_G^\infty(\mathcal{E}^\#/\mathcal{F}^\#) & \xrightarrow{D'} & C_G^\infty(\mathcal{E}^\#/\mathcal{F}^\#) \\ \theta^{-1}\downarrow & & \downarrow\theta^{-1} \\ C^\infty(\mathcal{E}) & \xrightarrow{\overline{D}} & C^\infty(\mathcal{E}) \end{array} .$$

• Now, since G is compact and commutes with D' the Hodge decomposition for D' induces a Hodge decomposition for this same operator on the space $C_G^\infty(\mathcal{E}^\#/\mathcal{F}^\#)$. Using the diagram $(\mathcal{D})$ one obtain a Hodge decomposition for D acting on $C^\infty(\mathcal{E}/\mathcal{F})$. This ends the sketch of the proof. □

3. Examples

3.1. The basic de Rham complex

We suppose as in Theorem 2.3 that $\mathcal{F}$ is Riemannian of codimension n, transversely oriented and that M is compact. For every $r \in \{0, \cdots, n\}$, let $\mathcal{E}^r$ denote the vector bundle $\Lambda^r(\nu^*\mathcal{F})$. Then $\mathcal{E}^r$ is a Hermitian $\mathcal{F}$-bundle. Its basic sections are exactly the basic forms of degree r which form a vector space denoted $\Omega^r(M/\mathcal{F})$. The exterior differential: $d : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^{r+1}(M/\mathcal{F})$ is a basic differential operator of order 1. The differential complex:

$$\text{(II.5)} \qquad 0 \longrightarrow \Omega^0(M/\mathcal{F}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r(M/\mathcal{F}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M/\mathcal{F}) \longrightarrow 0$$

is called the *basic de Rham complex* of $\mathcal{F}$; its homology is the basic cohomology $H^*(M/\mathcal{F})$ of the foliation $\mathcal{F}$.

To make things more simple, we suppose that $\mathcal{F}$ is *homologically orientable*, that is, the vector space $H^n(M/\mathcal{F})$ is non trivial, so it is necessarily one dimensional (*cf.* [EH]). This condition is equivalent to the existence of a (real) volume form on the leaves χ which is $\mathcal{F}$-relatively closed, that is, $d\chi(X_1, \cdots, X_d, Y) = 0$ for $X_1, \cdots, X_d \in \Gamma(\mathcal{F})$ (*cf.* [Mas]). (In that case, we can complete the transverse metric by a Riemannian metric along the leaves to obtain a Riemannian metric on the whole manifold for which the leaves are minimal and χ is associated to this metric.) This hypothesis will enable one to define an inner product on $\Omega^r(M/\mathcal{F})$ without using the basic manifold W. As in the classical case we define the Hodge star operator:

$$\text{(II.6)} \qquad * : \Omega^*(M/\mathcal{F}) \longrightarrow \Omega^*(M/\mathcal{F})$$

in the following way. Let U be an open set on which the foliation is trivial. Let $\theta_1, \cdots, \theta_n$ be real 1-forms such that $(\theta_1, \cdots, \theta_n)$ is an orthonormal basis of the free module $\Omega^1(U/\mathcal{F})$ (over the algebra of basic functions on U). Then define $*$ by:

$$*(\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}) = \varepsilon \theta_{j_1} \wedge \cdots \wedge \theta_{j_{n-r}}$$

where $\{j_1, \cdots, j_{n-r}\}$ is the increasing complementary sequence of $\{i_1, \cdots, i_r\}$ in the set $\{1, \cdots, n\}$ and ε is the signature of the permutation $\{i_1, \cdots, i_r, j_1, \cdots, j_{n-r}\}$. A straightforward calculation shows that $*$ satisfies the identity $** = (-1)^{r(n-r)}\mathrm{id}$. On $\Omega^r(M/\mathcal{F})$ we define a Hermitian product by:

$$\text{(II.7)} \qquad \langle \alpha, \beta \rangle = \int_M \alpha \wedge \overline{*\beta} \wedge \chi.$$

Then it is easy to see that the operator $\delta : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^{r-1}(M/\mathcal{F})$ defined by the formula $\delta = (-1)^{n(r-1)-1} * d*$ is the formal adjoint of $d : \Omega^{r-1}(M/\mathcal{F}) \longrightarrow \Omega^r(M/\mathcal{F})$ *i.e.* for every $\alpha \in \Omega^{r-1}(M/\mathcal{F})$ and every $\beta \in \Omega^r(M/\mathcal{F})$ we have $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$. Indeed:

$$\begin{aligned} d(\alpha \wedge *\beta \wedge \chi) &= d\alpha \wedge *\beta \wedge \chi + (-1)^{r-1} \alpha \wedge d(*\beta) \wedge \chi + (-1)^{n-1} \alpha \wedge *\beta \wedge d\chi \\ &= d\alpha \wedge *\beta \wedge \chi + (-1)^{(2-r)(r-1)-1} \alpha \wedge *(\delta\beta) \wedge \chi + (-1)^{n-1} \alpha \wedge *\beta \wedge d\chi \\ &= d\alpha \wedge *\beta \wedge \chi - \alpha \wedge *(\delta\beta) \wedge \chi + (-1)^{n-1} \alpha \wedge *\beta \wedge d\chi. \end{aligned}$$

Integrating the two members and using the fact that χ is $\mathcal{F}$-relatively closed, we obtain the desired equality. In the more general case in which the leaves are not minimal, the formula for the adjoint has a correction term involving the mean curvature of the foliation (*cf.* [Alv], [Ton] or [PR]). Let $\Delta_b : \Omega^r(M/\mathcal{F}) \longrightarrow \Omega^r(M/\mathcal{F})$ be the operator $\Delta_b = \delta d + d\delta$; Δ_b is selfadjoint; it is called the *basic Laplacian* (on the basic r-forms); a simple calculation, using local coordinates, proves that Δ_b is strongly transversely elliptic and therefore the complex (II.5) is transversely elliptic. Let:

$$\mathcal{H}^r(M/\mathcal{F}) = \mathrm{Ker}\Delta_b = \{\alpha \in \Omega^r(M/\mathcal{F}) : d\alpha = 0 \text{ and } \delta\alpha = 0\}.$$

An element of $\mathcal{H}^r(M/\mathcal{F})$ is called a *basic harmonic form* (of degree r). Then, applying Theorem 2.3, we obtain:

(i) dim $\mathcal{H}^r(M/\mathcal{F}) < +\infty$;

(ii) *we have orthogonal decompositions*:

$$\text{(II.8)} \qquad \Omega^r(M/\mathcal{F}) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(\Delta_b) = \mathcal{H}^r(M/\mathcal{F}) \oplus R(d) \oplus R(\delta).$$

As a consequence, the basic cohomology $H^r(M/\mathcal{F})$ is finite dimensional and is represented by $\mathcal{H}^r(M/\mathcal{F})$. Moreover the Hermitian map:

$$(\alpha, \beta) \in \Omega^r(M/\mathcal{F}) \times \Omega^{n-r}(M/\mathcal{F}) \longmapsto \int_M \alpha \wedge \overline{\beta} \wedge \chi \in \mathbb{C}$$

induces a non degenerate pairing $\Psi : H^r(M/\mathcal{F}) \times H^{n-r}(M/\mathcal{F}) \longrightarrow \mathbb{C}$ *i.e.* the basic cohomology $H^*(M/\mathcal{F})$ satisfies the Poincaré duality.

These results were originally obtained by B. Reinhart in [Rei2] without the assumption that $H^n(M/\mathcal{F})$ is nonzero. But in 1981, Y. Carrière [Car] constructed an example of a foliation whose basic cohomology does not satisfy Poincaré duality; this makes false a part of Reinhart's theorem. One year later F. Kamber and P. Tondeur [KT2] proved the same result as B. Reinhart for Riemannian foliations with minimal leaves (this is equivalent by [Mas] to $H^n(M/\mathcal{F}) \neq \{0\}$). We can easily observe that, with this hypothesis, Reinhart's proof is still valid. The general case (without any assumption) was completely established in [EH]. But as we have already pointed, these results are direct consequences of Theorem 2.3.

If $n = 2k = 4\ell$, Ψ defines a non degenerate quadratic form on $H^k(M/\mathcal{F})$; its signature is called the *signature* of $\mathcal{F}$ and is denoted Sign($\mathcal{F}$).

Now let $\mathcal{E}$ and $\mathcal{E}'$ be the vector bundles $\mathcal{E} = \bigoplus_{i\geq 0} \Lambda^{2i}(\nu^*\mathcal{F})$ and $\mathcal{E}' = \bigoplus_{i\geq 0} \Lambda^{2i+1}(\nu^*\mathcal{F})$. They are Hermitian $\mathcal{F}$-bundles and we have:

$$C^\infty(\mathcal{E}/\mathcal{F}) = \bigoplus_{i\geq 0} \Omega^{2i}(M/\mathcal{F}) \quad \text{and} \quad C^\infty(\mathcal{E}'/\mathcal{F}) = \bigoplus_{i\geq 0} \Omega^{2i+1}(M/\mathcal{F}).$$

The operator $d+\delta : C^\infty(\mathcal{E}/\mathcal{F}) \longrightarrow C^\infty(\mathcal{E}'/\mathcal{F})$ is basic and transversely elliptic, then it is a Fredholm operator. Its index:

$$\text{ind}_{\mathcal{F}}(d+\delta) = \sum_{i=0}^{n}(-1)^i \dim\, H^i(M/\mathcal{F}). \tag{II.9}$$

is the *basic Euler-Poincaré number* $\chi(M/\mathcal{F})$ of the foliation $\mathcal{F}$. As in the classical case, it is an obstruction to the existence of a nonsingular foliated vector field transverse to $\mathcal{F}$ (*cf.* [BPR]).

3.2. The basic Dolbeault complex

We suppose that $\mathcal{F}$ is Hermitian and, for simplicity, homologically orientable. Let ν be the complexified normal bundle $\nu\mathcal{F}\otimes_{\mathbb{R}}\mathbb{C}$ of $\nu\mathcal{F}$. Let J be the automorphism of ν associated to the complex structure; J satisfies $J^2 = -\text{id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles respectively denoted ν^{10} and ν^{01}. We have a splitting $\nu = \nu^{10}\oplus\nu^{01}$ which gives rise to a decomposition $\Lambda^r\nu^* = \bigoplus_{p+q=r}\Lambda^{p,q}$ where $\Lambda^{p,q} = \Lambda^p\nu^{10^*}\otimes\Lambda^q\nu^{01^*}$. Basic sections of $\Lambda^{p,q}$ are called *basic forms of type* (p,q). They form a vector space denoted $\Omega^{p,q}(M/\mathcal{F})$. We have:

$$\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r}\Omega^{p,q}(M/\mathcal{F}).$$

The exterior differential decomposes into a sum of two operators:

$$\partial : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p+1,q}(M/\mathcal{F}) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M/\mathcal{F}) \longrightarrow \Omega^{p,q+1}(M/\mathcal{F})$$

as in the classical case of a complex manifold. We have $\overline{\partial}^2 = 0$; so we obtain a differential complex:

$$\cdots \xrightarrow{\overline{\partial}} \Omega^{p,q}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M/\mathcal{F}) \xrightarrow{\overline{\partial}} \cdots \tag{II.10}$$

called the *basic Dolbeault complex* of $\mathcal{F}$; its homology $H^{p,q}(M/\mathcal{F})$ is the *basic Dolbeault cohomology* of the foliation $\mathcal{F}$: even though the leaf space is bad, it can be considered as a "complex manifold" whose Dolbeault cohomology is $H^{p,*}(M/\mathcal{F})$!

The star operator $*$ defined in (II.6) induces an isomorphism from the vector space $\Omega^{p,q}(M/\mathcal{F})$ to $\Omega^{n-q,n-p}(M/\mathcal{F})$. Moreover the restriction of the operator δ to the space $\Omega^{p,q}(M/\mathcal{F})$ decomposes into a sum of two operators $\delta' = - * \overline{\partial} *$ and $\delta'' = - * \partial *$ respectively of types $(-1,0)$ and $(0,-1)$. We can easily verify that δ'' is the formal adjoint of $\overline{\partial}$ for the inner product (II.7). Then the operator $\Delta''_b = \delta''\overline{\partial} + \overline{\partial}\delta''$ is self adjoint; a simple computation in local coordinates, like for the basic Laplacian, shows that Δ_b'' is strongly transversely elliptic and that the complex (II.10) is transversely elliptic. Let:

$$\mathcal{H}^{p,q}(M/\mathcal{F}) = \mathrm{Ker}\Delta_b'' = \{\alpha \in \Omega^{p,q}(M/\mathcal{F}) : \overline{\partial}\alpha = 0 \text{ and } \delta''\alpha = 0\}.$$

Applying again Theorem 2.3, we obtain:

(i) dim $\mathcal{H}^{p,q}(M/\mathcal{F}) < +\infty$;

(ii) *we have orthogonal decompositions*:

$$\Omega^{p,q}(M/\mathcal{F}) = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\Delta_b'') = \mathcal{H}^{p,q}(M/\mathcal{F}) \oplus R(\overline{\partial}) \oplus R(\delta''). \tag{II.11}$$

Consequently, the basic Dolbeault cohomology $H^{p,q}(M/\mathcal{F})$ is finite dimensional and is represented by $\mathcal{H}^{p,q}(M/\mathcal{F})$. Moreover the star operator induces an unitary isomorphism (of real vector spaces) $\overline{*} : \alpha \in \mathcal{H}^{p,q}(M/\mathcal{F}) \longmapsto \overline{*\alpha} \in \mathcal{H}^{n-p,n-q}(M/\mathcal{F})$ and then an isomorphism:

$$\overline{*} : H^{p,q}(M/\mathcal{F}) \longrightarrow H^{n-p,n-q}(M/\mathcal{F}) \tag{II.12}$$

i.e. the basic Dolbeault cohomology $H^{*,*}(M/\mathcal{F})$ satisfies the Serre duality.

Suppose now that $\mathcal{F}$ is transversely Kählerian with Kähler form ω (it is a basic differential form of degree 2; it is closed and non degenerate). In this case, we can prove that $\Delta_b = 2\Delta_b''$. Because of the decomposition $\Omega^r(M/\mathcal{F}) = \bigoplus_{p+q=r} \Omega^{p,q}(M/\mathcal{F})$, every basic differential r-form can be uniquely written $\alpha = \sum_{p+q=r} \alpha_{pq}$ where $\alpha_{pq} \in \Omega^{p,q}(M/\mathcal{F})$. Then we have the following assertions.

(iii) *α is Δ_b-harmonic if, and only if, each component α_{pq} is Δ_b''-harmonic. So we have a direct decomposition*

$$H^r(M/\mathcal{F}) = \bigoplus_{p+q=r} H^{p,q}(M/\mathcal{F}). \tag{II.13}$$

(iv) *The complex conjugation induces an isomorphism (of real vector spaces)*:

$$H^{p,q}(M/\mathcal{F}) \simeq H^{q,p}(M/\mathcal{F}).$$

(v) *For every odd $r \in \{0, \cdots, 2n\}$, the dimension of the space $H^r(M/\mathcal{F})$ is even. In particular if $n = 1$ we have $b_1(M/\mathcal{F}) = 2\mathrm{dim}\ H^{01}(M/\mathcal{F})$.*

The integer dim $H^{01}(M/\mathcal{F})$ will be denoted $g(\mathcal{F})$ and called the *genus* of the foliation $\mathcal{F}$. It is similar to the genus of a compact Riemann surface; it counts the number of linearly independent basic holomorphic 1-forms.

(vi) *Por every $p \in \{0, \cdots, n\}$ the differential form $\omega^p = \omega \wedge \cdots \wedge \omega$ (wedge product p times) is harmonic. So, the space $H^{p,p}(M/\mathcal{F})$ is non reduced to zero.*

PART III

SOME OPEN QUESTIONS

1. Transversely elliptic operators

1.1. Towards a basic index theory

Theorem 2.3 in Part II says that a basic transversely elliptic operator $D : \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{E}}'$ acting on basic sections is Fredholm over a manifold equipped with a Riemannian foliation; then it has an index defined as usual by the formula:

$$\mathrm{ind}_{\mathcal{F}}(D) = (\dim\ \mathrm{Ker}D - \dim\ \mathrm{Ker}D^*) \in \mathbb{Z}.$$

Problem. *Compute this integer in terms of invariants of the bundles $\mathcal{E}$ and $\mathcal{E}'$ and transverse topological invariants of $\mathcal{F}$. More precisely, is there an Atiyah-Singer Index Theorem for a transversely elliptic operator on a Riemannian foliation on a compact manifold?*

For example, in [EN] it was shown that the basic cohomology $H^*(M/\mathcal{F})$ is invariant by homeomorphism in the category of complete Riemannian foliations. So the basic signature and the Poincaré-Euler number of $\mathcal{F}$ (defined in subsection 3.1 Part II) are topological invariants. This reinforces the idea that it is certainly interesting to attack Problem 1.2.

Some progress were made in [GL] in solving Problem 1.2 in the particular case of Riemannian foliations whose Molino's central sheaf is Abelian, that is, the foliation in the leaf closure of $\mathcal{F}^{\#}$ is an Abelian Lie foliation.

1.2. Existence of transversely elliptic operators

Differential operators on a open set of the Euclidean space $\mathbb{R}^n$ abound while globally differential operators on a given manifolds are not so easy to found except the classical well know (Laplacian, Dirac operator...)

During Alberto's Fest in Cuernava in January 2003, we used to make the trip from the hotel to the institute by bus. Once I sat next to Dennis Sullivan and I talked little with him. I told him about transversally elliptic operators on Riemannian foliations. His first reaction was to let me out that such operators may actually exist only if the foliation is Riemannian. This was the origin of the following:

Question. *Let M be a compact manifold with a foliation $\mathcal{F}$ which admits non constant basic functions. Suppose that there exists a transversely elliptic operator acting on these functions. Is the foliation $\mathcal{F}$ Riemannian?*

Suppose that foliation is defined by a suspension $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ and let $\Gamma = \rho(\pi_1(B))$ and G its closure in $\mathrm{Diff}(F)$ with respect to the C^0-topology. In that case the operator D is an elliptic operator acting on the C^∞-functions on F and commuting with the action of G. Then by [Fur] the group G is compact; hence there exists a Riemannian metric on F for which G is a group of isometries that is $\mathcal{F}$ is Riemannian.

2. Complex foliations

Let M be a differentiable manifold of dimension $2m+n$ endowed with a codimension n foliation $\mathcal{F}$ (then the dimension of $\mathcal{F}$ is $2m$).

2.1. Definition. *The foliation $\mathcal{F}$ is said to be **complex** if it can be defined by an open cover $\{U_i\}$ of M and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \longrightarrow U_i$ (where Ω_i is an open polydisc in $\mathbb{C}^m$ and $\mathcal{O}_i$ is an open ball in $\mathbb{R}^n$) such that, for every pair (i,j) with $U_i \cap U_j \neq \emptyset$, the coordinate change $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \longrightarrow \phi_j^{-1}(U_i \cap U_j)$ is of the form $(z', t') = \left(\phi_{ij}^1(z,t), \phi_{ij}^2(t)\right)$ with $\phi_{ij}^1(z,t)$ holomorphic in z for t fixed.*

An open set U of M like one of the cover $\mathcal{U}$ is called *adapted* to the foliation. Any leaf of $\mathcal{F}$ is a complex manifold of dimension m. The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold.

2.2. Question

Does an odd sphere $\mathbb{S}^{2n+1}$ support a complex codimension one foliation?

The case of the sphere $\mathbb{S}^3$ is immediate. Indeed, a codimension one foliation is of dimension 2 and has a complex structure if in addition it is orientable. It is well known that such foliations exist in $\mathbb{S}^3$.

For the sphere $\mathbb{S}^5$ the question was answered by L. Meersseman and A. Verjovsky in [MV].

2.3. The $\overline{\partial}_{\mathcal{F}}$-cohomology

Let $(M,\mathcal{F})$ be a complex foliation of dimension m. Let $\Omega^{pq}(\mathcal{F})$ be the space of foliated differential forms of type (p,q) that is, differential forms on M which can be written in local coordinates adapted to the foliation $(z,t) = (z_1, \ldots, z_m, t_1, \ldots, t_n)$: $\alpha = \sum \alpha_{JK} dZ_J \wedge d\overline{Z}_K$ where $J = (j_1, \cdots, j_p)$, $K = (k_1, \cdots, k_q)$, $dZ_J = dz_{j_1} \wedge \ldots \wedge dz_{j_p}$ and $d\overline{Z}_K = dz_{k_1} \wedge \ldots \wedge dz_{k_q}$ and α_{JK} is a C^∞-function on (z,t). Let $\overline{\partial}_{\mathcal{F}} : \Omega^{pq}(\mathcal{F}) \longrightarrow \Omega^{p,q+1}(\mathcal{F})$ be the Cauchy-Riemann operator along the leaves defined by:

$$\overline{\partial}_{\mathcal{F}} \left(\sum \alpha_{JK} dZ_J \wedge d\overline{Z}_K \right) = \sum_{s=1}^{m} \frac{\partial \alpha_{JK}}{\partial \overline{z}_s}(z,t) d\overline{z}_s \wedge dZ_J \wedge d\overline{Z}_K$$

where $\frac{\partial}{\partial \bar{z}_s} = \frac{1}{2}\left\{\frac{\partial}{\partial x_s} + i\frac{\partial}{\partial y_s}\right\}$ with $z_s = x_s + iy_s$. It satifies $\overline{\partial}_{\mathcal{F}}^2 = 0$; hence we obtain a differential complex:

$$0 \longrightarrow \Omega^{p0}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{p,m-1}(\mathcal{F}) \xrightarrow{\overline{\partial}_{\mathcal{F}}} \Omega^{pm}(\mathcal{F}) \longrightarrow 0$$

called the $\overline{\partial}_{\mathcal{F}}$*-complex* of $(M, \mathcal{F})$; its homology $H^{pq}_{\mathcal{F}}(M)$ is the *foliated Dolbeault cohomology* (or the $\overline{\partial}_{\mathcal{F}}$*-cohomology*) of the complex foliation $(M, \mathcal{F})$. Compute this cohomology is equivalent to determine the conditions for solving the:

The $\overline{\partial}_{\mathcal{F}}$-problem. *Let $q \geq 1$ and $\omega \in \Omega^{pq}(\mathcal{F})$ such that $\overline{\partial}_{\mathcal{F}}\omega = 0$. Does there exists $\alpha \in \Omega^{p,q-1}(\mathcal{F})$ such that $\overline{\partial}_{\mathcal{F}}\alpha = \omega$?*

Question 1

Let $(M, \mathcal{F})$ be a complex foliation such that every leaf is a Stein manifold and closed in M. Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

Question 2 (weaker version)

Let $(M, \mathcal{F})$ be a complex complete Riemannian foliation such that every leaf is a Stein manifold and closed in M. Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

Question 3 can be reduced to the following one:

Question 3 (more weaker version)

Let $(M, \mathcal{F})$ be a complex foliation. Suppose that $\mathcal{F}$ is a differentiable product of a Stein manifold Σ by the intervalle $]-\varepsilon, \varepsilon[$ (where $\varepsilon > 0$) but the complex structure of each leaf $\Sigma \times \{t\}$ may depend on $t \in]-\varepsilon, \varepsilon[$. Is $H^{0q}_{\mathcal{F}}(M) = 0$ for $q \geq 1$?

The hypothesis "leaves are Stein" alone is not sufficient to solve the $\overline{\partial}_{\mathcal{F}}$-problem. For explicit computations see for example [ES] and [Sℓi].

Proposition. *If the answer to question 3 is positive so is to question 2.*

Proof. Let $\mathcal{F}$ be as in question 2. Let $\mathrm{O}(n) \longrightarrow \widehat{M} \longrightarrow M$ be the principal bundle of orthonormal frames transverse to $\mathcal{F}$. By [Mol], the foliation $\mathcal{F}$ lifts to $\widehat{M}$ to a transversely parallelizable foliation $\widehat{\mathcal{F}}$ with closed leaves whose dimension is the same as the dimension of $\mathcal{F}$; $\widehat{M}$ is just a fibration over a complete manifold W. Let π be the projection of $\widehat{M}$ over W. Since the restriction of π to a leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$ is a covering over a leaf L of $\mathcal{F}$, $\widehat{L}$ inherits naturally a complex structure for which it is also a Stein manifold [Ste]. Since $G = \mathrm{O}(n)$ acts on $\widehat{M}$ by automorphisms of $\widehat{\mathcal{F}}$, the foliated differential forms of type $(0, q)$ on M are forms of type $(0, q)$ on $\widehat{M}$ which are invariant by G, that is, we have a canonical isomorphism $A^{0q}(\mathcal{F}) \simeq A^{0q}_G(\widehat{\mathcal{F}})$; then the cohomologiy $H^{0q}_{\mathcal{F}}(M)$ is canonically isomorphic to the cohomology $H^{0q}_{\widehat{\mathcal{F}},G}(\widehat{M})$ of the complex:

$$0 \longrightarrow A^{00}_G(\widehat{\mathcal{F}}) \xrightarrow{\overline{\partial}_{\widehat{\mathcal{F}}}} A^{01}_G(\widehat{\mathcal{F}}) \xrightarrow{\overline{\partial}_{\widehat{\mathcal{F}}}} \cdots \xrightarrow{\overline{\partial}_{\widehat{\mathcal{F}}}} A^{0m}_G(\widehat{\mathcal{F}}) \longrightarrow 0.$$

Now, because G is compact, there exists a continuous linear map $A^{0q}(\widehat{\mathcal{F}}) \xrightarrow{\sigma} A^{0q}_G(\widehat{\mathcal{F}})$ (called the *averaging map*) defined by $\sigma(\alpha) = \int_G g^*(\alpha)d\mu(g)$ where μ is the normalized Haar measure on G. This map induces an injection:

$$\sigma : H^{0*}_{\mathcal{F}}(M) \hookrightarrow H^{0*}_{\widehat{\mathcal{F}}}(\widehat{M}).$$

To prove the nullity of $H^{0*}_{\mathcal{F}}(M)$, it is sufficient to prove that of $H^{0*}_{\widehat{\mathcal{F}}}(\widehat{M})$. Consider a cover of W by open sets V_j diffeomorphic to an open ball of $\mathbb{R}^n$. Let $\{\rho_j\}$ be a differentiable partition of 1 subordinated to the this cover. For any j, we set $U_j = \pi^{-1}(V_j)$ and $\psi_j = \rho_j \circ \pi$; then U_j is a differentiable product $F \times V_j$ (each factor $F \times \{t\}$ is a Stein manifold), $\{U_j\}$ is an open cover of $\widehat{M}$ and $\{\psi_j\}$ is a differentiable partition of 1 subordinated to $\{U_j\}$; each function ψ_j is constant on the leaves of $\widehat{\mathcal{F}}$.

For $q \geq 1$ let $\alpha \in A^{0q}(\widehat{\mathcal{F}})$ such that $\overline{\partial}_{\widehat{\mathcal{F}}}\alpha = 0$. Denote by α_j the restriction of α to U_j; then α_j is $\overline{\partial}_{\hat{\mathcal{F}}}$-closed. Since we have supposed $H^{0q}_{\mathcal{F}}(U_j) = 0$ for $q \geq 1$, there exists β_j of type $(0, q-1)$ defined on U_j such that $\overline{\partial}_{\hat{\mathcal{F}}}\beta_j = \alpha_j$. Let $\beta = \sum_j \psi_j\beta_j$. Then β is a foliated form of type $(0, q-1)$ defined globally on $\widehat{M}$; moreover since $\overline{\partial}_{\widehat{\mathcal{F}}}\psi_j = 0$ and $\overline{\partial}_{\widehat{\mathcal{F}}}$ is continuous (with respect to the C^∞-topology) we have:

$$\overline{\partial}_{\widehat{\mathcal{F}}}\beta = \overline{\partial}_{\widehat{\mathcal{F}}}\left(\sum_j \psi_j\beta_j\right) = \sum_j \psi_j\overline{\partial}_{\widehat{\mathcal{F}}}\beta_j = \sum_j \psi_j\alpha_j = \left(\sum_j \psi_j\right)\alpha = \alpha.$$

This shows that, for any $q \geq 1$, the vector space $H^{0q}_{\hat{\mathcal{F}}}(\widehat{M})$ is zero; then $H^{0q}_{\mathcal{F}}(M) = 0$. $\diamondsuit$

Question 4

Let M be an open set of $\mathbb{C} \times \mathbb{R}$ and $\mathcal{F}$ be the complex foliation whose leaves are the (non empty) sections $M_t = M \cap \mathbb{C} \times \{t\}$. This foliation on M is called the complex ***canonical foliation*** *on M. For which open sets M of $\mathbb{C} \times \mathbb{R}$ we have $H^{01}_{\mathcal{F}}(\widehat{M}) = 0$?*

For instance this is the case for the following class of open sets (cf. [E2]). An open set of $\mathbb{C}$ is said to be a *crown* if it is of type $C(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ where $r \in \mathbb{R}$ and $R \in]0, +\infty]$. Open crowns of $\mathbb{C}$ are of six *types*:

- (i) $C(r, R) = \mathbb{C}$ if $r < 0$ and $R = +\infty$;
- (ii) $C(r, R)$ is a disc if $r < 0$ and $R < +\infty$;
- (iii) $C(r, R)$ is a punctured disc if $r = 0$ and $R < +\infty$;
- (iv) $C(r, R) = \mathbb{C}^*$ if $r = 0$ and $R = +\infty$;
- (v) $C(r, R)$ is the complement of a closed disc if $r > 0$ and $R = +\infty$;
- (vi) $C(r, R)$ is an annulus if $0 < r < R < +\infty$. Two annulus $C(r, R)$ and $C(r', R')$ are holomorphically equivalent if and only if $\frac{R}{r} = \frac{R'}{r'}$.

An open set M of $\mathbb{C}\times B$ (B is a differentiable manifold) equipped with its canonical foliation $\mathcal{F}$ is called *$\mathcal{F}$-crowned* if each leaf M_t is an open crown of $\mathbb{C}$.

3. Deformations of Lie foliations

Almost all the contents of this section is extracted from the paper [EGN] which is a joint work with Gregori Guasp and Marcel Nicolau.

We take the example 2.5 in Part I with $\mathcal{H}=0$. Then we have a Lie algebra $\mathcal{G}$ of dimension n, $(e_1,\ldots,e_n)$ a basis of $\mathcal{G}$ and $(\theta^1,\ldots,\theta^n)$ the corresponding dual basis. One has $[e_i,e_j]=\sum_k K_{ij}^k e_k$, where the *structure constants* K_{ij}^k fulfill the relations (C1) and (C2).

We suppose that we are given a $\mathcal{G}$-valued 1-form $\omega=\sum_k \omega^k\otimes e_k$ on a connected manifold M defining a codimension n $\mathcal{G}$-foliation $\mathcal{F}$ on M. Let $(T,0)$ be the germ at 0 of a real analytic set T defined in a neighbourhood of the origin of an Euclidean space $\mathbb{R}^\ell$.

3.1. Definition. *A family of deformations $\mathcal{F}_t$ of the $\mathcal{G}$-foliation $\mathcal{F}$ parametrized by $(T,0)$ is given by a collection of 1-forms $\omega_t^1,\ldots,\omega_t^n$ on M, depending smoothly on $t\in T$, and a set of smooth functions $K_{ij}^k(t)$ such that conditions* (C1), (C2), *and* (C4) *are fulfilled for each $t\in T$. So for every $t\in T$ the set of constants $K_{ij}^k(t)$ defines a Lie algebra $\mathcal{G}_t$ and the forms $\omega_t=(\omega_t^1,\ldots,\omega_t^n)$ define a $\mathcal{G}_t$-foliation $\mathcal{F}_t$ on M. Moreover we require $\omega_0=\omega$.*

A family of deformations of $\mathcal{F}$ parametrized by $(T,0)$ is called *trivial* if it is equivalent to the constant family.

Let Ω^r be the space of differential forms on M of degree r. We denote by $\mathcal{R}=(\mathcal{R}^1,\ldots,\mathcal{R}^m)$ the linear map from $\bigwedge^r\mathcal{G}^*\otimes\mathcal{G}$ into $(\Omega^r)^m$ given by:

(III.1) $$\mathcal{R}^k(\theta^J\otimes e_i)=\delta_i^k\omega^J$$

where $J=(j_1,\ldots,j_r)$ and $\theta^J=\theta^{j_1}\wedge\ldots\wedge\theta^{j_r}$.

Given an element $\sigma=(\sigma^1,\ldots,\sigma^m)\in(\Omega^r)^m$ we denote by $\widehat{d}_M\sigma$ the element of $(\Omega^{r+1})^m$ whose components are given by:

(III.2) $$(\widehat{d}_M\sigma)^k=d\sigma^k+\sum_{i,j}K_{ij}^k\,\omega^i\wedge\sigma^j.$$

In a similar way we introduce an operator $\widehat{d}_{\mathcal{G}}:\bigwedge^r\mathcal{G}^*\otimes\mathcal{G}\to\bigwedge^{r+1}\mathcal{G}^*\otimes\mathcal{G}$ acting on an element $\psi\in\bigwedge^r\mathcal{G}^*\otimes\mathcal{G}$ by:

(III.3) $$\widehat{d}_{\mathcal{G}}\psi=\sum_k\left(d\psi^k+\sum_{i,j}K_{ij}^k\theta^i\wedge\psi^j\right)\otimes e_k$$

where here d denotes the exterior derivative on the Lie group G. Notice that the operators $\widehat{d}_M$ and $\widehat{d}_{\mathcal{G}}$ are formally the same.

For $r \in \mathbb{N}$, let V^r denote the space of elements $\xi \in \bigwedge^r \mathcal{G}^* \otimes \mathcal{G}$. We set $\mathcal{A}^r = (\Omega^r)^m \oplus V^{r+1}$ and define $D : \mathcal{A}^r \to \mathcal{A}^{r+1}$by $D(\sigma, \psi) = (\widehat{d}_M \sigma - \mathcal{R}\psi, -\widehat{d}_{\mathcal{G}}\psi)$. We can easily prove that $D^2 = 0$; therefore we obtain the differential complex $\mathcal{A}$:

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{D} \mathcal{A}^1 \xrightarrow{D} \mathcal{A}^2 \longrightarrow \dots$$

whose cohomolgy will be denoted by $H^*(\mathcal{A})$. Elements of $H^1(\mathcal{A})$ are called *infinitesimal deformations* of $\mathcal{F}$. This vector space space is very crucial in the determination of the space of deformations of the foliation.

3.2. Definition. *A family of deformations $\mathcal{F}_s$ of $\mathcal{F}$ parametrized by a smooth space of parameters $(S, 0)$ will be called* **versal** *if for any other family $\mathcal{F}_t$ of deformations of $\mathcal{F}$ parametrized by $(T, 0)$ there is a smooth map $\varphi : (T, 0) \to (S, 0)$ such that $\mathcal{F}_t$ and $\mathcal{F}_{\varphi(t)}$ are equivalent. Moreover the differential $d_0\varphi$ of φ at 0 is unique. Such a map φ, which need not to be unique, will be called* **versal**.

3.3. Example of a deformation of an Abelian foliation. We give here an example of an Abelian Lie foliation with a nilpotent deformation. Let H be the nilpotent Lie group of real matrices:

$$\begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}.$$

The vector fields $Z = \frac{\partial}{\partial z}$, $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and $X_3 = \frac{\partial}{\partial t}$ are a basis of the Lie algebra of left invariant vector fields on H with dual basis:

$$\beta = dz - xdy, \qquad \omega^1 = dx, \qquad \omega^2 = dy \qquad \text{and} \qquad \omega^3 = dt.$$

Let Γ be the discrete subroup of H whose elements are the matrices with $x, y, z, t \in \mathbb{Z}$ and denote by M the compact manifold $\Gamma \setminus H$. We still denote by Z, X_1, X_2, X_3 and β, ω^1, ω^2, ω^3 the respective projections of the above vector fields and 1-forms. The vector field Z defines on M an Abelian Lie foliation $\mathcal{F}$ of codimension 3 which is also defined by the differential system $\omega^1 = \omega^2 = \omega^3 = 0$. This foliation can be deformed into the family $\mathcal{F}_s$ (with $s \in \mathbb{R}$) defined by the vector field $Z + sX_3$. For $s \neq 0$, $\mathcal{F}_s$ is a Lie foliation modeled on the 3-dimensional Heisenberg group.

It was proved in [EGN] that the vector space $H^1(\mathcal{A})$ of infinitesimal deformations of $\mathcal{F}$ is of dimension 3; it is a direct sum of three copies of the 1-dimensional space generated by $(-\beta, \omega^1 \wedge \omega^2)$.

3.4. Question

Is the family $(\mathcal{F}_s)_{s\in\mathbb{R}}$ a versal deformation of $\mathcal{F}$?

As we have seen a deformation of a Lie $\mathcal{G}$-foliation $(\mathcal{F}_t)_{t\in T}$ (in the space of Lie foliations) gives rise to a deformation $\mathcal{G}_t$ of the Lie algebra $\mathcal{G}$. What about the converse? More precisely:

3.5. Question

Given a deformation $\mathcal{G}_t$ of the Lie algebra $\mathcal{G}$, does there exist a compact manifold M supporting a family of foliations $\mathcal{F}_t$ such that for $t \in T$, $\mathcal{F}_t$ is a Lie $\mathcal{G}_t$-foliation?

In subsection 3.3. we have seen that the deformation of the Abelian algebra $\mathcal{G} = \mathbb{R}^3$ into the Heinsenberg algebra is realized by a deformation a Lie foliation on a 4-compact manifold. Of course a necessary condition is that every Lie algebra have to be realized individually. This is the case for instance in the following simple question which is far to be trivial.

3.6. Question

Let $\mathcal{G}_t$ be the Lie algebra generated by two vectors X an Y satisfying the bracket relation $[X, Y] = tY$ where $t \in \mathbb{R}$. This is a deformation of the Abelian 2-dimensional algebra into the affine algebra. Can this deformation be realized by a deformation of Lie foliation in the sense of question 3.6?

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