Stabilization of non-homogeneous elastic materials with voids

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Abstract

We study the asymptotic behavior of the solution of the non-homogeneous elastic system with voids and a thermal effect. We first prove the well-posedness of this system under some realistic assumptions on the coefficients. Since this system suffers of exponential stability (as shown in dimension 1 in [18]), our main results concern strong and polynomial stabilities again under some assumptions on the coefficients. These stabilities are obtained in a closed subspace of the natural Hilbert space. Hence we characterize its orthogonal and further show that in the whole space the energy tends strongly or polynomially to the energy of the projection of the initial datum on this orthogonal space. In this respect we extend and precise former results obtained in one dimension in [18].

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1 Introduction and main results

There is a large literature devoted to the stabilization of the elasticity systems set in bounded domains of $\mathbb{R}^d, d \geq 1$ by boundary and/or internal dampings, see [1, 5, 7, 10] and the references cited there. As alternative damping we can couple the elasticity systems with the heat equation (elasticity with thermal effects) and it is well known that the thermal effects provokes the exponential decay of the solution [13, 21]. In this paper we are interested in porous elastic materials and in that case it was shown in [20] that the porous viscosity was not strong enough to obtain exponential decay of the solutions and that the decay can be very weak. Hence other dissipative mechanisms were considered recently in order to restore such an exponential decay, see for instance [16, 17, 18]. Here we want to consider the thermal and viscoelastic effects on the decay of the multi-dimensional problem (see [8, 11, 12] for the modelisation). Since in dimension 1, this system suffers of exponential stability [18], we concentrate on weak stability results by proving some strong and polynomial stabilities under some realistic conditions on the coefficients. Note that the main aim of this paper is to generalize the results from [18] to the multi-dimensional case and to non constant coefficients.

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Accordingly we consider the stabilization of the following coupled elastic solids with voids set in a bounded domain Ω of $\mathbb{R}^d$, $d = 1, 2$ or 3 with a Lipschitz boundary Γ (for the model, see section 5 of [8], [11] or [12]):

$$(1) \qquad \begin{cases} \rho u_{tt} = \operatorname{div}\left[C(\epsilon(u) + \gamma\epsilon(u_t)) + (b\varphi - \beta\theta)Id\right], \\ J\varphi_{tt} = \operatorname{div}(\delta\nabla\varphi) - b \operatorname{div} u - \xi\varphi + m\theta, \\ c\theta_t = \operatorname{div}(k\nabla\theta) - \beta \operatorname{div} u_t - m\varphi_t \end{cases} \quad \text{in } \Omega \times (0, +\infty),$$

with the boundary conditions (n being the unit outward normal vector along Γ)

$$(2) \qquad u = 0, \quad \delta\nabla\varphi \cdot n = 0, \quad k\nabla\theta \cdot n = 0 \quad \text{on } \Gamma \times (0, +\infty),$$

and, finally, the initial conditions

$$(3) \qquad \begin{cases} u(x, 0) = u^0(x) \\ u_t(x, 0) = u^1(x) \end{cases}, \quad \begin{cases} \varphi(x, 0) = \varphi^0(x) \\ \varphi_t(x, 0) = \varphi^1(x) \end{cases}, \quad \theta(x, 0) = \theta^0(x) \quad \text{in } \Omega.$$

Here the variables $u = (u_i)_{i=1}^d$, φ and θ are the (vectorial) displacement of the solid elastic material, the volume fraction and the temperature respectively. The coefficients ρ, b, β, γ, J, ξ, m and c belongs to $L^\infty(\Omega)$ and are related to the constitutive material. Similarly k and δ are $d \times d$ symmetric matrices and are assumed to belong to $L^\infty(\Omega)^{d\times d}$. Finally $C = (c_{ijk\ell})$ is a tensor such that

$$c_{ijk\ell} = c_{ji\ell k} = c_{k\ell ij} \in L^\infty(\Omega),$$

all indices running over the integers $1, \cdots, d$. As usual for $u = (u_i)_{i=1}^d$, $\epsilon(u)$ is the linear strain tensor defined by

$$\epsilon(u) = (\epsilon_{ij}(u))_{i,j=1}^d \quad \text{with } \epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

For a $d \times d$ matrix $\epsilon = (\epsilon_{ij})_{i,j=1}^d$ the product $C\epsilon = ((C\epsilon)_{ij})_{i,j=1}^d$ is the $d \times d$ matrix given by

$$(C\epsilon)_{ij} = \sum_{k,\ell=1}^d c_{ijk\ell}\epsilon_{k\ell}.$$

Finally for a (smooth enough) vector valued function $v : \Omega \to \mathbb{R}^d$, $\operatorname{div} v$ is its standard divergence, namely

$$\operatorname{div} v = \sum_{j=1}^d \partial_j v_j,$$

while for a (smooth enough) matrix-valued function $w = (w_{ij}) : \Omega \to \mathbb{R}^{d\times d}$, $\operatorname{div} w$ is its divergence line by line, i.e.,

$$\operatorname{div} w = (\sum_{j=1}^d \partial_j w_{ij})_{i=1}^d.$$

For well-posedness reason we assume that the first two equations of our system is of hyperbolic type while the third one is of parabolic type. Hence we require that there exist a positive function μ and positive real numbers $k_0, \delta_0, \rho_0, J_0, c_0, \xi_0$ and μ_0 such that for almost all $x \in \Omega$

$$(4) \qquad \rho(x) \geq \rho_0, \quad J(x) \geq J_0, \quad c(x) \geq c_0, \quad \xi(x) \geq \xi_0,$$

$$k(x)X \cdot X \geq k_0|X|^2, \quad \delta(x)X \cdot X \geq \delta_0|X|^2, \forall X \in \mathbb{R}^d, \tag{5}$$

and

$$C(x)\epsilon : \epsilon \geq \mu(x)|\epsilon|^2 \geq \mu_0|\epsilon|^2, \forall \epsilon \in \mathbb{R}^{d\times d}, \tag{6}$$

where $|\epsilon|^2 = \sum_{i,j=1}^d |\epsilon_{ij}|^2$ for all $\epsilon \in \mathbb{R}^{d\times d}$ and $\epsilon : \tau$ denotes the contraction of the two matrices, i.e.,

$$\epsilon : \tau = \sum_{i,j=1}^d \epsilon_{ij}\tau_{ij},$$

and finally

$$\gamma(x) \geq 0. \tag{7}$$

This paper is organized as follows. In Section 2 assuming

$$\int_\Omega (c\theta^0 + m\varphi^0 + \beta \operatorname{div} u^0)dx = 0, \tag{8}$$

we will prove that the system (1)-(3) is well-posed under some assumptions on the coefficients. We then find in Section 3 sufficient conditions that garantee the strong stability of the system, these conditions are mainly based on some spectral properties of a system coupling the elasticity system with a diffusion equation. In Section 4, we prove some polynomial stability by using a frequency domain approach and by taking the initial data in an appropriate subspace $\mathcal{H}_0$ of the natural space $\mathcal{H}$. If γ is positive definite and $m \not\equiv 0$, the orthogonal of the space $\mathcal{H}_0$ is at most of dimension 2, on the contrary the situation is more delicate as seen in Section 5, where we characterize this space $\mathcal{H}_0$ when all the coefficients are constants and when $\gamma = 0$.

Let us finish this introduction with some notation used in the remainder of the paper: The $L^2(\Omega)$-inner product (resp. norm) will be denoted by $(\cdot,\cdot)$ (resp. $\|\cdot\|$). The usual norm and semi-norm of $H^s(\Omega)$ $(s > 0)$ are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. For shortness, we will use the same notation in $H^s(\Omega)^d$.

2 Well-posedness of the system

We consider the Hilbert space

$$\mathcal{H} = \{(u, v, \varphi, \phi, \theta) \in H_0^1(\Omega)^d \times L^2(\Omega)^d \times H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \text{ satisfying (9) below}\}$$

$$\int_\Omega (c\theta + m\varphi + \beta \operatorname{div} u)dx = 0. \tag{9}$$

On $\mathcal{H}$, we introduce the sesquilinear form

$$\langle U, U^*\rangle_{\mathcal{H}} = \int_\Omega \left(C\epsilon(u) : \epsilon(\bar{u}^*) + \rho v \cdot \bar{v}^* + \delta\nabla\varphi \cdot \nabla\bar{\varphi}^* + \xi\varphi\bar{\varphi}^* + J\phi\bar{\phi}^* + c\theta\bar{\theta}^* + b(\operatorname{div} u\,\bar{\varphi}^* + \operatorname{div} \bar{u}^*\varphi)\right) dx$$

with $U = (u, v, \varphi, \phi, \theta)^\top$, $U^* = (u^*, v^*, \varphi^*, \phi^*, \theta^*)^\top \in \mathcal{H}$.

Lemma 2.1 *Assume that*

$$\sup_{x\in\Omega} \frac{b(x)^2}{\mu(x)\xi(x)} < 2^{1-d}. \tag{10}$$

Then $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ *is an inner product on* $\mathcal{H}$.

Proof. By Young's inequality with $\alpha(x) > 0$ for all $x \in \Omega$, we have

$$\begin{aligned} 2b\Re\left(\operatorname{div} u\, \bar{\varphi}\right) &\leq \alpha\,|b|\,|\operatorname{div} u|^2 + \frac{|b|}{\alpha}\,|\varphi|^2 \\ &\leq 2^{d-1}\alpha\,|b|\,|\epsilon(u)|^2 + \frac{|b|}{\alpha}\,|\varphi|^2\,, \end{aligned}$$

and then

$$\begin{aligned} \langle U, U\rangle_{\mathcal{H}} \geq \int_\Omega \left(\left(\mu - 2^{d-1}\,|b|\,\alpha\right)|\epsilon(u)|^2 + \left(\xi - \frac{|b|}{\alpha}\right)|\varphi|^2\right) dx \\ + \int_\Omega \left(\rho\,|v|^2 + \delta\nabla\varphi\cdot\nabla\varphi + J\,|\phi|^2 + c\,|\theta|^2\right) dx. \end{aligned} \tag{11}$$

Then by setting

$$M = 2^{d-1} \sup_{x\in\Omega} \frac{b(x)^2}{\mu(x)\xi(x)},$$

that is in $[0, 1)$ by the assumption (10) we take

$$\alpha(x) = 2^{1-d}(1-\eta)\frac{\mu(x)}{|b(x)|} \quad \forall x \in \Omega_b = \{y \in \Omega : b(y) \neq 0\},$$

where $\eta \in (0, 1]$ is fixed such that $M \leq (1-\eta)^2$ and $\alpha(x) = 1$ else. With that choice we check that

$$\frac{|b(x)|}{(1-\eta)\xi(x)} \leq \alpha(x) \leq 2^{1-d}(1-\eta)\frac{\mu(x)}{|b(x)|} \quad \forall x \in \Omega_b.$$

Since these estimates are equivalent to

$$\mu(x) - 2^{d-1}|b(x)|\alpha(x) \geq \eta\mu(x) \text{ and } \xi(x) - \frac{|b(x)|}{\alpha} \geq \eta\xi(x) \quad \forall x \in \Omega_b,$$

and since these two estimates trivially hold outside Ω_b the estimate (11) becomes

$$\langle U, U\rangle_{\mathcal{H}} \geq \int_\Omega \left(\eta\mu\,|\epsilon(u)|^2 + \eta\xi\,|\varphi|^2\right) dx + \int_\Omega \left(\rho\,|v|^2 + \delta\nabla\varphi\cdot\nabla\varphi + J\,|\phi|^2 + c\,|\theta|^2\right) dx.$$

By the assumption (4)-(6) on the coefficients and Korn's inequality, we deduce that there exists a positive constant c such that

$$\langle U, U\rangle_{\mathcal{H}} \geq c(\|u\|_{1,\Omega}^2 + \|v\|^2 + \|\varphi\|_{1,\Omega}^2 + \|\phi\|^2 + \|\theta\|^2) \quad \forall U \in \mathcal{H}.$$

Consequently $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product on $\mathcal{H}$ whose associated norm is equivalent to the natural norm of $\mathcal{H}$. ■

By a standard reduction order method, (1)-(3) can be rewritten as the first order evolution equation

$$\left\{\begin{array}{l} U' = \mathcal{A}U \\ U(0) = U_0 = (u^0, u^1, \varphi^0, \varphi^1, \theta^0)^\top \end{array}\right. \tag{12}$$

where U is the vector $U = (u, u_t, \varphi, \varphi_t, \theta)^\top$ and the operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \varphi \\ \phi \\ \theta \end{pmatrix} := \begin{pmatrix} v \\ \rho^{-1}(\operatorname{div}[C(\epsilon(u) + \gamma\epsilon(v)) + (b\varphi - \beta\theta)Id] \\ \phi \\ J^{-1}(\operatorname{div}(\delta\nabla\varphi) - b \operatorname{div} u - \xi\varphi + m\theta) \\ c^{-1}(\operatorname{div}(k\nabla\theta) - \beta \operatorname{div} v - m\phi) \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}) := \big\{&(u, v, \varphi, \phi, \theta) \in \mathcal{H} \cap (H_0^1(\Omega)^d \times H_0^1(\Omega)^d \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)); \\ &\operatorname{div}[C(\epsilon(u) + \gamma\epsilon(v)) + (b\varphi - \beta\theta)Id] \in L^2(\Omega)^d,\ \operatorname{div}(\delta\nabla\varphi) \in L^2(\Omega),\ \operatorname{div}(k\nabla\theta) \in L^2(\Omega) \text{ and} \\ &\delta\nabla\varphi \cdot n = k\nabla\theta \cdot n = 0 \text{ on } \Gamma\big\}. \end{aligned}$$

We now prove that the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions over $\mathcal{H}$. For that purpose we need the two following lemmas.

Lemma 2.2 *The operator $\mathcal{A}$ is dissipative and satisfies, for all $U = (u, v, \varphi, \phi, \theta)^\top \in D(\mathcal{A})$,*

$$\Re \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\int_\Omega (\gamma\,|\epsilon(v)|^2 + k\nabla\theta \cdot \nabla\bar{\theta})dx \le 0. \tag{13}$$

Proof. Take $U = (u, v, \varphi, \phi, \theta)^\top \in D(\mathcal{A})$. Then, we have

$$\begin{aligned} \Re \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \Re\Big(\int_\Omega &(C\epsilon(v) : \epsilon(\bar{u}) + \operatorname{div}[C\epsilon(u) + \gamma\epsilon(v) + (b\varphi - \beta\theta)Id] \cdot \bar{v} + \delta\nabla\phi \cdot \nabla\bar{\varphi} + \xi\phi\bar{\varphi} \\ &+(\operatorname{div}(\delta\nabla\varphi) - b \operatorname{div} u - \xi\varphi + m\theta)\bar{\phi} + (\operatorname{div}(k\nabla\theta) - \beta \operatorname{div} v - m\phi)\bar{\theta} + b(\operatorname{div} v\bar{\varphi} + \operatorname{div}\bar{u}\phi)\big)\ dx\Big). \end{aligned}$$

By integration by parts and recalling that $v \in H_0^1(\Omega)$, that $\delta\nabla\varphi \cdot n = k\nabla\theta \cdot n = 0$ on Γ, we obtain

$$\begin{aligned} \Re \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \Re\Big(\int_\Omega &(C\epsilon(v) : \epsilon(\bar{u}) - [C\epsilon(u) + \gamma\epsilon(v) + (b\varphi - \beta\theta)Id] : \epsilon(\bar{v}) + \delta\nabla\phi \cdot \nabla\bar{\varphi} + \xi\phi\bar{\varphi} \\ &-\delta\nabla\varphi\nabla\bar{\phi} + (-b \operatorname{div} u - \xi\varphi + m\theta)\bar{\phi} - k\nabla\theta \cdot \nabla\bar{\theta} - (\beta \operatorname{div} v + m\phi)\bar{\theta} + b(\operatorname{div} v\bar{\varphi} + \operatorname{div}\bar{u}\phi)\big)\ dx\Big). \end{aligned}$$

After simplification we get

$$\Re \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\int_\Omega (\gamma\,|\epsilon(v)|^2 + k\nabla\theta \cdot \nabla\bar{\theta})dx,$$

which leads to the conclusion with (5). ■

Lemma 2.3 *Assume that (10) holds. If $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$, then $0 \in \rho(\mathcal{A})$.*

Proof. Let $F = (f^1, f^2, g^1, g^2, h)^\top \in \mathcal{H}$. We look for $U = (u, v, \varphi, \phi, \theta)^\top \in D(\mathcal{A})$ solution of

$$\mathcal{A}U = F,$$

or equivalently

$$\begin{cases} v = f^1 \in H_0^1(\Omega)^d \\ \phi = g^1 \in H^1(\Omega) \\ \operatorname{div}[C(\epsilon(u) + \gamma\epsilon(v)) + (b\varphi - \beta\theta)Id] = \rho f^2 \\ \operatorname{div}(\delta\nabla\varphi) - b\operatorname{div} u - \xi\varphi + m\theta = Jg^2 \\ \operatorname{div}(k\nabla\theta) - \beta\operatorname{div} v - m\phi = ch \in L^2(\Omega). \end{cases} \tag{14}$$

Eliminating v and ϕ in this last equation and taking into account the boundary condition, we are first looking for a solution $\theta \in H^1(\Omega)$ of

$$\begin{cases} \operatorname{div}(k\nabla\theta) = ch + \beta\operatorname{div} f^1 + mg^1 & \text{in } \Omega, \\ k\nabla\theta \cdot n = 0 & \text{on } \Gamma. \end{cases} \tag{15}$$

Multiplying this identity by a test function θ^*, integrating in space and using formal integration by parts, we obtain the weak formulation

$$-\int_\Omega k\nabla\theta \cdot \nabla\bar{\theta}^*\, dx = \int_\Omega (ch + \beta\operatorname{div} f^1 + mg^1)\bar{\theta}^*\, dx \quad \forall\theta^* \in H^1(\Omega).$$

Since by assumption $ch + \beta\operatorname{div} f^1 + mg^1$ belongs to $L^2(\Omega)$ and has a zero mean in Ω, there exists a unique solution $\theta_0 \in H_*^1(\Omega) = \{w \in H^1(\Omega) : \int_\Omega w\, dx = 0\}$ of

$$-\int_\Omega k\nabla\theta_0 \cdot \nabla\bar{\theta}^*\, dx = \int_\Omega (ch + \beta\operatorname{div} f^1 + mg^1)\bar{\theta}^*\, dx \quad \forall\theta^* \in H_*^1(\Omega).$$

This solution is a solution of (15) because the condition $\int_\Omega (ch + \beta\operatorname{div} f^1 + mg^1)\, dx = 0$ implies that

$$-\int_\Omega k\nabla\theta_0 \cdot \nabla\bar{\theta}^*\, dx = \int_\Omega (ch + \beta\operatorname{div} f^1 + mg^1)\bar{\theta}^*\, dx \quad \forall\theta^* \in H^1(\Omega).$$

Note further that for any $\alpha \in \mathbb{C}$, the function $\theta_\alpha = \theta_0 + \alpha$ is still solution of the above problem, namely $\theta_\alpha \in H^1(\Omega)$ is a solution of

$$-\int_\Omega k\nabla\theta_\alpha \cdot \nabla\bar{\theta}^*\, dx = \int_\Omega (ch + \beta\operatorname{div} f^1 + mg^1)\bar{\theta}^*\, dx \quad \forall\theta^* \in H^1(\Omega),$$

and hence is a solution of (15). The parameter α will be fixed later on.

Now we are looking for $u_\alpha \in H_0^1(\Omega)^d$ and $\varphi_\alpha \in H^1(\Omega)$ solution of (compare with the third and four equation of (14), where v, ϕ are eliminated and θ is replaced by θ_α solution of (15)):

$$\begin{cases} \operatorname{div}[C\epsilon(u_\alpha) + b\varphi_\alpha Id] = \rho f^2 - \operatorname{div}(\gamma\epsilon(f^1) - \beta\theta_\alpha Id) & \text{in } \Omega, \\ \operatorname{div}(\delta\nabla\varphi_\alpha) - b\operatorname{div} u_\alpha - \xi\varphi_\alpha = Jg^2 - m\theta_\alpha & \text{in } \Omega, \\ \delta\nabla\varphi_\alpha \cdot n = 0 & \text{on } \Gamma. \end{cases} \tag{16}$$

Again multiplying these identities by test functions (u^*, φ^*), integrating in space and using integration by parts, we obtain

$$\int_\Omega ((C\epsilon(u_\alpha) + b\varphi_\alpha Id) : \epsilon(\bar{u^*}) + \delta\nabla\varphi_\alpha \cdot \nabla\bar{\varphi^*} + b\operatorname{div} u_\alpha\bar{\varphi^*} + \xi\varphi_\alpha\bar{\varphi^*})\, dx \tag{17}$$

$$= -\int_\Omega \left(\rho f^2\bar{u^*} + (\gamma\epsilon(f^1) - \beta\theta_\alpha Id) : \epsilon(\bar{u^*}) + (Jg^2 - m\theta_\alpha)\bar{\varphi^*}\right) dx, \quad \forall(u^*, \varphi^*) \in H_0^1(\Omega)^d \times H^1(\Omega).$$

Writing for shortness

$$a((u,\varphi),(u^*,\varphi^*)) = \int_\Omega ((C\epsilon(u)+b\varphi Id):\epsilon(\bar{u^*}) + \delta\nabla\varphi\cdot\nabla\bar{\varphi^*} + b\operatorname{div} u\bar{\varphi^*} + \xi\varphi\bar{\varphi^*})\,dx$$
$$= \int_\Omega (C\epsilon(u):\epsilon(\bar{u^*}) + b\varphi\operatorname{div}\bar{u^*} + \delta\nabla\varphi\cdot\nabla\bar{\varphi^*} + b\operatorname{div} u\bar{\varphi^*} + \xi\varphi\bar{\varphi^*})\,dx,$$

we see that a is a continuous sesquilinear form on $H_0^1(\Omega)^d\times H^1(\Omega)$ which is coercive because

$$a((u,\varphi),(u,\varphi)) = \int_\Omega \left(C\epsilon(u):\epsilon(\bar u) + 2b\Re\operatorname{div} u\bar\varphi + \delta\nabla\varphi\cdot\nabla\bar\varphi + \xi|\varphi|^2\right)\,dx.$$

Hence by Young's inequality and the arguments of the beginning of this section, the assumption (10) guarantees that

$$a((u,\varphi),(u,\varphi)) \geq c(\|u\|_{1,\Omega}^2 + \|\varphi\|_{1,\Omega}^2)\quad \forall (u,\varphi)\in H_0^1(\Omega)^d\times H^1(\Omega),$$

for some $c>0$. Since the right-hand side of (17) is clearly a continuous linear form on $H_0^1(\Omega)^d\times H^1(\Omega)$, by Lax-Milgram's lemma problem (17) has a unique solution $(u_\alpha,\varphi_\alpha)\in H_0^1(\Omega)^d\times H^1(\Omega)$.

Clearly this solution satisfies (16) by choosing appropriated test functions. Now we want to fix α such that (9) holds, namely

$$\int_\Omega (c\theta_\alpha + m\varphi_\alpha + \beta\operatorname{div} u_\alpha)dx = 0. \tag{18}$$

But in view of the splitting $\theta_\alpha = \theta_0 + \alpha$, we have

$$(u_\alpha,\varphi_\alpha) = (u_0,\varphi_0) + \alpha(u_1,\varphi_1),$$

where (u_0,φ_0) is the unique solution in $H_0^1(\Omega)^d\times H^1(\Omega)$ of problem (17) with $\theta_\alpha=\theta_0$, while (u_1,φ_1) is the unique solution in $H_0^1(\Omega)^d\times H^1(\Omega)$ of

$$a((u_1,\varphi_1),(u^*,\varphi^*)) = \int_\Omega (\beta\operatorname{div}\bar u^* + m\bar\varphi^*)\,dx,\quad \forall (u^*,\varphi^*)\in H_0^1(\Omega)^d\times H^1(\Omega). \tag{19}$$

With the help of these decompositions, (18) is equivalent to

$$\int_\Omega (c\theta_0 + m\varphi_0 + \beta\operatorname{div} u_0)dx + \alpha\int_\Omega (c + m\varphi_1 + \beta\operatorname{div} u_1)dx = 0. \tag{20}$$

Hence such a α exists if and only if

$$\int_\Omega (c + m\varphi_1 + \beta\operatorname{div} u_1)dx \neq 0. \tag{21}$$

Now looking at (19) and taking the test function (u^*,φ^*) equal to (u_1,φ_1) we find

$$a((u_1,\varphi_1),(u_1,\varphi_1)) = \int_\Omega (\beta\operatorname{div}\bar u_1 + m\bar\varphi_1)\,dx.$$

Since this left-hand side is a positive real number we find that

$$\int_\Omega (\beta\operatorname{div} u_1 + m\varphi_1)\,dx > 0.$$

Since $c\geq c_0>0$ in Ω, we deduce that (21) holds.

In summary fixing α such that (20) holds we have found $U=(u_\alpha,v,\varphi_\alpha,\phi,\theta_\alpha)^\top\in D(\mathcal{A})$ solution of $\mathcal{A}U=F$. ■

Corollary 2.4 *If (10) holds, then* $[0,\infty)\subset\rho(\mathcal{A})$.

Proof. As the previous lemma guarantees that $0\in\rho(\mathcal{A})$, $\mathcal{A}$ is closed and consequently $\rho(\mathcal{A})$ is open (see Theorem III.6.7 of [14]). Hence there exists a positive real number λ_0 in $\rho(\mathcal{A})$. The conclusion follows by Theorem I.4.5 of [19]. ■

Therefore (1)-(3) is well-posed in $\mathcal{H}$.

Theorem 2.5 *Assume that (4)-(7) and (10) hold. Then the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions over $\mathcal{H}$, and thus for an initial datum $U_0\in\mathcal{H}$, there exists a unique solution $U\in C([0,+\infty),\mathcal{H})$ to problem (12). Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C([0,+\infty),D(\mathcal{A}))\cap C^1([0,+\infty),\mathcal{H}).$$

Proof. Theorem I.4.6 of [19], Lemma 2.2 and Corollary 2.4 imply that the domain of $\mathcal{A}$ is dense in $\mathcal{H}$. It then suffices to apply Lumer-Philips's Theorem (see Theorem I.4.3 of [19]). ■

3 Strong stability

It is proved in [18] in dimension $d=1$ and in the case of constant coefficients on Ω that the system (1)-(3) is not exponentially stable. Then in the multi-dimensional situation with variable coefficients we cannot expect to obtain an exponential stability but we may hope a strong stability or even better a polynomial stability.

For that purpose we define the energy of (1)-(3) by

(22)

$$E(t)=\frac{1}{2}\int_\Omega\Big(C\epsilon(u):\epsilon(\bar{u})+\rho\left|u_t\right|^2+\delta\nabla\varphi\cdot\nabla\bar{\varphi}+\xi\left|\varphi\right|^2+J\left|\varphi_t\right|^2+c\left|\theta\right|^2+2b\Re(\operatorname{div}u\bar{\varphi})\Big)\,dx,$$

which corresponds to the norm of $(u,u_t,\varphi,\varphi_t,\theta)$ in $\mathcal{H}$.

Proposition 3.1 *The solution (u,φ,θ) of (1)-(3) with initial datum in $D(\mathcal{A})$ satisfies*

$$E'(t)\leq-\int_\Omega(\gamma\left|\epsilon(u_t)\right|^2+k\nabla\theta\cdot\nabla\bar{\theta})dx\leq 0.$$

Therefore the energy is non-increasing.

Proof. It suffices to derive the energy (22) for regular solutions and to use systems (1)-(3). The calculations are analogous to those of the proof of the dissipativeness of $\mathcal{A}$ in Lemma 2.2, and then, are left to the reader. ■

To get strong stability results, we make use of the following result due to Arendt and Batty [2]:

Theorem 3.2 *Let $(T(t))_{t\geq0}$ be a bounded C_0-semigroup on a reflexive space X. Denote by A the generator of $(T(t))$ and by $\sigma(A)$ the spectrum of A. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then $\lim_{t\to+\infty}T(t)x=0$ for all $x\in X$.*

In view of this theorem we now need to characterize the spectrum of $\mathcal{A}$ on the imaginary axis. For that purpose we introduce the following operator L on the Hilbert space $L^2(\Omega)^{d+1}$, that is here equipped with the inner product $(\cdot,\cdot)_{\rho,J}$ defined by

$$((u,\varphi),(v,\chi))_{\rho,J}:=\int_\Omega(\rho(x)u(x)\cdot\bar{v}(x)+J(x)\varphi(x)\bar{\chi}(x))\,dx\quad\forall(u,\varphi),(v,\chi)\in L^2(\Omega)^{d+1}.$$

From the assumption (4), its associated norm is equivalent to the usual norm of $L^2(\Omega)^{d+1}$. Then L is defined by

$$D(L) = \{(u,\varphi) \in H_0^1(\Omega)^d \times H^1(\Omega) : \quad \operatorname{div}(\delta\nabla\varphi) \in L^2(\Omega),$$
$$\operatorname{div}\left[C\epsilon(u) + (b\varphi + \frac{\beta}{c}(\beta \operatorname{div} u + m\varphi))Id\right] \in L^2(\Omega)^d \text{ and}$$
$$\delta\nabla\varphi \cdot n = 0 \text{ on } \Gamma\},$$

and

$$L(u,\varphi) := \quad \Big(-\rho^{-1} \operatorname{div}\left[C\epsilon(u) + (b\varphi + \frac{\beta}{c}(\beta \operatorname{div} u + m\varphi))Id\right],$$
$$J^{-1}\left[-\operatorname{div}(\delta\nabla\varphi) + b \operatorname{div} u + \xi\varphi + \frac{m}{c}(\beta \operatorname{div} u + m\varphi)\right]\Big) \quad \forall (u,\varphi) \in D(L).$$

We see that L is the Friedrichs extension of the symmetric, continuous sesquilinear form b defined by

$$b((u,\varphi),(u^*,\varphi^*)) = \int_\Omega \Big([C\epsilon(u) + (b\varphi + \frac{\beta}{c}(\beta \operatorname{div} u + m\varphi))Id] : \epsilon(\bar{u}^*)$$
$$+\delta\nabla\varphi \cdot \nabla\bar{\varphi}^* + \Big(b \operatorname{div} u + \xi\varphi + \frac{m}{c}(\beta \operatorname{div} u + m\varphi)\Big)\bar{\varphi}^*\Big)\, dx, \forall (u,\varphi),(u^*,\varphi^*) \in H_0^1(\Omega)^d \times H^1(\Omega),$$

in the sense that

$$b((u,\varphi),(u^*,\varphi^*)) = (L(u,\varphi),(u^*,\varphi^*))_{\rho,J} \quad \forall (u,\varphi) \in D(L), (u^*,\varphi^*) \in H_0^1(\Omega)^d \times H^1(\Omega).$$

Using Young's inequality we see that

$$b((u,\varphi),(u,\varphi)) \geq \int_\Omega \Big((\mu - \alpha 2^{d-1})|\epsilon(u)|^2 + \frac{\beta^2}{c}|\operatorname{div} u|^2$$
$$+\delta_0|\nabla\varphi|^2 + \left(\xi + \frac{m^2}{c} - \frac{1}{\alpha}\left(b + \frac{\beta m}{c}\right)^2\right)|\varphi|^2\Big)\, dx, \forall (u,\varphi) \in H_0^1(\Omega)^d \times H^1(\Omega),$$

for all $\alpha > 0$. Hence chosing α small enough we deduce that

$$b((u,\varphi),(u,\varphi)) \geq \alpha_0(\|u\|_{1,\Omega}^2 + \|\varphi\|_{1,\Omega}^2) + \tilde{m}(\|u\|^2 + \|\varphi\|^2),$$

where α_0 is a positive real number and $\tilde{m}$ is a real number (that is positive if $b + \frac{\beta m}{c}$ is small enough).

This property and the compact embedding of $H_0^1(\Omega)^d \times H^1(\Omega)$ into $L^2(\Omega)^{d+1}$ imply that L is a self-adjoint operator with a compact resolvent bounded from below. Therefore there exist a sequence of eigenvalues $\lambda_n \in [\tilde{m}, \infty), n \in \mathbb{N}^*$ (repeated according to their multiplicity) and of eigenvectors $(u_n, \varphi_n) \in D(L), n \in \mathbb{N}^*$ such that

$$L(u_n,\varphi_n) = \lambda_n(u_n,\varphi_n) \qquad \forall n \in \mathbb{N}^*,$$

or equivalently

$$-\operatorname{div}\left[C\epsilon(u_n) + \left(b\varphi_n + \frac{\beta}{c}(\beta \operatorname{div} u_n + m\varphi_n)\right)Id\right] = \rho\lambda_n u_n, \tag{23}$$

$$-\operatorname{div}(\delta\nabla\varphi_n) + b \operatorname{div} u_n + \xi\varphi_n + \frac{m}{c}(\beta \operatorname{div} u_n + m\varphi_n) = J\lambda_n\varphi_n, \quad \forall n \in \mathbb{N}^*. \tag{24}$$

Note that the eigenvectors can be chosen in order to form an orthonormal basis of $L^2(\Omega)^{d+1}$ for the inner product $(\cdot,\cdot)_{\rho,J}$, i.e.,

$$((u_n,\varphi_n),(u_{n'},\varphi_{n'}))_{\rho,J} = \delta_{n,n'}, \qquad \forall n,n' \in \mathbb{N}^*.$$

We are now ready to check if the spectrum of $\mathcal{A}$ contains points on the imaginary axis or not.

Lemma 3.3 *Assume that (4)-(7) and (10) hold. Then*
i) If there exists $n \in \mathbb{N}^*$ such that $\lambda_n > 0$ and if the associated eigenvector (u_n,φ_n) satisfies

(25) $$\exists k_1 \in \mathbb{C} : \frac{1}{c}(\beta \operatorname{div} u_n + m\varphi_n) = k_1 \text{ in } \Omega,$$

and

(26) $$\gamma\epsilon(u_n) = 0 \text{ in } \Omega,$$

then $i\sqrt{\lambda_n}$ and $-i\sqrt{\lambda_n}$ belong to the point spectrum $Sp(\mathcal{A})$ of $\mathcal{A}$, their associated eigenvector being respectively

(27) $$U_{n,\pm} = (u_n, \pm i\sqrt{\lambda_n}u_n, \varphi_n, \pm i\sqrt{\lambda_n}\varphi_n, -\frac{1}{c}(\beta \operatorname{div} u_n + m\varphi_n))^\top.$$

ii) If for all $n \in \mathbb{N}^*$ such that $\lambda_n > 0$, either (25) does not hold or (26) does not hold, then the point spectrum of $\mathcal{A}$ contains no point on the imaginary axis.

Proof. Since in Lemma 2.3 we have already shown that 0 belongs to the resolvent set of $\mathcal{A}$, we only need to look at its eventual eigenvalue in $i\mathbb{R}\setminus\{0\}$. For that purpose let $U = (u,v,\varphi,\phi,\theta)^\top \in D(\mathcal{A})$ be a solution of

$$\mathcal{A}U = i\omega U,$$

where $\omega \in \mathbb{R}\setminus\{0\}$, or equivalently

(28) $$\begin{cases} v = i\omega u \\ \phi = i\omega\varphi \\ \operatorname{div}\left[C(\epsilon(u)+\gamma\epsilon(v)) + (b\varphi - \beta\theta)Id\right] = i\rho\omega v \\ \operatorname{div}(\delta\nabla\varphi) - b\operatorname{div} u - \xi\varphi + m\theta = iJ\omega\phi \\ \operatorname{div}(k\nabla\theta) - \beta\operatorname{div} v - m\phi = ic\omega\theta, \end{cases}$$

with the boundary conditions

(29) $$u = 0, \quad \delta\nabla\varphi\cdot n = 0, \quad k\nabla\theta\cdot n = 0 \quad \text{on } \Gamma.$$

First taking the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ between $\mathcal{A}U$ and U, by (13), we have

$$0 = \Re\langle\mathcal{A}U,U\rangle_{\mathcal{H}} = -\int_\Omega (\gamma\,|\epsilon(v)|^2 + k\nabla\theta\cdot\nabla\bar{\theta})dx.$$

By the assumptions on γ and k, this is equivalent to

(30) $$\gamma\epsilon(v) = 0, \quad \theta = \theta_c \text{ in } \Omega,$$

for some constant θ_c. Hence (28) reduces to (reminding that $\omega \neq 0$)

$$\left\{\begin{array}{l} v = i\omega u \\ \phi = i\omega\varphi \\ \operatorname{div}[C\epsilon(u) + (b\varphi - \beta\theta_c)Id] = -\rho\omega^2 u \\ \operatorname{div}(\delta\nabla\varphi) - b\operatorname{div} u - \xi\varphi + m\theta_c = -J\omega^2\varphi \\ -\beta\operatorname{div} u - m\varphi = c\theta_c, \end{array}\right. \tag{31}$$

with the boundary condition

$$u = 0, \quad \delta\nabla\varphi\cdot n = 0 \text{ on } \Gamma. \tag{32}$$

As c is different from zero (see (4)), eliminating θ_c in the last equation we find that

$$\theta_c = -\frac{1}{c}(\beta\operatorname{div} u + m\varphi).$$

Replacing θ_c by this expression in the third and fourth equations of (31) we arrive at

$$\left\{\begin{array}{l} v = i\omega u \\ \phi = i\omega\varphi \\ \operatorname{div}\left[C(\epsilon(u)) + (b\varphi + \frac{\beta}{c}(\beta\operatorname{div} u + m\varphi))Id\right] = -\rho\omega^2 u \\ \operatorname{div}(\delta\nabla\varphi) - b\operatorname{div} u - \xi\varphi - \frac{m}{c}(\beta\operatorname{div} u + m\varphi) = -J\omega^2\varphi \\ \theta_c = -\frac{1}{c}(\beta\operatorname{div} u + m\varphi). \end{array}\right. \tag{33}$$

We recognize in the third and fourth equations the eigenvalue problem (23)-(24) with the appropriated boundary conditions (32). Hence these two equations have a solution if $\omega^2 = \lambda_n$ (hence λ_n has to be positive), $u = u_n$ and $\varphi = \varphi_n$ for some $n \in \mathbb{N}^*$.

Since the last condition of (33) and (30) are equivalent to (25)-(26), if both constraints are satisfied, we find two non trivial solutions $U_n \in D(\mathcal{A})$ given as in the statement of the Lemma (notice that the condition (9) trivially holds because $\theta = -\frac{1}{c}(\beta\operatorname{div} u + m\varphi)$), on the contrary case no solution exists and we deduce that $Sp(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. ■

If we assume that there exists $\gamma_0 > 0$ such that

$$\gamma(x) \geq \gamma_0 > 0 \quad \forall \text{ a.e.} x \in \Omega, \tag{34}$$

then Lemma 3.3 can be reformulated in the following way. First (26) is equivalent to $u_n = 0$ (since $u_n = 0$ on Γ) and therefore (25) reduces to

$$\exists k_1 \in \mathbb{C} : \frac{m}{c}\varphi_n = k_1 \text{ in } \Omega. \tag{35}$$

Moreover the eigenvalue problem (23)-(24) becomes

$$\nabla((b + \frac{\beta m}{c})\varphi_n) = 0, \tag{36}$$

$$-\operatorname{div}(\delta\nabla\varphi_n) + (\xi + \frac{m^2}{c})\varphi_n = J\lambda_n\varphi_n, \quad \forall n \in \mathbb{N}^*. \tag{37}$$

Hence we introduce the operator L_1 on the Hilbert space $L^2(\Omega)$, here equipped with the inner product $(\cdot,\cdot)_J$ defined by

$$(f,g)_J := \int_\Omega J(x)f(x)g(x)\,dx \quad \forall f,g \in L^2(\Omega).$$

From the assumption (4), its associated norm is equivalent to the usual norm of $L^2(\Omega)$. Then L_1 is defined by

$$D(L_1) = \left\{\varphi \in H^1(\Omega) : \operatorname{div}(\delta\nabla\varphi) \in L^2(\Omega) \text{ and } \delta\nabla\varphi \cdot n = 0 \text{ on } \Gamma\right\},$$

and

$$L_1\varphi := J^{-1}\Big(-\operatorname{div}(\delta\nabla\varphi) + (\xi + \frac{m^2}{c})\varphi\Big) \quad \forall\varphi \in D(L).$$

As before L_1 is the Friedrichs extension of the symmetric, continuous and coercive sesquilinear form b_1 defined by

$$b_1(\varphi, \varphi^*) = \int_\Omega (\delta\nabla\varphi \cdot \nabla\bar{\varphi}^* + (\xi + \frac{m^2}{c})\varphi\bar{\varphi}^*)\, dx, \forall\varphi, \varphi^* \in H^1(\Omega),$$

in the sense that

$$b_1(\varphi, \varphi^*) = (L\varphi, \varphi^*)_J \quad \forall\varphi \in D(L_1), \varphi^* \in H^1(\Omega).$$

This property and the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ imply that L_1 is a positive self-adjoint operator with a compact resolvent. Therefore there exist a sequence of eigenvalues $\lambda_n^{(1)} \in (0, \infty), n \in \mathbb{N}^*$ (repeated according to their multiplicity) and of eigenvectors $\varphi_n^{(1)} \in D(L_1), n \in \mathbb{N}^*$ such that

$$L_1\varphi_n^{(1)} = \lambda_n\varphi_n^{(1)} \qquad \forall n \in \mathbb{N}^*,$$

or equivalently

$$-\operatorname{div}(\delta\nabla\varphi_n^{(1)}) + (\xi + \frac{m^2}{c})\varphi_n^{(1)} = J\lambda_n^{(1)}\varphi_n^{(1)} \quad \forall n \in \mathbb{N}^*. \tag{38}$$

By the above argument we have characterized the spectrum on the imaginary axis if (34) holds.

Lemma 3.4 *Assume that (4)-(6), (10) and (34) hold. Then*
i) If there exists $n \in \mathbb{N}^*$ such that

$$\exists k_1 \in \mathbb{C} : \frac{m}{c}\varphi_n^{(1)} = k_1 \text{ in } \Omega, \tag{39}$$

and

$$\exists k_2 \in \mathbb{C} : (b + \frac{\beta m}{c})\varphi_n^{(1)} = k_2 \text{ in } \Omega, \tag{40}$$

then $i\sqrt{\lambda_n^{(1)}}$ and $-i\sqrt{\lambda_n^{(1)}}$ belong to $Sp(\mathcal{A})$, their associated eigenvector being respectively

$$U_{n,\pm} = (0, 0, \varphi_n^{(1)}, \pm i\sqrt{\lambda_n^{(1)}}\varphi_n^{(1)}, -\frac{m}{c}\varphi_n^{(1)})^\top. \tag{41}$$

ii) If for all $n \in \mathbb{N}^*$, either (25) does not hold or (26) does not hold, the point spectrum of $\mathcal{A}$ contains no point on the imaginary axis.

On the contrary if $\gamma = 0$ then (26) trivially holds and the existence of an eigenvalue of $\mathcal{A}$ on the imaginary axis is reduced to the existence of a pair of eigenvector $(u_n, \varphi_n) \in D(L)$ satisfying (25). We refer to section 5 for an illustration in dimension 1.

In the first situation of Lemma 3.3, system (1)-(3) is clearly not stable in $\mathcal{H}$, but, as the next result shows, it turns out that $\mathcal{A}$ let invariant the closed subspace

$$\mathcal{H}_0 = \{U \in \mathcal{H} : \langle U, U_{n,+}\rangle_{\mathcal{H}} = \langle U, U_{n,-}\rangle_{\mathcal{H}} = 0 \quad \forall n \in \mathbb{N}^* \text{ such that } (25)-(26) \text{ hold}\},$$

where $U_{n,\pm}$ are defined by (27). Hence we will reduce problem (1)-(3) to $\mathcal{H}_0$. Note that the vectors $U_{n,\pm}$ and $U_{n',\pm}$ are orthogonal in $\mathcal{H}$ for $n \neq n'$ as well as $U_{n,+}$ and $U_{n,-}$.

Lemma 3.5 *Let $n \in \mathbb{N}^*$ be such that (25)-(26) hold. Then $U \in D(\mathcal{A})$ is orthogonal to $U_{n,+}$ (resp. $U_{n,-}$) if and only if $\mathcal{A}U$ is orthogonal to $U_{n,+}$ (resp. $U_{n,-}$).*

Proof. Let $U = (u, v, \varphi, \phi, \theta)^\top \in D(\mathcal{A})$ be fixed and denote by $\mathcal{A}U = (f^1, f^2, g^1, g^2, h)^\top$. By the definition (27) of $U_{n,\pm}$, we see that (for shortness we write $\omega = \sqrt{\lambda_n}$)

$$\langle U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega (C\epsilon(u) : \epsilon(\bar{u}_n) \mp \rho i\omega v \cdot \bar{u}_n + \delta\nabla\varphi \cdot \nabla\bar{\varphi}_n + \xi\varphi\bar{\varphi}_n \mp Ji\omega\phi\bar{\varphi}_n$$
$$-\theta(\beta \operatorname{div} \bar{u}_n + m\bar{\varphi}_n) + b \operatorname{div} u\bar{\varphi}_n + b \operatorname{div} \bar{u}_n\varphi)\, dx.$$

Since $\operatorname{div}[C\epsilon(u) + \gamma\epsilon(v) + (b\varphi - \beta\theta)Id] = \rho f^2$, $-\operatorname{div}(\delta\nabla\varphi) + \xi\varphi + b \operatorname{div} u - m\theta = -Jg^2$, $v = f^1$ and $\phi = g^1$, we find that

$$\langle U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega \left(\rho(-f^2 \mp i\omega f^1) \cdot \bar{u}_n + J(-g^2 \mp i\omega g^1)\bar{\varphi}_n\right) dx - \int_\Omega \gamma\epsilon(v) : \epsilon(\bar{u}_n)\, dx. \tag{42}$$

By (26) we conclude that

$$\langle U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega \left(\rho(-f^2 \mp i\omega f^1) \cdot \bar{u}_n + J(-g^2 \mp i\omega g^1)\bar{\varphi}_n\right) dx. \tag{43}$$

In the same manner we have

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega \big(C\epsilon(f^1) : \epsilon(\bar{u}_n) \mp i\rho\omega f^2 \cdot \bar{u}_n + \delta\nabla g^1 \cdot \nabla\bar{\varphi}_n + \xi g^1\bar{\varphi}_n \mp iJ\omega g^2\bar{\varphi}_n$$
$$-h(\beta \operatorname{div} \bar{u}_n + m\bar{\varphi}_n) + b(\operatorname{div} f^1\bar{\varphi}_n + \operatorname{div} \bar{u}_n g^1)\big)\, dx.$$

Again by (23)-(24) (in a weak form) and reminding that $\beta \operatorname{div} u_n + m\varphi_n = -c\theta_n$ we find that

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega \big(\beta\bar{\theta}_n \operatorname{div} f^1 + \rho\omega^2 f^1 \cdot \bar{u}_n) \mp i\rho\omega f^2 \cdot \bar{u}_n + g^1(J\omega^2\bar{\varphi}_n + m\bar{\theta}_n) \mp iJ\omega g^2\bar{\varphi}_n + ch\bar{\theta}_n)\big)\, dx,$$

or equivalently

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \pm i\omega \int_\Omega \left(\mp i\omega\rho f^1 \cdot \bar{u}_n - \rho f^2 \cdot \bar{u}_n \mp i\omega J g^1\bar{\varphi}_n - Jg^2\bar{\varphi}_n\right) dx$$
$$+ \int_\Omega (\beta \operatorname{div} f^1 + mg^1 + ch)\bar{\theta}_n\, dx.$$

But

$$\int_\Omega (\beta \operatorname{div} f^1 + mg^1 + ch)\bar{\theta}_n\, dx = 0,$$

because θ_n is constant and due to the condition (9) satisfied by $\mathcal{A}U$, therefore the last identity becomes

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \pm i\omega \int_\Omega \left(\mp i\omega\rho f^1\cdot \bar{u}_n - \rho f^2\cdot \bar{u}_n \mp i\omega J g^1\bar{\varphi}_n - Jg^2\bar{\varphi}_n\right)\,dx.$$

Comparing this identity with (43) we have shown that

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \pm i\omega \langle U, U_{n,\pm}\rangle_{\mathcal{H}},$$

and the conclusion follows (since $\omega \neq 0$). ■

The same phenomenon occurs in the first situation of Lemma 3.4 as (34) holds.

Lemma 3.6 *Assume that (34) holds. Let $n\in\mathbb{N}^*$ be such that (39)-(40) hold. Then $U\in D(\mathcal{A})$ is orthogonal to $U_{n,+}$ (resp. $U_{n,-}$) if and only if $\mathcal{A}U$ is orthogonal to $U_{n,+}$ (resp. $U_{n,-}$).*

Proof. For shortness we drop the index $^{(1)}$. Let $U=(u,v,\varphi,\phi,\theta)^\top\in D(\mathcal{A})$ be fixed and denote by $\mathcal{A}U=(f^1,f^2,g^1,g^2,h)^\top$. By the definition (41) of $U_{n,\pm}$, we see that (for shortness we write $\omega=\sqrt{\lambda_n}$)

$$\langle U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega (\delta\nabla\varphi\cdot\nabla\bar{\varphi}_n + \xi\varphi\bar{\varphi}_n \mp Ji\omega\phi\bar{\varphi}_n - m\theta\bar{\varphi}_n + b\,\mathrm{div}\,u\bar{\varphi}_n)\,dx.$$

Since $-\mathrm{div}(\delta\nabla\varphi)+\xi\varphi+b\,\mathrm{div}\,u-m\theta=-Jg^2$ and $\phi=g^1$, we find that

$$\langle U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega J\left(-g^2\mp i\omega g^1\right)\bar{\varphi}_n\,dx. \tag{44}$$

In the same manner we have

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega \left(\delta\nabla g^1\cdot\nabla\bar{\varphi}_n + \xi g^1\bar{\varphi}_n \mp iJ\omega g^2\bar{\varphi}_n - mh\bar{\varphi}_n + b\,\mathrm{div}\,f^1\bar{\varphi}_n\right)\,dx.$$

By (37) we find that

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \int_\Omega \left(g^1(J\omega^2-\frac{m^2}{c})\bar{\varphi}_n \mp iJ\omega g^2\bar{\varphi}_n - mh\bar{\varphi}_n + b\,\mathrm{div}\,f^1\bar{\varphi}_n\right)\,dx,$$

or equivalently

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \omega\int_\Omega J(g^1\omega\mp ig^2)\bar{\varphi}_n\,dx - \int_\Omega\left(\frac{m^2}{c}g^1+mh-b\,\mathrm{div}\,f^1\right)\bar{\varphi}_n\,dx.$$

If we show that

$$\int_\Omega\left(\frac{m^2}{c}g^1+mh-b\,\mathrm{div}\,f^1\right)\bar{\varphi}_n\,dx = 0, \tag{45}$$

then the last identity becomes

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \omega\int_\Omega J(g^1\omega\mp ig^2)\bar{\varphi}_n\,dx = \pm i\omega\int_\Omega J(\mp i\omega g^1-g^2)\bar{\varphi}_n\,dx.$$

Comparing this identity with (44) we have shown that

$$\langle \mathcal{A}U, U_{n,\pm}\rangle_{\mathcal{H}} = \pm i\omega \langle U, U_{n,\pm}\rangle_{\mathcal{H}},$$

and the conclusion follows (since $\omega \neq 0$).

It then remains to check (45). Let us denote by I the left-hand side of (45). First writing $\theta_n = -\frac{m}{c}\varphi_n$ we see that

$$I = -\int_\Omega \big(mg^1\bar{\theta}_n + ch\bar{\theta}_n + b\,\mathrm{div}\, f^1\bar{\varphi}_n\big)\, dx.$$

Reminding that $\mathrm{div}(k\nabla\theta) - \beta\,\mathrm{div}\, v - m\phi = ch$ (in a weak form) and $\phi = g^1$ we find

$$I = \int_\Omega \big((k\nabla\theta\cdot\nabla\bar{\theta}_n + \beta\,\mathrm{div}\, v\bar{\theta}_n) - b\,\mathrm{div}\, f^1\bar{\varphi}_n\big)\, dx.$$

Hence recalling that θ_n is constant we find

$$I = \int_\Omega \big(\beta\,\mathrm{div}\, v\bar{\theta}_n - b\,\mathrm{div}\, f^1\bar{\varphi}_n\big)\, dx = -\int_\Omega (b + \frac{\beta m}{c})\,\mathrm{div}\, v\bar{\varphi}_n\, dx,$$

recalling that $v = f^1$. Hence an application of Green's formula yields (recall that $v \in H^1_0(\Omega)^d$)

$$I = \int_\Omega v\cdot\nabla[(b + \frac{\beta m}{c})\bar{\varphi}_n]\, dx = 0$$

due to (40). ■

Remark 3.7 *If all coefficients are constant, if $\gamma > 0$ (i.e. if (34) holds) and if $m \neq 0$, we see that the unique eigenvector of L_1 satisfying (39)-(40) is the constant function $\varphi_1^{(1)} = 1$ with eigenvalue $\lambda_1^{(1)} = J^{-1}(\xi + \frac{m^2}{c})$. In that case $\mathcal{A}$ has two eigenvectors*

$$U_\pm = (0, 0, 1, \pm i\sqrt{\lambda_1^{(1)}}, -\frac{m}{c})^\top$$

of purely imaginary eigenvalue $\omega = \pm i\sqrt{\lambda_1^{(1)}}$. Consequently the subspace $\mathcal{H}_0$ is reduced to the one introduced in [18] (and used in a more restrictive setting than ours), namely

$$\mathcal{H}_0 = H^1_0(\Omega)^d \times L^2(\Omega)^d \times H^1_*(\Omega) \times L^2_*(\Omega) \times L^2_*(\Omega),$$

where $L^2_*(\Omega) = \{w \in L^2(\Omega) : \int_\Omega w\, dx = 0\}$ and $H^1_*(\Omega) = \{w \in H^1(\Omega) : \int_\Omega w\, dx = 0\}$. Indeed we directly see that

$$\langle U, U_\pm\rangle_{\mathcal{H}} = \int_\Omega \left(\xi\varphi \mp iJ\sqrt{\lambda_1^{(1)}}\phi - m\theta\right)\, dx.$$

Hence the conditions $\langle U, U_\pm\rangle_{\mathcal{H}} = 0$ combined with (9) that here takes the form

$$\int_\Omega (c\theta + m\varphi)dx = 0$$

is equivalent to

$$\int_\Omega \varphi dx = \int_\Omega \phi dx = \int_\Omega \theta dx = 0.$$

Remark 3.8 *i) If γ satisfies (34) and if $m \not\equiv 0$ in the sense that*

$$m(x) \neq 0 \quad \forall a.e.\, x \in \Omega,$$

then there exists at most one eigenvector $\varphi_n^{(1)}$ satisfying (39)-(40). Consequently in that case $\mathcal{A}$ has at most two eigenvalues on the imaginary axis.
ii) If $\gamma = \beta = b = 0$, $m \not\equiv 0$ and $\frac{c}{m}$ is a constant function, then $\varphi_n = 1$ of eigenvalue $J^{-1}(\xi + \frac{m^2}{c})$ and therefore at most one u_n exists. Again in that case $\mathcal{A}$ has at most two eigenvalues on the imaginary axis.

The two above lemmas characterize the point spectrum of $\mathcal{A}$ on the imaginary axis, our next goal is to show that the remaining set is in the resolvent set. Usually this is obtained by the fact that $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$ but here the presence of the term $\gamma\epsilon(v)$ does not allow to prove this compactness property.

Lemma 3.9 *If $d = 1$ we assume that Ω is a finite union of intervals $I_i, i \in \{1, \cdots, I\}$ such that $C_{|I_i}, \delta_{|I_i}, b_{|I_i}, \beta_{|I_i} \in W^{1,\infty}(I_i)$, for all $i = 1, \cdots, I$ (later on we will say that C, δ, b and β are piecewise $W^{1,\infty}$), on the contrary if $d \geq 2$, we assume that $b, \beta \in W^{1,\infty}(\Omega)$ and that*

$$\begin{aligned} D(E) &:= \left\{u \in H_0^1(\Omega)^d; \operatorname{div}[C(\epsilon(u))] \in L^2(\Omega)^d\right\}, \\ D(D_\delta) &:= \left\{\varphi \in H^1(\Omega); \operatorname{div}(\delta\nabla\varphi) \in L^2(\Omega), \delta\nabla\varphi \cdot n = 0 \text{ on } \Gamma\right\}, \end{aligned}$$

are compactly embedded into $H_0^1(\Omega)^d$ and $H^1(\Omega)$ respectively (equipped with their natural norm). If $d \geq 2$, $\gamma \neq 0$ and $\beta \neq 0$, we also require that $D(D_k)$ is compactly embedded into $H^1(\Omega)$. For any space dimension we further suppose that $\gamma \in W^{1,\infty}(\Omega)$. If (10) holds, then recalling that $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$, we have the identity

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \textit{Sp}\ (\mathcal{A}) \cap i\mathbb{R}.$$

Proof. *First case:* $\gamma \equiv 0$. We then show that $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$. Indeed let $U_n = (u_n, v_n, \varphi_n, \phi_n, \theta_n)^\top \in D(\mathcal{A})$ such that $\|U_n\|^2_{\mathcal{H}} + \|\mathcal{A}U_n\|^2_{\mathcal{H}} = 1$ for all $n \in \mathbb{N}$, which is equivalent to

$$\begin{aligned} &\|u_n\|^2_{1,\Omega} + \|v_n\|^2 + \|\varphi_n\|^2_{1,\Omega} + \|\phi_n\|^2 + \|\theta_n\|^2 + \|v_n\|^2_{1,\Omega} + \|\operatorname{div}\left(C(\epsilon(u_n) + \gamma\epsilon(v_n)) + (b\varphi_n - \beta\theta_n)Id\right)\|^2 + \|\phi_n\|^2_{1,\Omega} \\ &\quad + \|\operatorname{div}(\delta\nabla\varphi_n) - b \operatorname{div} u_n - \xi\varphi_n + m\theta_n\|^2 + \|\operatorname{div}(k\nabla\theta_n) - \beta \operatorname{div}(v_n) - m\phi_n\|^2 = 1. \end{aligned}$$

This implies that

$$\|v_n\|_{1,\Omega} + \|\operatorname{div}\left(C(\epsilon(u_n)) + (b\varphi_n - \beta\theta_n)Id\right)\| + \|\operatorname{div}(\delta\nabla\varphi_n)\| + \|\operatorname{div}(k\nabla\theta_n)\| \leq C,$$

where $C > 0$.

The estimates $\|\theta_n\| \leq C_1$, for some $C_1 > 0$ (consequence of $\|U_n\|_{\mathcal{H}} \leq 1$) and $\|\operatorname{div}(k\nabla\theta_n)\| \leq C$ imply that

$$\|\theta_n\|_{1,\Omega} \leq C_2,$$

for some $C_2 > 0$.

If $d \geq 2$, our assumptions on b and β, the two previous estimates and the fact that $(\varphi_n)_n$ are uniformly bounded in $H^1(\Omega)$ lead to

$$\|\operatorname{div} C(\epsilon(u_n))\| + \|\operatorname{div}(\delta\nabla\varphi_n)\| \leq C_3,$$

where $C_3 > 0$. Hence we conclude by the compact embedding of $D(E)$ into $H^1(\Omega)^d$, of $D(D_\delta)$ into $H^1(\Omega)$ and of $H^1(\Omega)$ into $L^2(\Omega)$.

If $d = 1$, the assumptions on b and β imply here that

$$\|(Cu_{nx})_x\|_{0,I_i} + \|(\delta\varphi_{nx})_x\|_{0,I_i} \leq C_4, \forall i = 1, \cdots, I,$$

where $C_4 > 0$ and w_x means the derivative of w with respect to x. The assumptions on C and δ guarantee that $(u_n)_n$ and $(\varphi_n)_n$ are uniformly bounded in $H^2(I_i)$ and we conclude as before.

Second case: $\gamma \not\equiv 0$. The above arguments fail because if $d \geq 2$ we will only obtain that

$$\|u_n + \gamma v_n\|_{D(E)} \leq C,$$

where $C > 0$, while we only have the information $\|v_n\|_{1,\Omega} \leq C$. As a consequence we would get a convergent subsequence of $(u_n + \gamma v_n)$ in $H^1(\Omega)^d$, and a convergent subsequence of (v_n) in $L^2(\Omega)^d$, that do not yield a convergent subsequence of (u_n) in $H^1(\Omega)^d$.

Hence we have to find an alternative argument. Namely we try to characterize the set $\rho(\mathcal{A}) \cap i\mathbb{R}^*$. For $\omega \in \mathbb{R}^*$ and $F = (f_1, f_2, g_1, g_2, h)^\top) \in \mathcal{H}$, we look for $U = (u, v, \varphi, \phi, \theta)^\top \in D(\mathcal{A})$ solution of

$$(\mathcal{A} - i\omega)U = F, \tag{46}$$

or equivalently

$$\begin{cases} v = i\omega u + f_1 \\ \phi = i\omega\varphi + g_1 \\ \operatorname{div}(C(\epsilon(u) + \gamma\epsilon(v)) + (b\varphi - \beta\theta)Id) - i\rho\omega v = \rho f_2 \\ \operatorname{div}(\delta\nabla\varphi) - b \operatorname{div} u - \xi\varphi + m\theta - i\omega J\phi = Jg_2 \\ \operatorname{div}(k\nabla\theta) - \beta \operatorname{div} v - m\phi - i\omega c\theta = ch. \end{cases} \tag{47}$$

The main idea is to introduce the new unknown

$$\tilde{u} = u + \gamma v. \tag{48}$$

Since $v = i\omega u + f_1$, we deduce that

$$\tilde{u} = (1 + i\omega\gamma)u + \gamma f_1.$$

Therefore

$$u = \frac{\tilde{u} - \gamma f_1}{1 + i\omega\gamma}, \tag{49}$$

and consequently

$$v = i\omega u + f_1 = \frac{i\omega}{1 + i\omega\gamma}\tilde{u} + \frac{1}{1 + i\omega\gamma}f_1. \tag{50}$$

The identity (49) allows to recover u if $\tilde{u}$ is known (and f_1 given). Now using (48) and (50) into (47) yield (equivalently)

$$\begin{cases} v = \frac{i\omega}{1+i\omega\gamma}\tilde{u} + \frac{1}{1+i\omega\gamma}f_1 \\ \phi = i\omega\varphi + g_1 \\ \operatorname{div}(C\epsilon(\tilde{u}) + (b\varphi - \beta\theta)Id) - i\rho\omega\left(\frac{i\omega}{1+i\omega\gamma}\tilde{u} + \frac{1}{1+i\omega\gamma}f_1\right) = \rho f_2 \\ \operatorname{div}(\delta\nabla\varphi) - b \operatorname{div}\left(\frac{\tilde{u}-\gamma f_1}{1+i\omega\gamma}\right) - \xi\varphi + m\theta - i\omega J(i\omega\varphi + g_1) = Jg_2 \\ \operatorname{div}(k\nabla\theta) - \beta \operatorname{div}\left(\frac{i\omega}{1+i\omega\gamma}\tilde{u} + \frac{1}{1+i\omega\gamma}f_1\right) - m(i\omega\varphi + g_1) - i\omega c\theta = ch. \end{cases}$$

Hence we are reduced to look for $\tilde{u} \in H_0^1(\Omega)^d$, $\varphi \in H^1(\Omega)$, $\theta \in H^1(\Omega)$ solution of
(51)

$$\left\{\begin{array}{l} \operatorname{div}\left(C\epsilon(\tilde{u}) + (b\varphi - \beta\theta)Id\right) + \frac{\omega^2\rho}{1+i\omega\gamma}\tilde{u} = \rho f_2 + \frac{i\omega\rho}{1+i\omega\gamma} f_1 =: \rho\tilde{f_2} \in L^2(\Omega) \\ \operatorname{div}(\delta\nabla\varphi) - b\operatorname{div}\left(\frac{\tilde{u}}{1+i\omega\gamma}\right) - \xi\varphi + m\theta + \omega^2 J\varphi = Jg_2 + b\operatorname{div}\left(\frac{\gamma f_1}{1+i\omega\gamma}\right) + i\omega J g_1 =: J\tilde{g_2} \in L^2(\Omega) \\ \operatorname{div}(k\nabla\theta) - i\beta\omega\operatorname{div}\left(\frac{\tilde{u}}{1+i\omega\gamma}\right) - i\omega m\varphi - i\omega c\theta = ch + mg_1 + \beta\operatorname{div}\left(\frac{f_1}{1+i\omega\gamma}\right) =: c\tilde{h} \in L^2(\Omega), \end{array}\right.$$

with the following boundary conditions

$$\delta\nabla\varphi\cdot n = k\nabla\theta\cdot n = 0 \text{ on } \Gamma.$$

If $d \geq 2$, we rewrite this system in the form

$$\left\{\begin{array}{l} \operatorname{div}\left(C\epsilon(\tilde{u})\right) - \rho\tilde{u} + \operatorname{div}((b\varphi - \beta\theta)Id) + \frac{\omega^2\rho}{1+i\omega\gamma}\tilde{u} + \rho\tilde{u} = \rho\tilde{f_2} \\ \operatorname{div}(\delta\nabla\varphi) - J\varphi - b\operatorname{div}\left(\frac{\tilde{u}}{1+i\omega\gamma}\right) - \xi\varphi + m\theta + (\omega^2+1)J\varphi = J\tilde{g_2} \\ \operatorname{div}(k\nabla\theta) - c\theta - i\omega\beta\operatorname{div}\left(\frac{\tilde{u}}{1+i\omega\gamma}\right) - i\omega m\varphi - (-1+i\omega)c\theta = c\tilde{h}, \end{array}\right. \tag{52}$$

or in operator form

$$L\begin{pmatrix}\tilde{u}\\ \varphi\\ \theta\end{pmatrix} + R_\omega\begin{pmatrix}\tilde{u}\\ \varphi\\ \theta\end{pmatrix} = \tilde{F} \text{ in } H_1, \tag{53}$$

where

$$H_1 = L^2(\Omega)^d \times L^2(\Omega) \times L^2(\Omega)$$

with inner product

$$\left(\begin{pmatrix}\tilde{u}\\ \varphi\\ \theta\end{pmatrix}, \begin{pmatrix}\tilde{u}^*\\ \varphi^*\\ \theta^*\end{pmatrix}\right)_{H_1} = \int_\Omega (\rho\tilde{u}\bar{\tilde{u}}^* + J\varphi\bar{\varphi}^* + c\theta\bar{\theta}^*)dx,$$

and

$$L\begin{pmatrix}\tilde{u}\\ \varphi\\ \theta\end{pmatrix} = \begin{pmatrix}\rho^{-1}\operatorname{div}(C\epsilon(\tilde{u})) - \tilde{u}\\ J^{-1}\operatorname{div}(\delta\nabla\varphi) - \varphi\\ c^{-1}\operatorname{div}(k\nabla\theta) - \theta\end{pmatrix}$$

with domain

$$D(L) = D(E) \times D(D_\delta) \times D(D_k)$$

and R_ω is the remainder defined by

$$R_\omega\begin{pmatrix}\tilde{u}\\ \varphi\\ \theta\end{pmatrix} = \begin{pmatrix}\rho^{-1}\operatorname{div}((b\varphi - \beta\theta)Id) + \frac{1+\omega^2}{1+i\omega\gamma}\tilde{u}\\ -bJ^{-1}\operatorname{div}(\frac{\tilde{u}}{1+i\omega\gamma}) - J^{-1}\xi\varphi + J^{-1}m\theta + (\omega^2+1)\varphi\\ -i\omega\beta c^{-1}\operatorname{div}(\frac{\tilde{u}}{1+i\omega\gamma}) + (1-i\omega)\theta - i\omega m c^{-1}\varphi\end{pmatrix}.$$

It turns out that L is an isomorphism from $D(L)$ into H_1 and since $D(L)$ is compact embedded into H_1, L^{-1} is a compact operator from H_1 into H_1. Now we set

$$V = L\begin{pmatrix}\tilde{u}\\ \varphi\\ \theta\end{pmatrix}.$$

Notice that find $(\tilde{u}, \varphi, \theta)^\top \in D(L)$ is equivalent to find $V \in H_1$. From the expression

$$\begin{pmatrix} \tilde{u} \\ \varphi \\ \theta \end{pmatrix} = L^{-1}V,$$

the identity (53) is equivalent to

$$V + R_\omega L^{-1}V = \tilde{F} \text{ in } H_1. \tag{54}$$

Now we see that $R_\omega L^{-1}$ is a compact operator from H_1 into itself. Indeed

$$H_1 \overset{L^{-1}\, continuous}{\longrightarrow} D(L) \overset{Id\, compact}{\longrightarrow} H^1(\Omega)^d \times H^1(\Omega) \times H^1(\Omega) \overset{R_\omega\, continuous}{\longrightarrow} H_1.$$

By the Fredholm alternative, we deduce that $I + R_\omega L^{-1}$ is a Fredholm operator of index 0 from H_1 into itself. Hence the inversibility of (54) is reduced to the nullity of the kernel of $I + R_\omega L^{-1}$. Therefore $V \in \ker(I + R_\omega L^{-1})$ if and only if $L^{-1}V = (\tilde{u}, \varphi, \theta)^\top$ satisfies

$$L \begin{pmatrix} \tilde{u} \\ \varphi \\ \theta \end{pmatrix} + R_\omega \begin{pmatrix} \tilde{u} \\ \varphi \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the equivalence between (52) and (53), we deduce that $U = (u, i\omega u, \varphi, i\omega\varphi)^\top$ with u given by $u = \frac{\tilde{u}}{1+i\omega\gamma}$ satisfies

$$(\mathcal{A} - i\omega)U = 0.$$

In other words, if $i\omega \notin Sp(\mathcal{A})$, $U \equiv 0$ and therefore $V = 0$. In conclusion, if $i\omega \notin Sp(\mathcal{A})$, for all $\tilde{F} \in H_1$, (54) has a unique solution and consequently coming back to (46), for all $F \in \mathcal{H}$, there exists a unique solution of (46). This shows that

$$i\mathbb{R} \backslash (Sp(\mathcal{A}) \cap i\mathbb{R}) \subset \rho(\mathcal{A}),$$

and the conclusion follows.

If $d = 1$, then the system (51) is a system of differential equations on each subdomain I_i with a complete set of boundary conditions. On each subdomain using a system of fundamental solutions, we are reduced to a system of homogeneous differential equations with non homogeneous boundary conditions. For this last system, using again a basis of fundamental solutions we are reduced to solve a square system of $N = 5I$ linear equations with $N = 5I$ unknowns. Hence the existence of a solution is reduced to its uniqueness, and again this mean that if $i\omega \notin Sp(\mathcal{A})$, the system (51) has a unique solution. ■

Remark 3.10 *The assumption that $D(E)$ (resp. $D(D_\delta)$) is compactly embedded into $H_0^1(\Omega)^d$ (resp. $H^1(\Omega)$) is very weak and holds in the many situations. For instance it holds if the coefficients of C are $C^2(\bar{\Omega})$ (resp. of δ are $C^2(\bar{\Omega})$) and if the boundary of Ω is $C^{1,1}$. It also hold for piecewise smooth coefficients, we refer to the book [9] for some illustrations.*

The previous results and Theorem 3.2 yield to the following theorem:

Theorem 3.11 *Assume that (4)-(7) and (10) hold. If the assumptions of Lemma 3.9 are satified, then we can distinguish the following cases:*
i) In the first case of Lemma 3.3, for all $U_0 \in \mathcal{H}_0$, the solution of system (1)-(3) satisfies $\lim_{t\to+\infty} E(t) = 0$.
ii) In the second case of Lemma 3.3, for all $U_0 \in \mathcal{H}$, the solution of system (1)-(3) satisfies $\lim_{t\to+\infty} E(t) = 0$.

In the first situation of Lemma 3.3, we denote by $\mathcal{H}_1$ the vectorial space spanned by $B := \{U_{n,\pm} : \lambda_n > 0,\ n \in \mathbb{N}^*$ such that (25)-(26) hold$\}$. By definition, $\mathcal{H}_0$ and $\mathcal{H}_1$ are orthogonal in $\mathcal{H}$ and B forms an orthonormal basis of $\mathcal{H}_1$. Consequently if we denote by $U_{proj,0}$ the orthogonal projection of the initial datum $U_0 \in \mathcal{H}$ on $\mathcal{H}_1$, then the solution U of (1)-(3) with an initial datum U_0 can be split up as follows:

$$U(t) = U^{(0)}(t) + U^{(1)}(t),$$

where $U^{(0)}(t) = e^{t\mathcal{A}}(U_0 - U_{proj,0})$ and

$$U^{(1)}(t) = e^{t\mathcal{A}}U_{proj,0} = \sum_{U_{n,\pm}\in B} e^{\pm it\sqrt{\lambda_n}}(U_{proj,0}, U_{n,\pm})_{\mathcal{H}}\, U_{n,\pm}.$$

As

$$\left\|U^{(1)}(t)\right\|_{\mathcal{H}}^2 = \|U_{proj,0}\|_{\mathcal{H}}^2\,,$$

and since $U_0 - U_{proj,0}$ belongs to $\mathcal{H}_0$, applying Theorem 3.11 to the term $e^{t\mathcal{A}}(U_0 - U_{proj,0})$, we have obtained the next result:

Corollary 3.12 *Under the assumptions of Theorem 3.11, the energy of the solution U of (1)-(3) with an initial datum $U_0 \in \mathcal{H}$ satisfies*

$$\lim_{t\to+\infty} E(t) = \frac{1}{2}\|U_{proj,0}\|_{\mathcal{H}}^2\,.$$

4 Polynomial stability

Our main goal is here to prove the polynomial decay of the energy of solutions of (1)-(3). For that purpose we use the following result from Theorem 2.4 of [6] (see also [3, 4, 15] for weaker variants).

Lemma 4.1 *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space satisfies*

$$||e^{t\mathcal{L}}U_0|| \le C\,t^{-\frac{1}{l}}||U_0||_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in D(\mathcal{L}), \quad \forall t > 1,$$

for some constant $C > 0$ and for $l > 0$ if

(55) $$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

(56) $$\limsup_{|\beta|\to\infty} \frac{1}{\beta^l}\,\|(i\beta - \mathcal{L})^{-1}\| < \infty,$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

In view of this Lemma we need to check the properties (55) (see section 3) and (56).

The next lemmas show that (56) holds with $\mathcal{L} = \mathcal{A}$ and $l \ge 2$.

Lemma 4.2 *Assume that (4)-(5), (10) and (34) hold. Assume that m does not change of sign, in the sense that there exist $m_0 \in \mathbb{R}^*$ and $m_1 > 0$ such that*

$$\frac{m(x)}{m_0} \ge m_1 \quad \forall \text{ a.e. } x \in \Omega.$$

If $m = m_0 J$ a.e. in Ω or if there exists a positive real number K such that $c = KJ$ a.e. in Ω, then the resolvent operator of $\mathcal{A}$ satisfies condition (56) for $l \ge 2$ in $\mathcal{H}_0$ (resp. $\mathcal{H}$) in the first (resp. second) case of Lemma 3.3.

Proof. First assume that $\mathcal{A}$ has no eigenvalue on the imaginary axis (second case of Lemma 3.3). Then we need to check (56) in $\mathcal{H}$. For that purpose we use a contradiction argument, i.e., we suppose that (56) is false for $l \geq 2$. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (u_n, v_n, \varphi_n, \phi_n, \theta_n)^\top$ in $D(\mathcal{A})$ with $\|z_n\|_{\mathcal{H}} = 1$ such that

$$\beta_n^l \|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}} \to 0 \text{ as } n \to \infty. \tag{57}$$

This directly implies that, by (4)-(6),

$$\beta_n^l \|v_n - i\beta_n u_n\|_{1,\Omega} \to 0, \tag{58}$$

$$\beta_n^l \|i\beta_n \varphi_n - \phi_n\|_{1,\Omega} \to 0, \tag{59}$$

$$\beta_n^l \|i\beta_n v_n - \rho^{-1}(\mathrm{div}[C\epsilon(u_n) + \gamma\epsilon(v_n) + (b\varphi_n - \beta\theta_n)Id]\| \to 0, \tag{60}$$

$$\beta_n^l \|i\beta_n \phi_n - J^{-1}(\mathrm{div}(\delta\nabla\varphi_n) - b\,\mathrm{div}\, u_n - \xi\varphi_n + m\theta_n)\| \to 0, \tag{61}$$

$$\beta_n^l \|i\beta_n \theta_n - c^{-1}(\mathrm{div}(k\nabla\theta_n) - \beta\,\mathrm{div}\, v_n - m\phi_n)\| \to 0. \tag{62}$$

We first notice that

$$\beta_n^l \Re \langle (i\beta_n - \mathcal{A})z_n, z_n\rangle_{\mathcal{H}} \leq \beta_n^l \|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}} \|z_n\|_{\mathcal{H}} = \beta_n^l \|(i\beta_n - \mathcal{A})z_n\|_{\mathcal{H}}$$

and, by (13),

$$\beta_n^l \Re \langle (i\beta_n - \mathcal{A})z_n, z_n\rangle_{\mathcal{H}} = \beta_n^l \int_\Omega \left(\gamma|\epsilon(v_n)|^2 + k\nabla\theta_n \cdot \nabla\bar{\theta}_n\right) dx.$$

By Korn's inequality, (5) and (34) we immediately deduce that

$$\beta_n^l (\|v_n\|_{1,\Omega}^2 + |\theta_n|_{1,\Omega}^2) \to 0. \tag{63}$$

By (58) and (63), we obtain

$$\beta_n^{l+2} \|u_n\|_{1,\Omega}^2 \to 0. \tag{64}$$

As $\|\phi_n\|$ is bounded due to $\|z_n\|_{\mathcal{H}} = 1$ and by (59) we see that there exists $C > 0$ (independent of n) such that

$$\beta_n \|\varphi_n\| \leq \|i\beta_n\varphi_n - \phi_n\| + \|\phi_n\| \leq C \quad \forall n \in \mathbb{N}, \tag{65}$$

and therefore

$$\|\varphi_n\| \to 0. \tag{66}$$

Moreover, as

$$\|\phi_n\|_{1,\Omega} \leq \|\phi_n - i\beta_n\varphi_n\|_{1,\Omega} + \beta_n \|\varphi_n\|_{1,\Omega},$$

with (59) and since $\|\varphi_n\|_{1,\Omega}$ is bounded ($\|z_n\|_{\mathcal{H}} = 1$), we obtain that

$$\beta_n^{-1} \|\phi_n\|_{1,\Omega} \text{ is bounded.} \tag{67}$$

As c is positive definite (see (4)) and using the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, we can show that there exists a positive constant C such that

$$\|\theta_n\| \leq C\Big(|\theta_n|_{1,\Omega} + \Big|\int_\Omega c\theta_n\,dx\Big|\Big), \forall n \in \mathbb{N}. \tag{68}$$

On the other hand the fact that $z_n \in \mathcal{H}$ implies that

$$\int_\Omega c\theta_n\,dx = -\int_\Omega (m\varphi_n + \beta\operatorname{div} u_n)\,dx. \tag{69}$$

Therefore by Cauchy-Schwarz's inequality and (66) and (64), we deduce that

$$\int_\Omega c\theta_n\,dx \to 0.$$

This property and (63) in the estimate (68) allow to conclude that

$$\|\theta_n\|_{1,\Omega} \to 0. \tag{70}$$

The same argument replacing (66) by (65) yields

$$\beta_n\|\theta_n\|_{1,\Omega} \text{ bounded} \tag{71}$$

since $l \geq 2$.

But this is not sufficient for our next purposes because we need that

$$\beta_n(c\theta_n, \phi_n) \to 0. \tag{72}$$

To prove this property, we first notice that (59) implies that

$$\int_\Omega m(i\beta_n\varphi_n - \phi_n)\,dx \to 0,$$

since $m \in L^2(\Omega)$ and therefore

$$i\beta_n\int_\Omega m\varphi_n\,dx - \int_\Omega m\phi_n\,dx \to 0. \tag{73}$$

Similarly as $J \in L^2(\Omega)$ by (61) we get

$$\int_\Omega (i\beta_n J\phi_n - (\operatorname{div}(\delta\nabla\varphi_n) - b\operatorname{div} u_n - \xi\varphi_n + m\theta_n))\,dx \to 0,$$

and hence by Green's formula and the boundary condition $\delta\nabla\varphi_n \cdot n = 0$ on Γ, we find

$$\int_\Omega (i\beta_n J\phi_n - (-b\operatorname{div} u_n - \xi\varphi_n + m\theta_n))\,dx \to 0.$$

Owing to $\|z_n\|_{\mathcal{H}} = 1$ we deduce that

$$\int_\Omega J\phi_n\,dx \to 0. \tag{74}$$

In the case when $m = m_0 J$, this condition guarantees that

$$\int_\Omega m\phi_n \, dx \to 0,$$

and owing to (73) we arrive at

$$\beta_n \int_\Omega m\varphi_n \, dx \to 0. \tag{75}$$

Coming back to (69) we obtain

$$\beta_n \left| \int_\Omega c\theta_n \, dx \right| \le \beta_n \left| \int_\Omega m\varphi_n \, dx \right| + \beta_n \, \|\beta\| \, \| \operatorname{div} u_n \|.$$

Therefore by (64) and (75) we deduce that

$$\beta_n \int_\Omega c\theta_n \, dx \to 0.$$

This property and (63) in the estimate (68) allow to conclude that

$$\beta_n \|\theta_n\|_{1,\Omega} \to 0. \tag{76}$$

Consequently

$$|\beta_n (c\theta_n, \phi_n)| \le \sup_{x\in\Omega} c(x) \, \beta_n \, \|\theta_n\| \, \|\phi_n\| \to 0,$$

by (76) and $\|z_n\|_{\mathcal{H}} = 1$.

In the case when $c = KJ$ for some positive real number K, we may write

$$\begin{aligned} \beta_n (c\theta_n, \phi_n) &= K\beta_n (J\theta_n, \phi_n) \\ &= K\beta_n \int_\Omega \theta_n (J\phi_n - \mathcal{M}_\Omega (J\phi_n)) \, dx + K\beta_n \mathcal{M}_\Omega (J\phi_n) \Big(\int_\Omega \theta_n \, dx \Big) \\ &= K\beta_n \int_\Omega (\theta_n - \mathcal{M}_\Omega \theta_n)(J\phi_n - \mathcal{M}_\Omega (J\phi_n)) \, dx + K\beta_n \mathcal{M}_\Omega (J\phi_n) \Big(\int_\Omega \theta_n \, dx \Big), \end{aligned} \tag{77}$$

where $\mathcal{M}_\Omega w = |\Omega|^{-1} \int_\Omega w \, dx$ is the mean in Ω of w. Hence by Cauchy-Schwarz's inequality we obtain

$$\beta_n |(c\theta_n, \phi_n)| \le C \Big(\beta_n \|\theta_n - \mathcal{M}_\Omega \theta_n\| \|\phi_n\| + \beta_n \|\theta_n\| \, |\mathcal{M}_\Omega (J\phi_n)| \Big),$$

for some positive constant C independent of n. By Friedrichs' inequality we deduce that

$$\beta_n |(c\theta_n, \phi_n)| \le C' \Big(\beta_n |\theta_n|_{1,\Omega} \|\phi_n\| + \beta_n \|\theta_n\| |\mathcal{M}_\Omega (J\phi_n)| \Big),$$

for some positive constant C' independent of n. This proves that (72) still holds in that case because the right-hand side of this estimate tends to zero as $n \to \infty$ owing to (63), $\|z_n\|_{\mathcal{H}} = 1$ for the first term, while for the second term we use (74) and (71).

By (62) and since $\|\phi_n\|$ is bounded ($\|z_n\|_{\mathcal{H}} = 1$), we have

$$(ic\beta_n\theta_n - (\operatorname{div}(k\nabla\theta_n) - \beta \operatorname{div} v_n - m\phi_n), \phi_n) \to 0,$$

i.e., by Green's formula,

$$i\beta_n(c\theta_n, \phi_n) + (\beta \operatorname{div} v_n, \phi_n) + (m\phi_n, \phi_n) + \int_\Omega k\nabla\theta_n \cdot \nabla\bar{\phi}_n \, dx \to 0, \tag{78}$$

recalling that $z_n \in D(\mathcal{A})$ implies the boundary condition $k\nabla\theta_n \cdot n = 0$ on Γ. First we notice that the first term of this left-hand side tends to zero due to (72). Moreover, we have

$$|(\beta \operatorname{div} v_n, \phi_n)| \leq 2^{\frac{d-1}{2}} \sup_{x\in\Omega} |\beta(x)| \, |v_n|_{1,\Omega} \, \|\phi_n\| \to 0,$$

by (63), since $\|z_n\|_{\mathcal{H}} = 1$, and finally

$$\left| \int_\Omega k\nabla\theta_n \cdot \nabla\bar{\phi}_n \, dx \right| \leq \sup_{x\in\Omega} \|k(x)\|_2 \, \beta_n |\theta_n|_{1,\Omega} \beta_n^{-1} \|\phi_n\|_{1,\Omega} \to 0,$$

by (63), (67) and since $l \geq 2$. These three properties in (78) imply that

$$(m\phi_n, \phi_n) \to 0.$$

Due to the assumption on m this guarantees that

$$\|\phi_n\| \to 0. \tag{79}$$

In the same manner, by (61) and since $\|J\varphi_n\|$ is bounded ($\|z_n\|_{\mathcal{H}} = 1$), we have

$$(iJ\beta_n\phi_n - (\operatorname{div}(\delta\nabla\varphi_n) - b \operatorname{div} u_n - \xi\varphi_n + m\theta_n), \varphi_n) \to 0,$$

i.e., by Green's formula,

$$i\beta_n(J\phi_n, \varphi_n) + (b \operatorname{div} u_n, \varphi_n) + \int_\Omega \delta\nabla\varphi_n \cdot \nabla\bar{\varphi}_n \, dx + \left\| \xi^{1/2}\varphi_n \right\|^2 - (m\theta_n, \varphi_n) \to 0, \tag{80}$$

recalling $z_n \in D(\mathcal{A})$ implies the boundary conditions $\delta\nabla\varphi_n \cdot n = 0$ on Γ. Moreover, we have

$$|\beta_n(J\phi_n, \varphi_n)| \leq \sup_{x\in\Omega} J(x)\beta_n \, \|\varphi_n\| \, \|\phi_n\| \to 0,$$

by (79) and (65). Similarly

$$|(b \operatorname{div} u_n, \varphi_n)| \leq 2^{\frac{d-1}{2}} \sup_{x\in\Omega} |b(x)| \; |u_n|_{1,\Omega} \, \|\varphi_n\| \to 0,$$

by (64), since $\|z_n\|_{\mathcal{H}} = 1$. Finally

$$|(m\theta_n, \varphi_n)| \leq \sup_{x\in\Omega} |m(x)| \, \|\theta_n\| \, \|\varphi_n\| \to 0,$$

by (66) and since $\|z_n\|_{\mathcal{H}} = 1$. These properties and (66) in (80) yield

$$|\varphi_n|_{1,\Omega} \to 0. \tag{81}$$

In conclusion, by (63), (64), (66), (70), (79) and (81) we obtain

$$\|z_n\|_{\mathcal{H}} \to 0,$$

which contradicts $\|z_n\|_{\mathcal{H}} = 1$.

In the first case of Lemma 3.3, $\mathcal{A}$ has some eigenvalues on the imaginary axis, but by the assumption on m and Remark 3.8, $\mathcal{A}$ has only a finite number of eigenvalues on the imaginary axis. As before to check (56) in $\mathcal{H}_0$, we use a contradiction argument, i.e., we suppose that (56) is false for $l \geq 2$. Then there exist a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $z_n = (u_n, v_n, \varphi_n, \phi_n, \theta_n)^\top$ in $D(\mathcal{A}) \cap \mathcal{H}_0$ with $\|z_n\|_{\mathcal{H}} = 1$ satisfying (57). Since for n large enough β_n will be greater than the largest eigenvalue in modulus of $\mathcal{A}$ in the imaginary axis, the previous arguments lead to the contradiction. ■

Since the two hypothesis of Lemma 4.1 are proved in Lemma 3.3 and Lemma 4.2 we deduce the main result of this paper.

Theorem 4.3 *Assume that (4)-(6), (10) and (34) hold. Assume that m satisfies the assumptions of the above Lemma 4.2. Then we can distinguish the following case:*
i) In the first case of Lemma 3.3, there exists a constant $C > 0$ such that for all $U_0 \in \mathcal{D}(\mathcal{A}) \cap \mathcal{H}_0$, the solution of system (1)-(3) satisfies the following estimate

$$E(t) \leq C\, t^{-1} \, \|U_0\|^2_{\mathcal{D}(\mathcal{A})}, \forall t > 1. \tag{82}$$

ii) In the second case of Lemma 3.3, there exists a constant $C > 0$ such that for all $U_0 \in \mathcal{D}(\mathcal{A})$, the solution of system (1)-(3) satisfies the estimate (82).

Remark 4.4 *This Theorem gives a correct proof of Theorem 3.4 of [18]. Indeed the proof given in [18] uses the wrong estimate (3.6) of [18].*

Corollary 4.5 *Under the assumptions of Theorem 4.3, the energy of the solution U of (1)-(3) with an initial datum $U_0 \in D(\mathcal{A})$ tends polynomially at the speed $1/t$ to the energy of the orthogonal projection of the initial datum on $\mathcal{H}_1$, more precisely the following estimate holds*

$$0 \leq E(t) - \frac{1}{2} \|U_{proj,0}\|^2_{\mathcal{H}} \leq C\, t^{-1} \, \|U_0\|^2_{\mathcal{D}(\mathcal{A})}, \forall t > 1,$$

recalling that $U_{proj,0}$ is the orthogonal projection of the initial datum $U_0 \in \mathcal{D}(\mathcal{A})$ on $\mathcal{H}_1$.

Proof. The proof is the same as the one of Corollary 3.12 by noticing that for an initial datum $U_0 \in D(\mathcal{A})$, its projection $U_{proj,0}$ is still in $D(\mathcal{A})$. Indeed we may write

$$U_{proj,0} = \sum_{U_{n,\pm} \in B} (U_0, U_{n,\pm})_{\mathcal{H}}\, U_{n,\pm},$$

and consequently $U_{proj,0}$ belongs to $D(\mathcal{A})$ if and only if

$$\sum_{U_{n,\pm} \in B} |(U_0, U_{n,\pm})_{\mathcal{H}}|^2 \lambda_n < \infty.$$

But according to the proof of Lemma 3.5, we have $(\mathcal{A}U_0, U_{n,\pm})_{\mathcal{H}} = \pm i\sqrt{\lambda_n}(U_0, U_{n,\pm})_{\mathcal{H}}$ and therefore

$$\sum_{U_{n,\pm} \in B} |(U_0, U_{n,\pm})_{\mathcal{H}}|^2 \lambda_n = \sum_{U_{n,\pm} \in B} |(\mathcal{A}U_0, U_{n,\pm})_{\mathcal{H}}|^2 \leq \|\mathcal{A}U_0\|^2_{\mathcal{H}}.$$

■

Remark 4.6 *Note that the assumptions on m in Lemma 4.2 are quite weak and are satisfied, for instance, if c and J are positive constant functions and m is a non zero constant function.*

5 Characterization of $\mathcal{H}_0$ for constant coefficients when $\gamma = 0$

In Remark 3.7 we have characterized the space $\mathcal{H}_0$ in the case when the coefficients are constant and when γ is positive and $m \neq 0$. Here we want to made a similar characterization when γ is zero. Especially in dimension 1 we make a full characterization.

We assume now that all coefficients are constant and $\gamma = 0$.

First we notice that system (23)-(24) has an eigenvector $(u_*, \varphi_*) = (0, 1)$ of eigenvalue $\lambda_* = J^{-1}(\xi + \frac{m^2}{c})$. Hence for all $n \in \mathbb{N}^*$, with $\lambda_n \neq \lambda_*$, we have also

$$\int_\Omega J\varphi_n dx = \left(\begin{pmatrix} u_* \\ \varphi_* \end{pmatrix}, \begin{pmatrix} u_n \\ \varphi_n \end{pmatrix} \right)_{\rho,J} = 0. \tag{83}$$

It is easy to verify that (u_*, φ_*) satisfy (25)-(26). Therefore $\pm i\sqrt{\lambda_*} \in \rho(\mathcal{A})$ with $U_{*,\pm} = (0, 0, 1, \pm i\sqrt{\lambda_*}, -\frac{m}{c})^\top$.

Note that λ_* is not necessarily the smallest eigenvalue of L.

We now distinguish the following four cases.

5.1 First case: $m \neq 0$ and $\beta \neq 0$.

Let $n \in \mathbb{N}^*$, with $\lambda_n \neq \lambda_*$. Assume that (25) holds, namely there exists $k_1 \in \mathbb{C}$ such that

$$\beta \operatorname{div} u_n + m\varphi_n = ck_1. \tag{84}$$

By (83), we deduce that

$$\beta \int_\Omega \operatorname{div} u_n dx = ck_1 \left|\Omega\right|,$$

where $|\Omega|$ is the measure of Ω. By the Green's formula, since $u_n = 0$ on Γ,

$$\int_\Omega \operatorname{div} u_n dx = 0,$$

and therefore $k_1 = 0$ by (4). This means that (84) is reduced to

$$\beta \operatorname{div} u_n + m\varphi_n = 0,$$

and therefore, as $m \neq 0$,

$$\varphi_n = -\frac{\beta}{m} \operatorname{div} u_n. \tag{85}$$

Now using this identity in (23)-(24) allows to obtain two disjoint eigenvalue problems

$$-\operatorname{div}\left[C\epsilon(u_n) - \frac{b\beta}{m} (\operatorname{div} u_n)\, Id\right] = \rho\lambda_n u_n, \tag{86}$$

$$-\operatorname{div}(\delta\nabla\varphi_n) + \left(\xi - \frac{bm}{\beta}\right)\varphi_n = J\lambda_n\varphi_n. \tag{87}$$

Hence we first introduce the two symmetric operators

$$L_2 u = -\rho^{-1} \operatorname{div} \left[C\epsilon(u) - \frac{b\beta}{m} (\operatorname{div} u)\, Id \right],$$

$$L_3 \varphi = -J^{-1} \operatorname{div}(\delta \nabla \varphi) + J^{-1} \left(\xi - \frac{bm}{\beta} \right) \varphi,$$

with natural domains. They are symmetric if $L^2(\Omega)^d$ (resp. $L^2(\Omega)$) is equipped with the inner product

$$\int_\Omega \rho u \cdot \bar{u}^* dx \quad (\text{resp.} \int_\Omega J \varphi \bar{\varphi}^* dx).$$

The operator L_3 is a selfadjoint operator with a compact resolvent since its associated sesquilinear form given by

$$\int_\Omega \delta \nabla \varphi \cdot \bar{\varphi}' + (\xi - \frac{bm}{\beta}) \varphi \bar{\varphi}' \, dx,$$

is weakly coercive. Hence it has a discrete spectrum $Sp(L_3)$. The situation is more delicate for L_2 due to the second term in (86). Therefore for the sake of simplicity and since only this case will be treated below, we assume that L_2 is selfadjoint with a discrete spectrum $Sp(L_2)$.

Now we have two possibilities:

1. If $Sp(L_2) \cap Sp(L_3) \cap]0, +\infty[= \emptyset$, then (25) never holds and

$$\mathcal{H}_0 = \{ U \in \mathcal{H} : \, (U, U_{*,\pm})_{\mathcal{H}} = 0 \}.$$

2. If $Sp(L_2) \cap Sp(L_3) \cap]0, +\infty[\neq \emptyset$, we denote by u_n (resp. φ_n) an eigenvector of L_2 (resp. L_3) of common eigenvalue λ_n. Then again we have two alternatives:

   (a) (85) does not hold, i.e. $\varphi_n \neq -\frac{\beta}{m} \operatorname{div} u_n$, and again $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \{ U \in \mathcal{H} : \, (U, U_{*,\pm})_{\mathcal{H}} = 0 \}.$$

   (b) (85) holds and therefore (25) holds.
   This shows that

$$\mathcal{H}_0 = \{ U \in \mathcal{H} : \, (U, U_{*,\pm})_{\mathcal{H}} = 0 \text{ and } (U, U_{n,\pm})_{\mathcal{H}} = 0, \, \forall \lambda_n \in Sp(L_2) \cap Sp(L_3) \cap]0, +\infty[\text{ s.t. (85) holds} \}.$$

This result holds in any dimension but an explicit calculation is only possible in one dimension or for some special geometries in dimension larger than two.

We first restrict ourselves to the dimension one, i.e. $\Omega =]0, L[$, with $L > 0$. In this case we can prove the following results:

Lemma 5.1 *If $\Omega = [0, L] \subset \mathbb{R}$, if $m \neq 0$, $\beta \neq 0$ and $\gamma = 0$ (and all coefficients are constant), then we have the following characterisation of $\mathcal{H}_0$:*

1. *If* $c - \frac{b\beta}{m} > 0$ *and if* $r := \frac{(\xi\beta - mb)L^2\rho m}{\pi^2\beta(Jcm - b\beta J - \delta\rho m)}$ *is positive and* $\sqrt{r} \in \mathbb{N}^*$*, then* $\mathcal{H}_0$ *is given by*

$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0 \text{ and } (U, U_{n_0,\pm})_{\mathcal{H}} = 0\},$$

where $n_0 \in \mathbb{N}^*$ *is the unique integer such that* $\lambda_{n_0} = \frac{l_1^2\pi^2}{\rho L^2}\left(c - \frac{b\beta}{m}\right)$ *with* $l_1 = \sqrt{r}$ *and*

$$U_{n_0,\pm} = (\alpha_{n_0} \sin\left(\frac{l_1\pi x}{L}\right), \pm i\sqrt{\lambda_{n_0}}\alpha_{n_0} \sin\left(\frac{l_1\pi x}{L}\right), \beta_{n_0} \cos\left(\frac{l_1\pi x}{L}\right), \pm i\sqrt{\lambda_{n_0}}\beta_{n_0} \cos\left(\frac{l_1\pi x}{L}\right), 0)^\top.$$

2. *In the other cases,* $\mathcal{H}_0$ *is given by*

$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0\}.$$

Proof. In $1-d$, (86)-(87) becomes

$$\begin{cases} \left(\frac{b\beta}{m} - c\right) u_n'' = \rho\lambda_n u_n \text{ in }]0, L[\\ u_n(0) = u_n(L) = 0 \end{cases}, \quad \begin{cases} -\delta\varphi_n'' = \left(J\lambda_n - \left(\xi - \frac{bm}{\beta}\right)\right)\varphi_n \text{ in }]0, L[\\ \varphi_n'(0) = \varphi_n'(L) = 0. \end{cases}$$

Consequently there exist $l_1 \in \mathbb{N}^*$, $l_2 \in \mathbb{N}$ such that

$$\frac{\rho\lambda_n}{c - \frac{b\beta}{m}} = \frac{l_1^2\pi^2}{L^2}, \quad \delta^{-1}\left(J\lambda_n + \left(\frac{bm}{\beta} - \xi\right)\right) = \frac{l_2^2\pi^2}{L^2},$$

with

$$u_n(x) = \alpha_n \sin\left(\frac{l_1\pi x}{L}\right), \quad \varphi_n(x) = \beta_n \cos\left(\frac{l_2\pi x}{L}\right),$$

for some $\alpha_n, \beta_n \in \mathbb{C}$. Hence such a solution exists if there exist $l_1 \in \mathbb{N}^*$, $l_2 \in \mathbb{N}$ such that

$$0 < \frac{l_1^2\pi^2}{L^2}\left(\frac{c - \frac{b\beta}{m}}{\rho}\right) = J^{-1}\left(\frac{\delta l_2^2\pi^2}{L^2} + \xi - \frac{bm}{\beta}\right).$$

We have then two cases:

1. $c - \frac{b\beta}{m} \leq 0$: there is no solution.
2. $c - \frac{b\beta}{m} > 0$: a solution exists if there exist $l_1 \in \mathbb{N}^*$ and $l_2 \in \mathbb{N}$ such that

$$\frac{l_1^2\pi^2}{L^2}J\left(\frac{c - \frac{b\beta}{m}}{\rho}\right) - \frac{\delta l_2^2\pi^2}{L^2} = \xi - \frac{bm}{\beta}. \tag{88}$$

   But recall that we also need that (85) holds. In our case, this reduces to

$$\beta\frac{l_1\pi}{L}\alpha_n \cos\left(\frac{l_1\pi x}{L}\right) + m\beta_n \cos\left(\frac{l_2\pi x}{L}\right) = 0 \quad \forall \text{ a.e. } x \in]0, L[.$$

   This property does not hold if $l_1 \neq l_2$. We then have $l_1 = l_2$ and the previous identity holds by choosing

$$\beta_n = -\frac{\beta l_1\pi}{mL}\alpha_n,$$

α_n being fixed to normalize the eigenvector (u_n, φ_n) of L. Now (88) reduces to

$$\frac{l_1^2\pi^2}{L^2}\left(\frac{J\left(c-\frac{b\beta}{m}\right)}{\rho}-\delta\right)=\xi-\frac{bm}{\beta},$$

which is only possible if there exists $l_1 \in \mathbb{N}^*$ such that

$$l_1^2=\frac{(\xi\beta-mb)L^2\rho m}{\pi^2\beta(Jcm-b\beta J-\delta\rho m)}=:r,$$

i.e. if r is positive and $\sqrt{r}\in\mathbb{N}^*$. These two conditions are quite oftenly not satisfied. Moreover only one l_1 is possible.

■

Remark 5.2 *In $1-d$ and if $m\neq 0$, $\beta\neq 0$, $\mathcal{A}$ has at most four eigenvalues on the imaginary axis.*

In higher dimension, we can state the following result.

Lemma 5.3 *Let $m \neq 0$, $\beta \neq 0$, $\gamma = 0$, $\delta = \delta_0 Id$, with $\delta_0 > 0$ and all other coefficients are constant. If $d \geq 2$ we also assume that the tensor C corresponds to the Lamé system, namely*

$$\operatorname{div}(C\epsilon(u))=\mu\Delta u+(\lambda^*+\mu)\nabla\operatorname{div}u,$$

with λ^* and μ two positive real numbers. Assume furthermore that $b\beta = m(\lambda^*+\mu)$ and that $\Omega\subset\mathbb{R}^d$ is such that there is no eigenvectors $u\in H_0^1(\Omega)^d$ of

$$-\Delta u=\lambda u \text{ in } \Omega,$$

for some positive real number λ that satifies

$$\frac{\partial}{\partial n}\operatorname{div}u=0 \text{ on } \Gamma. \tag{89}$$

Then $\mathcal{H}_0$ is characterized by

$$\mathcal{H}_0=\{U\in\mathcal{H}:\,(U,U_{*,\pm})_{\mathcal{H}}=0\}.$$

Proof. Our assumptions garantee that

$$L_2=-\rho^{-1}\mu\Delta.$$

Hence for $\lambda_n \in Sp(L_2)\cap Sp(L_3)\cap]0,+\infty[$, then $u_n\in H_0^1(\Omega)^d$ has to be an eigenvector of the Laplace operator with Dirichlet boundary condition, namely

$$-\Delta u_n=\rho\mu^{-1}\lambda_n u_n \text{ in } \Omega.$$

But then the identity (85) cannot hold because if it would hold, we would have

$$0=\frac{\partial}{\partial n}\varphi_n=-\frac{\beta}{m}\frac{\partial}{\partial n}\operatorname{div}u_n \text{ on } \Gamma,$$

which is a contradiction with our assumption on the eigenvector u_n. ■

Remark 5.4 *For any rectangle of the plane, it is easy to check that there is no eigenvector of the Laplace operator with Dirichlet boundary condition satisfying (89).*

5.2 Second case: $m = 0$ and $\beta \neq 0$.

First, as previously, (25) and Green's formula imply that $\operatorname{div} u_n = 0$. In that case, (23)-(24) is reduced to

$$\left\{ \begin{array}{l} -\operatorname{div}(C\epsilon(u_n) + b\varphi_n Id) = \rho\lambda_n u_n \\ -\operatorname{div}(\delta\nabla\varphi_n) + \xi\varphi_n = J\lambda_n\varphi_n. \end{array} \right. \tag{90}$$

We again start with the $1-d$ case:

Lemma 5.5 *If $\Omega = [0, L] \subset \mathbb{R}$, $m = 0$, $\beta \neq 0$ and $\gamma = 0$ (and all coefficients are constant), then we have the following characterisation of $\mathcal{H}_0$:*

1. *If $b \neq 0$, then $\mathcal{H}_0$ is given by*

$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0\}.$$

2. *If $b = 0$, then $\mathcal{H}_0$ is given by*

$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0 \text{ and } (U, U_{n,\pm})_{\mathcal{H}} = 0, \forall n \in \mathbb{N}\},$$

where for all $n \in \mathbb{N}$, $\lambda_n = J^{-1}\left(\xi + \frac{n^2\pi^2\delta}{L^2}\right)$ and

$$U_{n,\pm} = \left(0, 0, \cos\left(\frac{n\pi x}{L}\right), \pm i\sqrt{\lambda_n}\cos\left(\frac{n\pi x}{L}\right), 0\right)^\top.$$

Proof. In $1-d$, as $\operatorname{div} u_n = u_n' = 0$ and $u_n(0) = u_n(L) = 0$, we obtain $u_n = 0$ in $[0, L]$ and the previous system reduces to

$$\left\{ \begin{array}{l} b\varphi_n' = 0 \text{ and } -\delta\varphi_n'' + \xi\varphi_n = J\lambda_n\varphi_n \text{ in }]0, L[, \\ \varphi_n'(0) = \varphi_n'(L) = 0. \end{array} \right.$$

Then we have two cases:

1. If $b = 0$ it is reduced to $$\left\{ \begin{array}{l} -\delta\varphi_n'' = (J\lambda_n - \xi)\varphi_n \text{ in }]0, L[\\ \varphi_n'(0) = \varphi_n'(L) = 0, \end{array} \right.$$
   and then there exists $l \in \mathbb{N}$ such that $J\lambda_n = \xi + \frac{l^2\pi^2\delta}{L^2}$. Consequently we have an infinite number of λ_n.
2. If $b \neq 0$ then $\varphi_n' = 0$ and $\varphi_n = C$. Therefore there is a unique eigenvalue $\lambda_* = J^{-1}\xi$.

■

Lemma 5.6 *Let $m = 0$, $\beta \neq 0$, $\gamma = 0$, $\delta = \delta_0 Id$, with $\delta_0 > 0$ and all other coefficients are constant. If $d \geq 2$ we also assume that the tensor C corresponds to the Lamé system. Assume furthermore that $\Omega \subset \mathbb{R}^d$ is such that there is no eigenvector $u \in H_0^1(\Omega)^d$ of*

$$-\Delta u = \lambda u \text{ in } \Omega,$$

for some positive real number λ that is divergence free, i.e., that satifies

$$\operatorname{div} u = 0 \text{ in } \Omega. \tag{91}$$

Then we have the following characterisation of $\mathcal{H}_0$:

1. *If $b \neq 0$, then $\mathcal{H}_0$ is given by*

$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0\}.$$

2. *If $b = 0$, then $\mathcal{H}_0$ is given by*

$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0 \text{ and } (U, U_{n,\pm})_{\mathcal{H}} = 0, \forall n \in \mathbb{N}\},$$

where for all $n \in \mathbb{N}$, $\lambda_n = J^{-1}\left(\xi + \lambda^2_{Neu,n}\right)$ and

$$U_{n,\pm} = (0, 0, \varphi_{Neu,n}, \pm i\sqrt{\lambda_n}\varphi_{Neu,n}, 0)^\top,$$

the sequence $\varphi_{Neu,n}$ is the eigenvector of the Laplace operator with Neumann boundary condition of associated eigenvalue $\lambda^2_{Neu,n}$, in other words

$$\begin{cases} -\operatorname{div}(\delta\nabla\varphi_{Neu,n}) = \lambda^2_{Neu,n}\varphi_{Neu,n} \text{ in } \Omega, \\ \delta\nabla\varphi_{Neu,n} \cdot n = 0 \text{ on } \Gamma. \end{cases}$$

Proof. Coming back to (90) and remembering that u_n is divergence free, we here get

$$\text{(92)} \qquad \begin{cases} -(\mu\Delta u_n + b\nabla\varphi_n) = \rho\lambda_n u_n \\ -\operatorname{div}(\delta\nabla\varphi_n) + \xi\varphi_n = J\lambda_n\varphi_n. \end{cases}$$

Taking the divergence of the first identity we find

$$b\Delta\varphi_n = 0 \text{ in } \Omega.$$

Hence if $b \neq 0$, we find that φ_n is constant in the whole of Ω, and coming back to the first identity of (92) we find that

$$-\mu\Delta u_n = \rho\lambda_n u_n \text{ in } \Omega.$$

Since u_n is divergence free we find that the only possibility is $u_n = 0$, which is a contradiction.

On the contrary if $b = 0$, then u_n has to be zero by the same arguments as before and the second identity of (92) yields an infinite number of solutions φ_n, since

$$-\operatorname{div}(\delta\nabla\varphi_n) = (J\lambda_n - \xi)\varphi_n.$$

■

Remark 5.7 *For any rectangle of the plane, it is easy to check that there is no eigenvector of the Laplace operator with Dirichlet boundary condition satisfying (91).*

5.3 Third case: $m \neq 0$ and $\beta = 0$.

In that case, (25) and (83) imply that $\varphi_n = 0$, and (23)-(24) is reduced to

$$\begin{cases} -\operatorname{div}(C\epsilon(u_n)) = \rho\lambda_n u_n \\ b \operatorname{div} u_n = 0. \end{cases}$$

Then we have two cases:

1. If $b = 0$ it is reduced to
$$\left\{ \begin{array}{l} -\operatorname{div}(C\epsilon(u_n)) = \rho\lambda_n u_n \\ u_n = 0 \text{ on } \Gamma, \end{array} \right.$$
that always admits an infinite number of solutions.

   In one dimension, they are fully explicit since the above system reduces to
$$\left\{ \begin{array}{l} -cu_n'' = \rho\lambda_n u_n \text{ in }]0, L[\\ u_n(0) = u_n(L) = 0. \end{array} \right.$$
   Hence there exists $l \in \mathbb{N}^*$ such that $\lambda_n = \frac{l^2\pi^2 c}{L^2\rho}$.

   In dimension $d \geq 2$, $\rho\lambda_n$ is simply the eigenvalue of the elasticity system with Dirichlet boundary condition.

2. If $b \neq 0$ then $\operatorname{div} u_n = 0$. In one dimension we obtain that $u_n = 0$, while if $d \geq 2$, assuming again that C corresponds to the Lamé system and under the assumption that no eigenvectors of the Laplace operator with Dirichlet boundary condition is divergence free, we still conclude that $u_n = 0$.

We have then proved the following lemmas:

Lemma 5.8 *If $\Omega = [0, L] \subset \mathbb{R}$, if $m \neq 0$, $\beta = 0$ and $\gamma = 0$ (and all coefficients are constant), then we have the following characterisation of $\mathcal{H}_0$:*

1. *If $b \neq 0$, then $\mathcal{H}_0$ is given by*
$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0\}.$$

2. *If $b = 0$, then $\mathcal{H}_0$ is given by*
$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0 \text{ and } (U, U_{n,\pm})_{\mathcal{H}} = 0, \forall n \in \mathbb{N}^*\},$$
   where for all $n \in \mathbb{N}^*$, $\lambda_n = \frac{n^2\pi^2 c}{L^2\rho}$ and
$$U_{n,\pm} = (\sin\left(\frac{n\pi x}{L}\right), \pm i\sqrt{\lambda_n}\sin\left(\frac{n\pi x}{L}\right), 0, 0, 0)^\top.$$

Lemma 5.9 *If $d \geq 2$, $m \neq 0$, $\beta = 0$ and $\gamma = 0$ (and all coefficients are constant), then we have the following characterisation of $\mathcal{H}_0$:*

1. *If $b \neq 0$, C corresponds to the Lamé system and if no eigenvector of the Laplace operator with Dirichlet boundary condition in Ω is divergence free, then $\mathcal{H}_0$ is given by*
$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0\}.$$

2. *If $b = 0$, then $\mathcal{H}_0$ is given by*
$$\mathcal{H}_0 = \{U \in \mathcal{H} : (U, U_{*,\pm})_{\mathcal{H}} = 0 \text{ and } (U, U_{n,\pm})_{\mathcal{H}} = 0, \forall n \in \mathbb{N}^*\},$$
   where for all $n \in \mathbb{N}^*$, $\lambda_n = \frac{\lambda_{E,n}^2}{\rho}$ and
$$U_{n,\pm} = (u_{E,n}, \pm i\sqrt{\lambda_n}u_{E,n}, 0, 0, 0)^\top,$$

$u_{E,n}$ *is an eigenvector of the elasticity system with Dirichlet boundary condition in* Ω *of associated eigenvalue* $\lambda^2_{E,n}$*:*

$$\begin{cases} -\operatorname{div}(C\epsilon(u_{E,n})) = \lambda^2_{E,n} u_{E,n} \\ u_{E,n} = 0 \text{ on } \Gamma. \end{cases}$$

5.4 Fourth case: $m = \beta = 0$.

In this case (25) always holds and $\mathcal{A}$ has an infinite number of eigenvalues given by $\pm i\sqrt{\lambda_n}$, where $\{\lambda_n\}_{n\in\mathbb{N}^*}$ is the spectrum of the operator L. According to the previous results, we get strong stability in $\mathcal{H}_0$, which means that we impose an infinite number of constraints on the initial datum.

References

[1] F. Alabau and V. Komornik. Boundary observability, controllability, and stabilization of linear elastodynamic systems. *SIAM J. Control Optim.*, 37(2):521–542 (electronic), 1999.

[2] W. Arendt and C. J. K. Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Amer. Math. Soc.*, 305(2):837–852, 1988.

[3] A. Bátkai, K.-J. Engel, J. Prüss, and R. Schnaubelt. Polynomial stability of operator semigroups. *Math. Nachr.*, 279(13-14):1425–1440, 2006.

[4] C. J. K. Batty and T. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8(4):765–780, 2008.

[5] R. Bey, A. Heminna, and J.-P. Lohéac. Boundary stabilization of the linear elastodynamic system by a Lyapunov-type method. *Rev. Mat. Complut.*, 16(2):417–441, 2003.

[6] A. Borichev and Y. Tomilov. Optimal polynomial decay of functions and operator semigroups. *Math. Ann.*, 347(2):455–478, 2010.

[7] R. Brossard and J.-P. Lohéac. Boundary stabilization of elastodynamic systems, II. The case of a linear feedback. *J. Dyn. Control Syst.*, 16(3):355–375, 2010.

[8] S. Chirita and I.-D. Ghiba. Strong ellipticity and progressive waves in elastic materials with voids. *Proceedings of the Royal Society A, in print.*, 2009.

[9] M. Costabel, M. Dauge, and S. Nicaise. *Corner Singularities and Analytic Regularity for Linear Elliptic Systems.* Book in preparation.

[10] M. A. Horn. Implications of sharp trace regularity results on boundary stabilization of the system of linear elasticity. *J. Math. Anal. Appl.*, 223(1):126–150, 1998.

[11] D. Ieşan. *Thermoelastic models of continua*, volume 118 of *Solid Mechanics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 2004.

[12] G. Iovane and A. Nasedkin. Finite element dynamic analysis of anisotropic elastic solids with voids. *Computers and Structures*, 87:981?989, 2009.

[13] S. Jiang and R. Racke. *Evolution equations in thermoelasticity*, volume 112 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics.* Chapman & Hall/CRC, Boca Raton, FL, 2000.

[14] T. Kato. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[15] Z. Liu and B. Rao. Characterization of polynomial decay rate for the solution of linear evolution equation. *Z. Angew. Math. Phys.*, 56:630–644, 2005.

[16] A. Magaña and R. Quintanilla. On the exponential decay of solutions in one-dimensional generalized porous-thermo-elasticity. *Asymptot. Anal.*, 49(3-4):183–187, 2006.

[17] J. Muñoz-Rivera and R. Quintanilla. On the time polynomial decay in elastic solids with voids. *J. Math. Anal. Appl.*, 338(2):1296–1309, 2008.

[18] P. X. Pamplona, J. E. Muñoz Rivera, and R. Quintanilla. Stabilization in elastic solids with voids. *J. Math. Anal. Appl.*, 350(1):37–49, 2009.

[19] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Math. Sciences.* Springer-Verlag, New York, 1983.

[20] R. Quintanilla. Slow decay for one-dimensional porous dissipation elasticity. *Appl. Math. Lett.*, 16(4):487–491, 2003.

[21] R. Quintanilla and R. Racke. Stability in thermoelasticity of type III. *Discrete Contin. Dyn. Syst. Ser. B*, 3(3):383–400, 2003.