Stability results for the approximation of weakly coupled wave equations

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Abstract

In this note, we consider the approximation of two coupled wave equations with internal damping. Our goal is to damp the spurious high frequency modes by introducing a numerical viscosity term in the approximation schemes. With this viscosity term, we show the exponential or polynomial decay of the discrete scheme when the continuous problem has such a decay and when the spectrum of the spatial operator associated with the undamped system satisfies the generalized gap condition.

Résultats de stabilité de l'approximation de deux équations des ondes faiblement couplées

Résumé

Dans cette note, nous considérons l'approximation de deux équations des ondes couplées avec dissipation interne. Notre but est d'amortir les modes étranges en introduisant un terme de viscosité numérique. Avec ce terme de viscosité, nous montrons la décroissance exponentielle ou polynomiale du schéma discret lorsque le problème continu a une telle décroissance et lorsque le spectre de l'opérateur spatial associé au système sans dissipation satisfait la condition du gap généralisé.

I- Version française abregée

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1. Introduction

Dans cette note, nous considérons le système de deux équations des ondes couplées suivant :

$$\begin{cases} u_{tt} - u_{xx} + \alpha y + \beta u_t = 0 & \text{dans } (0,1) \times \mathbb{R}_+, \\ y_{tt} - y_{xx} + \alpha u + \gamma y_t = 0 & \text{dans } (0,1) \times \mathbb{R}_+, \\ u(0,t) = u(1,t) = y(0,t) = y(1,t) = 0 & \forall t > 0, \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, y(\cdot,0) = y_0, y_t(\cdot,0) = y_1 & \text{dans } (0,1), \end{cases} \tag{1}$$

où $\alpha, \beta, \gamma \in \mathbb{R}$ sont des constantes données telles que $\beta \geq 0$, $\gamma \geq 0$ (non nulles simultanément) et $\alpha > 0$ suffisamment petite. Nous écrivons le système dans un cadre Hilbertien comme suit : Nous définissons l'opérateur A par

$$A(u,y) = (-u_{xx} + \alpha y, -y_{xx} + \alpha u)\,, \forall (u,y) \in \mathcal{D}(A) = H_0^1(0,1)^2 \cap H^2(0,1)^2. \tag{2}$$

Si $\alpha < \pi^2$, alors l'opérateur A est positif auto-adjoint dans $H := L^2(0,1)^2$, car il est l'extension de Friedrichs du triplet $(L^2(0,1)^2, H_0^1(0,1)^2, a)$, où la forme bilinéaire, continue, symmétrique et coercive a est définie par :

$$a(\omega, \omega^*) = \int_0^1 (u_x u_x^* + y_x y_x^* + \alpha u^* y + \alpha y^* u)\, dx, \forall \omega = (u,y)\,, \omega^* = (u^*, y^*) \in H_0^1(0,1)^2. \tag{3}$$

Nous introduisons l'opérateur linéaire borné B, sur $L^2(0,1)^2$, par

$$B\,(u,y) = \sqrt{\beta}\,(u,0) + \sqrt{\gamma}\,(0,y)\,. \tag{4}$$

Nous réécrivons alors (1) sous la forme

$$\begin{gathered} \ddot{\omega}(t) + A\omega(t) + BB^*\dot{\omega}(t) = 0 \text{ in } H, \\ \omega(0) = \omega_0,\, \dot{\omega}(0) = \omega_1. \end{gathered} \tag{5}$$

Sous les conditions précédentes sur les coefficients, le problème est bien posé et est même dissipatif.

Selon le Théorème 2.2 de [3], si $\beta \neq 0$ et $\gamma \neq 0$, alors le système continu (1) est exponentiellement stable. Dans le cas contraire, si β ou γ est nul (mais pas simultanément), le Théorème 2.4 de [3] (voir aussi [2,11]) donne une stabilité polynomiale de (1). Notre but, dans cette note, est d'établir le même taux de décroissance pour un schéma discrétisé par éléments finis bien choisi du système (1). Notons que l'approximation du système par éléments finis ou par différences finies n'est pas uniformément exponentiellement ou polynomialement stable par rapport au paramètre de discrétisation à cause des modes de haute fréquence. Dès lors plusieurs remèdes ont été proposés et analysés pour surmonter ces difficultés. Citons la régularisation de Tychonoff [9], l'algorithme bi-grille [7], l'utilisation de méthodes d'éléments finis mixtes [8,4], ou la filtration des hautes fréquences [10,12,13].

Comme [12,13] notre but est d'amortir les modes haute fréquence en introduisant une viscosité numérique dans les schémas d'approximation. Cependant, les résultats de [12] ne sont pas applicables au système (1) puisque les valeurs propres de l'opérateur $A^{1/2}$ ne satisfont pas la condition du gap standard (voir ci-dessous) et le problème continu n'est pas toujours exponentiellement stable. Donc notre but est d'étendre les résultats de [12] pour prendre en considération les cas où les valeurs propres satisfont la condition du gap généralisée et quand le système (1) est exponentiellement ou polynomiallement stable.

2. Résultats principaux

Nous considérons la famille standard d'espaces d'approximation $(V_h)_{h>0}$ de l'espace $H_0^1(0,1)^2$, plus précisément V_h est l'espace vectoriel engendré par la famille de fonctions $(e_i, e_j)_{i,j\in\{1,\dots,N\}}$ où

$$e_j(x) = \left[1 - \frac{|x - x_j|}{h}\right]^+, \text{pour } j = 1, ..., N, \tag{6}$$

$N \in \mathbb{N}$, $h = \frac{1}{N+1}$, et $x_j = jh$, $j = 0, 1, ..., N+1$ (voir [6]).

Maintenant, nous étudions la stabilité du système discrétisé par éléments finis (approximation de (1)) défini sur l'espace de dimension finie V_h suivant :

$$\begin{gathered} \ddot{\omega}_h(t) + A_h\omega_h(t) + B_hB_h^*\dot{\omega}_h(t) + hA_h^{1+\frac{l}{2}}\dot{\omega}_h(t) = 0, \\ \omega_h(0) = \omega_{0h} \in V_h,\, \dot{\omega}_h(0) = \omega_{1h} \in V_h, \end{gathered} \tag{7}$$

où $\omega_h = (u_h, y_h)$, $\omega_{0h} = (u_{0h}, y_{0h}) \in V_h$ (resp. $\omega_{1h} = (u_{1h}, y_{1h}) \in V_h$) est une approximation de $\omega_0 = (u_0, y_0)$ (resp. $\omega_1 = (u_1, y_1)$), l est un entier positif à déterminer, $A_h : V_h \to V_h$, est défini par

$$(A_h\varphi_h,\, \psi_h) = a(\varphi_h,\, \psi_h),\, \forall\varphi_h,\, \psi_h \in V_h, \tag{8}$$

tandis que $B_h = B_h^* = B$ (qui est une application de V_h dans lui même). Signalons que (7) peut être identifié à un système de $2N$ inconnues réelles.

Notons que le système discrétisé (7) est équivalent à

$$\dot{z_h}(t) = \tilde{A}_{l,h}z_h(t)$$

avec $z_h(t) = (\omega_h(t), \dot{\omega}_h(t))^\top$ où $\tilde{A}_{l,h}$ est l'opérateur défini sur $V_h \times V_h$ par

$$\tilde{A}_{l,h} = \begin{pmatrix} 0 & I \\ -A_h & -hA_h^{1+\frac{l}{2}} - B_hB_h^* \end{pmatrix}. \tag{9}$$

Théorème 2.1 *(taux de décroissance exponentiel)*
Si $\beta \neq 0$ et $\gamma \neq 0$, alors le système discrétisé (7) avec $l = 0$ est uniformément exponentiellement stable, dans le sens où il existe M, α, $h^* > 0$ (indépendants de h, ω_{0h}, ω_{1h}) tel que pour tout $h \in (0, h^*)$:

$$\|\dot{\omega}_h(t)\|^2_{L^2(0,1)^2} + a(\omega_h(t),\, \omega_h(t)) \leq Me^{-\alpha t}(\|\omega_{1h}\|^2_{L^2(0,1)^2} + a(\omega_{0h},\, \omega_{0h})), \quad \forall t \geq 0.$$

Théorème 2.2 *(taux de décroissance polynomial)*
Si $\beta = 0$ et $\gamma > 0$, alors le système discrétisé (7) avec $l = 2$ est uniformément polynomialement stable, dans le sens où il existe C, $h^* > 0$ (indépendants de h, ω_{0h}, ω_{1h}) tel que pour tout $h \in (0, h^*)$:

$$\|\dot{\omega}_h(t)\|^2_{L^2(0,1)^2} + a(\omega_h(t),\, \omega_h(t)) \leq \frac{C}{t}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{l,h})}, \quad \forall t > 0,$$

lorsque $\|.\|_{D(\tilde{A}_{l,h})}$ est la norme du graphe.

Remarque 1 Les résultats présentés dans cette note peuvent être généralisés à un problème abstrait (5) défini dans un espace de Hilbert reél H avec $A : \mathcal{D}(A) \to H$ opérateur autoadjoint positif à inverse compact et $B \in \mathcal{L}(U, H)$ où U est un autre espace de Hilbert réel. Le schéma numérique est défini de façon similaire à (7) où V_h est un espace de dimension finie qui approche $\mathcal{D}(A^{\frac{1}{2}})$. Si les valeurs propres de l'opérateur $A^{\frac{1}{2}}$ satisfont la condition du gap généralisé, si B satisfait une condition analogue à (14) et sous des propriétés d'approximation sur V_h, nous pouvons alors prouver que le schéma discret est exponentiellement (resp. polynomialement) stable (quand le problème continu l'est). Pour plus de détails, nous renvoyons à [1].

II- English Version

3. Introduction

In this note, we consider the following system coupling two wave equations

$$\begin{cases} u_{tt} - u_{xx} + \alpha y + \beta u_t = 0 & \text{in } (0,1) \times \mathbb{R}_+, \\ y_{tt} - y_{xx} + \alpha u + \gamma y_t = 0 & \text{in } (0,1) \times \mathbb{R}_+, \\ u(0,t) = u(1,t) = y(0,t) = y(1,t) = 0 & \forall t > 0, \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, y(\cdot,0) = y_0, y_t(\cdot,0) = y_1 & \text{in } (0,1), \end{cases} \tag{10}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are given such that $\beta \geq 0$, $\gamma \geq 0$ (and not equal to zero together) and $\alpha > 0$ small enough. This system enters into an abstract Hilbert framework as follows: If $\alpha < \pi^2$, the operator A defined by (2) is a positive self-adjoint operator in $H := L^2(0,1)^2$, since it is the Friedrichs extension of the triple $(L^2(0,1)^2, H_0^1(0,1)^2, a)$, where the continuous, symmetric, and coercive bilinear form a is given by (3). Introducing the bounded operator B, defined on $L^2(0,1)^2$, as in (4), we can rewrite (10) in the form (5). The assumptions on the coefficients directly lead to the well-posedness of this problem that is even dissipative.

According to Theorem 2.2 of [3], if $\beta \neq 0$ and $\gamma \neq 0$, then the continuous system (10) is exponentially stable. Otherwise, if β or γ vanishes (but not both), Theorem 2.4 of [3] (see also [2,11]) gives the polynomial stability of (10). Our aim here is to show the same decay results for a suitable finite element discretization of system (10). It is well known that the approximated systems by finite element or finite difference are not uniformly exponentially or polynomially stable with respect to the discretization parameter due to the spurious high frequency modes. Hence several remedies have been proposed and analyzed to overcome these difficulties. Let us quote the Tychonoff regularization [9], a bi-grid algorithm [7], a mixed finite element method [8,4], or filtering the high frequencies [10,12,13].

As in [12,13] our goal is to damp the spurious high frequency modes by introducing a numerical viscosity in the approximation schemes. However, the results from [12] are not applicable to system (10) since the eigenvalues of the operator $A^{1/2}$ do not satisfy the standard gap condition (see below) and the continuous problem is not always exponentially stable. Therefore our goal is to extend the results from [12] to take into consideration the cases when the eigenvalues satisfy the generalized gap condition and when the exponential or the polynomial stability of (10) holds.

4. Generalized gap condition.

In this section, we show that the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ of the operator $A^{1/2}$ satisfy the generalized gap condition, i.e, that there exists a constant $\gamma_0 > 0$ such that

$$\lambda_{k+2} - \lambda_k > 2\gamma_0, \ \forall k \in \mathbb{N}^*. \tag{11}$$

Recall that the standard gap condition is

$$\lambda_{k+1} - \lambda_k > \gamma_0, \ \forall k \in \mathbb{N}^*. \tag{12}$$

Indeed, we find out that the spectrum of A is given by

$$\text{Sp}(A) = \{\lambda_{+,k}^2\}_{k\in\mathbb{N}^*} \cup \{\lambda_{-,k}^2\}_{k\in\mathbb{N}^*},$$

where $\lambda_{+,k}^2 = k^2\pi^2 + \alpha$ with associated eigenvector $\omega_{+,k} = (\sin(k\pi\cdot), \sin(k\pi\cdot))$, and $\lambda_{-,k}^2 = k^2\pi^2 - \alpha$ with associated eigenvector $\omega_{-,k} = (\sin(k\pi\cdot), -\sin(k\pi\cdot))$. We then remark that each eigenvalue is simple and that the sequence $\{\omega_{+,k}\}_{k\in\mathbb{N}^*} \cup \{\omega_{-,k}\}_{k\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)^2$. To estimate the distance between the consecutive eigenvalues of $A^{1/2}$, we distinguish two different cases:

1. For all $k \in \mathbb{N}^*$, we need to look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k}$. Since

$$\lambda_{+,k} - \lambda_{-,k} = \sqrt{k^2\pi^2 + \alpha} - \sqrt{k^2\pi^2 - \alpha} = \frac{2\alpha}{\sqrt{k^2\pi^2 + \alpha} + \sqrt{k^2\pi^2 - \alpha}},$$

we see that this distance goes to zero as k goes to infinity.

2. For all $k \in \mathbb{N}^*$, we look at the distance between $\lambda_{+,k}$ and $\lambda_{-,k+1}$. Here we have

$$\lambda_{-,k+1} - \lambda_{+,k} = \sqrt{(k+1)^2\pi^2 - \alpha} - \sqrt{k^2\pi^2 + \alpha} = \frac{2k\pi^2 + \pi^2 - 2\alpha}{\sqrt{(k+1)^2\pi^2 - \alpha} + \sqrt{k^2\pi^2 + \alpha}},$$

which tends to π as k goes to infinity. In conclusion (11) is satisfied while (12) is not.

5. Exponential and polynomial energy decay rate of the discrete scheme

We consider the standard family of spaces $(V_h)_h$ which approximates $H_0^1(0,1)^2$, namely V_h is the linear span of the family of hat functions $(e_i, e_j)_{i,j\in\{1,\dots,N\}}$ where e_j is defined by (6), $N \in \mathbb{N}$, $h = \frac{1}{N+1}$, and $x_j = jh$, $j = 0, 1, \dots, N+1$ (see [6]).

Now we study the stability of the following (finite elements) discrete system (approximation of (10)) defined on the finite dimensional space V_h as follows:

$$\begin{gathered} \ddot{\omega}_h(t) + A_h\omega_h(t) + B_hB_h^*\dot{\omega}_h(t) + hA_h^{1+\frac{l}{2}}\dot{\omega}_h(t) = 0, \\ \omega_h(0) = \omega_{0h} \in V_h,\ \dot{\omega}_h(0) = \omega_{1h} \in V_h, \end{gathered} \tag{13}$$

where $\omega_h = (u_h, y_h)$, $\omega_{0h} = (u_{0h}, y_{0h}) \in V_h$ (resp. $\omega_{1h} = (u_{1h}, y_{1h}) \in V_h$) is an approximation of $\omega_0 = (u_0, y_0)$ (resp. $\omega_1 = (u_1, y_1)$), l is a non-negative integer to be determined, $A_h : V_h \to V_h$ is defined in (8), while $B_h = B_h^* = B$ (that maps V_h into itself). Obviously (13) can be identified to a system with $2N$ real unknowns but this is not useful later on.

We notice that the discrete system (13) is equivalent to

$$\dot{z_h}(t) = \tilde{A}_{l,h} z_h(t)$$

with $z_h(t) = (\omega_h(t), \dot{\omega}_h(t))^\top$ where $\tilde{A}_{l,h}$ is the operator defined on $V_h \times V_h$ by (9).

Theorem 5.1 *(exponential decay rate)*
If $\beta \neq 0$ and $\gamma \neq 0$, then the discrete system (13) with $l = 0$ is uniformly exponentially stable, in the sense that there exist constants M, α, $h^* > 0$ (independent of h, ω_{0h}, ω_{1h}) such that for all $h \in (0, h^*)$:

$$\|\dot{\omega}_h(t)\|^2_{L^2(0,1)^2} + a(\omega_h(t), \omega_h(t)) \le Me^{-\alpha t}(\|\omega_{1h}\|^2_{L^2(0,1)^2} + a(\omega_{0h}, \omega_{0h})), \quad \forall t \ge 0.$$

Theorem 5.2 *(polynomial decay rate)*
If $\beta = 0$ and $\gamma > 0$, then the discrete system (13) with $l = 2$ is uniformly polynomially stable, in the sense that there exist constants C, $h^* > 0$ (independent of h, ω_{0h}, ω_{1h}) such that for all $h \in (0, h^*)$:

$$\|\dot{\omega}_h(t)\|^2_{L^2(0,1)^2} + a(\omega_h(t), \omega_h(t)) \le \frac{C}{t}\|(\omega_{0h}, \omega_{1h})\|^2_{D(\tilde{A}_{l,h})}, \quad \forall t > 0,$$

where $\|.\|_{D(\tilde{A}_{l,h})}$ is the graph norm.

The proof of Theorem 5.1 is a direct extension of the results from [12], therefore we concentrate on the case $\beta = 0$ and $\gamma > 0$.

6. Proof of Theorem 5.2.

Step 1. First, we set for all $k \in \mathbb{N}^*$, $\alpha_k = \lambda_{+,k} - \lambda_{-,k}$, that behaves like k^{-1} or equivalently like $\lambda_{-,k}^{-1}$. We further introduce the matrices

$$B_k^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_k \end{pmatrix}, \quad \Phi_k = \begin{pmatrix} B\omega_{-,k} & 0 \\ 0 & B\omega_{+,k} \end{pmatrix},$$

where B is defined in (4). Hence for all $C = (c_1, c_2)^\top \in \mathbb{R}^2$, we have

$$\begin{aligned} \|B_k^{-1}\Phi_k C\|^2 &= \|c_1 B\omega_{-,k} + c_2 B\omega_{+,k}\|^2_{L^2(0,1)^2} + |\alpha_k|^2 |c_2|^2 \|B\omega_{+,k}\|^2_{L^2(0,1)^2} \\ &= \frac{\beta}{2}|c_1 + c_2|^2 + \frac{\gamma}{2}|c_2 - c_1|^2 + \left(\frac{\beta+\gamma}{2}\right)|\alpha_k|^2|c_2|^2. \end{aligned}$$

As $|\alpha_k| \sim \lambda_{-,k}^{-1}$, we deduce that there exists a positive constant α_0 (independent of k) such that

$$\|B_k^{-1}\Phi_k C\| \geq \alpha_0 \lambda_{-,k}^{-1}\|C\|_2. \tag{14}$$

Step 2. Using (14) and some approximation properties, we prove that there exist $\epsilon > 0$, $h^* > 0$ small enough and $c > 0$ such that for all $0 < h < h^*$ and for all $k \leq N^2 = dim V_h$ satisfying

$$h\lambda_{-,k}^2 \leq \epsilon,$$

it holds

$$\|B_k^{-1}\Phi_{k,h} C\|_{L^2(0,1)^4} \geq c\lambda_{-,k}^{-1}\|C\|_2, \quad \forall C \in \mathbb{R}^2, \tag{15}$$

where

$$\Phi_{k,h} = \begin{pmatrix} B\omega_{-,k,h} & 0 \\ 0 & B\omega_{+,k,h} \end{pmatrix},$$

$\omega_{\pm,k,h}$ being the eigenvector of A_h of eigenvalue $\lambda_{\pm,k,h}$ (that is close to $\lambda_{\pm,k}$ for h small enough).

Step 3. We prove by contradiction that the spectrum of the operator $\tilde{A}_{l,h}$ contains no point on the imaginary axis and that

$$\sup_{h\in(0,h^*)} \|(is - \tilde{A}_{l,h})^{-1}\| = O(|s|^2), \quad |s| \geq 1 \tag{16}$$

as well as

$$\sup_{h\in(0,h^*)} \|(is - \tilde{A}_{l,h})^{-1}\| = O(1), \quad |s| \leq 1. \tag{17}$$

For low frequency modes ($h\lambda_{-,k}^2 \leq \varepsilon$) we use Step 2, while the action of the added viscosity term plays its role for the high frequency modes ($h\lambda_{-,k}^2 > \varepsilon$).

Step 4. Adapting the results of [5], we prove that if (17) holds, then (16) is equivalent to

$$\sup_{h\in(0,h^*)} \|T_h(t)(\tilde{A}_{l,h})^{-1}\| = O(t^{-\frac{1}{2}}), \quad t \to +\infty \tag{18}$$

where $(T_h(t))_{t\geq 0}$ is the uniformly bounded C_0 semigroup generated by $\tilde{A}_{l,h}$ (in $V_h \times V_h$). □

Remark 1 The results presented in this note can be generalized to an abstract problem (5) set in a real Hilbert space H with $A : \mathcal{D}(A) \to H$ a positive self-adjoint operator with a compact inverse and $B \in \mathcal{L}(U, H)$ with U another real Hilbert space. The numerical scheme is defined similarly as (13) where V_h is a finite dimensional approximation of $\mathcal{D}(A^{\frac{1}{2}})$. Then if the eigenvalues of the operator $A^{\frac{1}{2}}$ satisfy the generalized gap condition, if B satisfies a condition like (14) and under some approximation properties *on V_h, we can prove that the discrete scheme is exponentially (resp. polynomially) stable (as for the continuous problem). This will be detailed in [1].*

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