OPTIMAL ENERGY DECAY RATE OF RAYLEIGH BEAM EQUATION WITH ONLY ONE BOUNDARY CONTROL FORCE

MAYA BASSAM, DENIS MERCIER, AND ALI WEHBE

Abstract. We consider a clamped Rayleigh beam equation subject to only one boundary control force. Using an explicit approximation, we first give the asymptotic expansion of eigenvalues and eigenfunctions of the underlying system. We next establish a polynomial energy decay rate for smooth initial data via an observability inequality of the corresponding undamped problem combined with a boundedness property of the transfer function of the associated undamped problem. Finally, we identify the obtained energy decay rate with the real part of the spectrum of the infinitesimal generator of the associated semigroup.

1. Introduction

We consider a clamped Rayleigh beam equation with only one control force:

\begin{align}
\alpha &> 0, \beta > 0, \gamma > 0 \\
\gamma y_{xxxx} + \gamma y_{xtt} - y_{tt} &= 0, \quad 0 < x < 1, \quad t > 0, \\
y(0, t) &= y_x(0, t) = 0, \quad t > 0, \\
y_x(1, t) + \alpha y_t(1, t) &= 0, \quad t > 0, \\
y_{xxxx}(1, t) - \gamma y_{xtt}(1, t) - \beta y_t(1, t) &= 0, \quad t > 0
\end{align}

where \( \gamma > 0 \) is the coefficient of moment of inertia, and where \( \beta > 0 \) is the coefficient of the control force \( \alpha > 0 \) is the coefficient of the control moment (in this paper we will consider the case \( \alpha = 0 \)). For more details concerning the modeling of the system, we refer to Russell [26].

If \( \gamma = 0 \) the Rayleigh beam equation simplifies to the Euler-Bernoulli beam equation. But, in the case of one control force \( \alpha = 0 \) and \( \beta > 0 \), the nature of the stabilization of the Rayleigh beam equation is different from that of Euler-Bernoulli beam equation. In fact, Chen et al. [19], [20] (see also [15]) proved the uniform stability of Euler-Bernoulli beam equation while Rao [19] proved the strong but nonuniform stability of Rayleigh beam equation if and only if the inertia coefficient \( \gamma \) is large enough. We refer the reader to the references [1], [5], [20], [22], [16] and [2] for the Euler-Bernoulli beam equation with different types of damping mechanisms.

There are number of publications concerning the stabilization of Rayleigh beam equation with different types of damping. For the internal stabilization, Rao [23] studied the stabilization of Rayleigh beam equation subject to a positive viscous damping and he established the optimal energy decay rate. Wehbe in [30], considered the Rayleigh beam equation with two boundary dynamical feedbacks and using a spectral method he obtained the optimal polynomial energy decay rate. In [14], Lagnese studied the stabilization of system (1.1)-(1.4) with two boundary control (the case \( \alpha > 0 \) and \( \beta > 0 \)), and proved that the energy decays exponentially. Rao [19] extended the results of [14] to the case of one boundary feedback (the case \( \alpha > 0, \beta = 0 \) or \( \alpha = 0, \beta > 0 \)). Using a compact perturbation theory due to Gibson [12], he established an exponential stability of system (1.1)-(1.4) in the case \( \alpha > 0, \beta = 0 \). Moreover, in the case \( \alpha = 0, \beta > 0 \), he first, proved the lack of exponential stability of system (1.1)-(1.4). Next he proved that the Rayleigh beam equation can be strongly stabilized by only one control force if and only if the inertia coefficient \( \gamma \) is large enough but no decay rate has been discussed.

Nevertheless, the energy decay rate and it’s optimality appears to be open problem. Then, in this paper, we consider the Rayleigh beam equation with only one boundary control force, \( \alpha = 0, \beta > 0 \). Using an explicit approximation, we first give the asymptotic expansion of eigenvalues and eigenfunctions of system (1.1)-(1.4). Using a methodology introduced in [1], we next establish a polynomial energy decay rate for smooth initial data via an observability inequality of the corresponding undamped problem combined with a boundedness property of the transfer function of the associated undamped problem. Finally, we identify the obtained energy decay rate with the real part of the spectrum of the infinitesimal generator of the associated semigroup.

We now outline briefly the content of this paper. In section 2, in a convenient Hilbert space, we formulate system (1.1)-(1.4) into an evolution equation. We recall the well-posedness of the problem by the semigroup approach (see [18], [19]). In section 3, we propose an explicit approximation of the characteristic determinant of the underlying system and we obtain an asymptotic expansion of eigenvalues and eigenfunctions of the corresponding operator. In section 4, Using a methodology introduced in [1], we establish a polynomial energy...
2. WELL-POSEDNESS OF THE PROBLEM

We first introduce the following spaces

\[ V = \{ y \in H^1(0,1) : y(0) = 0 \}, \quad \| y \|_V^2 = \int_0^1 (|y|^2 + \gamma |y_x|^2) \, dx, \]

\[ W = \{ y \in H^2(0,1) : y(0) = y_x(0) = 0 \}, \quad \| y \|_W^2 = \int_0^1 |y_{xx}|^2 \, dx. \]

We identify \( L^2(0,1) \) with its dual so that we have the following continuous embedding

\[ W \subset V \subset L^2(0,1) \subset V' \subset W'. \]

Let \( y \) a smooth solution of system (2.1). Then multiplying (2.1) by a function \( \varphi \in W \) and integrating by parts, we get

\[ \int_0^1 (y_{tt} \varphi + \gamma y_{txx} \varphi) \, dx + \int_0^1 y_{xx} \varphi_x \, dx + \beta y_t(1) \varphi(1) = 0. \]

Accordingly, we define the linear operators \( \tilde{A} \in L(W,W'), \tilde{B} \in L(V,V'), C \in L(V,V') \), by

\[ \langle \tilde{A} y, \varphi \rangle_{W' \times W} = (y, \varphi)_{W'}, \quad \forall y, \varphi \in W, \]

\[ \langle \tilde{B} y, \varphi \rangle_{V' \times V} = y(1) \varphi(1), \quad \forall y, \varphi \in V, \]

\[ \langle C y, \varphi \rangle_{V' \times V} = (y, \varphi)_{V'}, \quad \forall y, \varphi \in V. \]

By means of Lax-Milgram’s theorem (see [4]), we see that \( \tilde{A}, \tilde{B} \) is the canonical isomorphism from \( W \) onto \( W' \) and from \( V \) onto \( V' \) respectively. On the other hand, using the usual traces theorems, we check easily that \( \tilde{B} \) is continuous operator for the corresponding topology.

Assume that \( \tilde{A} y \in V' \), then we can formulate the variational equation (2.2) into the following form

\[ y_{tt} + C^{-1} \tilde{A} y + \beta C^{-1} \tilde{B} y_t = 0, \quad \text{in } V. \]

Now define the energy space as \( \mathcal{H} = W \times V \) which is endowed with the usual inner product

\[ \langle (y,z), (\tilde{y},\tilde{z}) \rangle_{\mathcal{H}} = (y, \tilde{y})_W + (z, \tilde{z})_V, \quad \forall (y,z), (\tilde{y},\tilde{z}) \in \mathcal{H}. \]

Next we introduce the linear unbounded operator \( A_0 \) by

\[ D(A_0) = \{ (y,z) \in \mathcal{H} : z \in W \text{ and } \tilde{A} y \in V' \}, \]

\[ A_0 u = (z, -C^{-1} \tilde{A} y), \quad \forall u = (y,z) \in D(A_0). \]

and the linear bounded operator \( B_\beta \) as follows

\[ B_\beta u = (0, -\beta C^{-1} \tilde{B} z), \quad \forall u = (y,z) \in \mathcal{H}. \]

Then, denoting by \( u = (y,y_t) \) the state of system (2.6), we can formulate (2.6) into an evolution equation

\[ \left\{ \begin{array}{l}
   u_t = (A_0 + B_\beta) u, \\
   u(0) = u_0 \in \mathcal{H}.
\end{array} \right. \]

It is easy to prove that \( A_0 \) is maximal dissipative and \( B_\beta \) is dissipative in the energy space \( \mathcal{H} \), therefore \( A = A_0 + B_\beta, \quad D(A) = D(A_0) \), generates a \( C_0 \)-semigroup \( e^{tA} \) of contractions on the energy space \( \mathcal{H} \) following Hille-Yosida’s theorem (see [4]). Then we have the following results concerning existence and uniqueness of solution of the problem (2.10).

**Theorem 2.1.** For any initial data \( u_0 \in \mathcal{H} \), the problem (2.10) has a unique weak solution \( u(t) \in C^0([0, \infty), \mathcal{H}) \). Moreover, if \( u_0 \in D(A) \), then the problem (2.10) has a strong solution \( u(t) \in C^1([0, \infty), \mathcal{H}) \cap C^0([0, \infty), D(A)). \)

In addition we have the following characterization of \( D(A) \) (see [19]).
Proposition 2.2. Let \( u = (y, z) \in \mathcal{H} \). Then \( u \in D(A) \) if and only if the following condition holds

\[
\begin{align*}
  (2.11) & \quad \begin{cases}
    y \in H^3(0, 1) \cap W, \\
    z \in W, \\
    y_{xx}(1) = 0.
  \end{cases}
\end{align*}
\]

In particular, the resolvent \((I - A)^{-1}\) of \((A)\) is compact on the energy space \( \mathcal{H} \) and the solution of the system \((2.1) - (2.4)\) satisfies

\[
\begin{align*}
  (2.12) & \quad y(t) \in C^0([0, \infty), V) \cap C^1([0, \infty), W) \cap C^0([0, \infty), H^3(0, 1) \cap W)).
\end{align*}
\]

Our goal is to establish a polynomial energy decay rate via an observability inequality for the conservative problem by a method introduced in [3, 1]. Then we give the following characterization of the linear bounded operator \( B_\beta \).

Proposition 2.3. Let \( y_0(x) = \gamma^{-1/2} \cosh^{-1}(\gamma^{-1/2}) \sinh(\gamma^{-1/2}x) \) and define the linear bounded operator \( B \) by

\[
(2.13) \quad B : C \rightarrow V, \quad \text{such that } B1 = y_0.
\]

Then we have

1. \( C\gamma_0 = \delta_1 \), where \( \delta_1 \) is the Dirac distribution at \( x = 1 \) and \( C \) defined in \((2.9)\).
2. For all \( y \in V \), \( B^* y = y(1) \) where \( B^* \) is the adjoint operator of \( B \) with respect to the pivot space \( V \).
3. For all \( y \in V \), \( C^{-1} \tilde{B} y = B^* y \).

Proof. (1) Let \( \varphi \in V \), using \((2.1) - (2.5)\), we get

\[
< C y_0, \varphi >_{C \times C} = (y_0, \varphi)_{V \times V} = \int_0^1 (y_0 \varphi + \gamma y_0 \varphi x) dx = \int_0^1 (y_0 - \gamma y_0 \varphi) dx + \varphi(1) = \varphi(1).
\]

This leads to the desired equality.

(2) Let \( v \in V \), then, using \((2.13)\), we have

\[
< 1, B^* v >_{C \times C} = (B1, v)_{V \times V} = (y_0, v)_{V \times V}.
\]

On the other hand, we have

\[
(y_0, v)_{V \times V} = \int_0^1 (y_0 v + \gamma y_0 v x) dx = \int_0^1 (y_0 - \gamma y_0 v x) dx + v(1) = v(1).
\]

This implies that \( B^* v = v(1) \) for all \( v \in V \).

(3) Let \( y \in V \). Then using \((2.9)\) and \((2.5)\), we get

\[
(C^{-1} \tilde{B} y, \varphi)_{V \times V} = < \tilde{B} y, \varphi >_{V \times V} = y(1) \varphi(1) = < y(1) \delta_1, \varphi >_{V \times V}, \quad \forall \varphi \in V.
\]

This implies that

\[
(2.14) \quad \tilde{B} y = y(1) \delta_1,
\]

On the other hand, using \((1), (2)\) and \((2.14)\), we get

\[
(2.15) \quad CBB^* y = C y(1) B1 = y(1) C y_0 = y(1) \delta_1 = \tilde{B} y.
\]

This leads to the desired equality. \( \square \)

Then, we will formulate problem \((2.7)\) into the following closed loop system

\[
\begin{align*}
  (2.16) & \quad \begin{cases}
    y_{tt} + Ay + \beta BB^* y_t = 0, \\
    y(0) = y_0, \quad y_t(0) = y_1.
  \end{cases}
\end{align*}
\]

We recall the following stability results (see [3, 1])

\[
(2.17) \quad \sqrt{\gamma_0} \sinh^{-1}(\sqrt{\gamma_0 \pi}) = 1.
\]

Then for any \( \gamma \geq \gamma_0 \) the semigroup of contractions \( S_1(t) \) is strongly asymptotically stable on the space \( \mathcal{H} \). For any \( u_0 \in \mathcal{H} \), we have

\[
(2.18) \quad \lim_{t \to +\infty} \| S_1(t) u_0 \|_{\mathcal{H}}^2 = 0.
\]

Remark 2.5. Using a numerical program we find an approximate value of \( \gamma_0 \) defined in \((2.11)\),

\[
\gamma_0 \approx 0.6708.
\]
3. Spectral Analysis

In this section, we will give the asymptotic form of eigenvalues and eigenfunctions of the operator \( A_0 \). Since \( A_0 \) is closed with compact resolvent, then the spectrum of \( A_0 \) consists entirely of isolated eigenvalues with finite multiplicities (see [11]). Moreover, it is easy to prove that \( A_0 \) is a skew-adjoint operator and \( \mu = 0 \) is not an eigenvalue of \( A_0 \). Also the coefficients of \( A_0 \) are real then the eigenvalues appears by conjugate pairs. Then we denote \( \sigma(A_0) = \{ \lambda_k = i\mu k, k \in \mathbb{Z}^+ \} \) with \( \mu_k = -\mu_k \) and \( U_k = (y, i\mu_k y_k) \) be an associated eigenvector.

Now, let \( \lambda = i\mu \) be an eigenvalue of and \( U = (y, z) \in D(A_0) \) be an associated eigenfunction. Then we have

\[
\begin{align*}
\text{(3.1)} & \quad \begin{cases} 
z = i\mu y, \\
-C^{-1}Ay = i\mu z.
\end{cases}
\end{align*}
\]

Then, using (3.2) and (2.3), we interpret (3.1) as the following variational equation:

\[
\begin{align*}
\text{(3.2)} & \quad -\int_0^1 y_{xxx} \varphi dx + \mu^2 \int_0^1 (y\varphi + \gamma y_x \varphi) dx = 0, \quad \forall \varphi \in W.
\end{align*}
\]

This gives that the function \( y \) is determined by the following system:

\[
\begin{align*}
\text{(3.3)} & \quad \begin{cases} 
y_{xxx} + \gamma \mu^2 y_{xx} - \mu^2 y = 0, \\
y(0) = y_x(0) = y_{xx}(1) = y_{xxx}(1) + \gamma \mu^2 y_x(1) = 0.
\end{cases}
\end{align*}
\]

We have found that \( \lambda \) is an eigenvalue of \( A_0 \) if and only if there is a non trivial solution of (3.3). The general solution of (3.3) is given by

\[
\begin{align*}
y(x) = \sum_{i=1}^4 c_i e^{t_i x},
\end{align*}
\]

where \( t_1(\mu) = \sqrt{-\gamma \mu^2 - \mu \gamma^2 \mu^2 + \frac{4}{2}}, t_2(\mu) = -t_1(\mu), t_3(\mu) = \sqrt{-\gamma \mu^2 + \mu \gamma^2 \mu^2 + \frac{4}{2}}, \) and \( t_4(\mu) = -t_3(\mu) \).

Here and below, for simplicity we denote \( t_i(\mu) \) by \( t_i \). Thus the boundary conditions in (3.3) may be written as the following system:

\[
\begin{align*}
\text{(3.5)} & \quad M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\
(t_1)^2 & t_2 & t_3 & t_4 \\
h_\lambda(1) & h_\lambda(2) & h_\lambda(3) & h_\lambda(4) \\
(\gamma \mu^2)^3 & (t_1)^3 & (t_2)^3 & (t_3)^3 & (t_4)^3 & (t_5)^3 & (t_6)^3 \\
\end{pmatrix} \begin{pmatrix} c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix} = 0,
\end{align*}
\]

where we have set \( h_\lambda(t) = (t^3 + \gamma \mu^2 t)e^t \). Hence a non trivial solution \( y \) exists if and only if the determinant of \( M(\lambda) \) vanishes. Set \( f(\lambda) = \det M(\lambda), \) thus the characteristic equation is \( f(\lambda) = 0 \).

\begin{proposition}
Let \( \{ \lambda_k = i\mu_k \}_{k \in \mathbb{Z}} \) the set of eigenvalues of \( A_0 \). Then there exists \( m \in \mathbb{Z} \) such that the following asymptotic behavior holds

\[
\mu_k = \frac{(k + m)\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + O(1), k \in \mathbb{N}^*, k \to +\infty.
\]

Moreover, there exists \( N \in \mathbb{N} \), such that for all \( |k| \geq N \), the eigenvalues \( \lambda_k \) are simple.
\end{proposition}

\begin{proof}
The proof is decomposed in two steps.

**Step 1.** We start by the expansion of \( t_1 \) and \( t_3 \):

\[
\begin{align*}
\text{(3.7)} & \quad t_1 = i\mu \sqrt{\gamma} + \frac{i}{2\gamma \sqrt{\gamma} \mu} - \frac{5i}{8\gamma^3 \sqrt{\gamma} \mu^4} + O(\frac{1}{\mu^5}), \\
\text{(3.8)} & \quad t_3 = \frac{1}{\sqrt{\gamma}} - \frac{1}{2\gamma^2 \sqrt{\gamma} \mu^2} + O(\frac{1}{\mu^4}).
\end{align*}
\]

Using (3.7)-(3.8), we find the asymptotic development of:

\[
\begin{align*}
\text{(3.9)} & \quad t_1^2 e^{t_1} = e^{i\mu \sqrt{\gamma}}(-\gamma \mu^2 - \frac{i\mu}{2\sqrt{\gamma}} + O(1)), \\
\text{(3.10)} & \quad t_2^2 e^{t_2} = e^{-i\mu \sqrt{\gamma}}(-\gamma \mu^2 + \frac{i\mu}{2\sqrt{\gamma}} + O(1)).
\end{align*}
\]
This gives
\begin{align}
  h_\lambda(t_1) &= e^{i\mu \sqrt{\gamma}(-i \frac{\mu}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} + O(\frac{1}{\mu^2}))}, \\
  h_\lambda(t_2) &= e^{-i\mu \sqrt{\gamma}(i \frac{\mu}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} + O(\frac{1}{\mu^2}))}, \\
  h_\lambda(t_3) &= e^{\sqrt{\gamma}(\sqrt{\gamma} \mu^2 + \frac{\sqrt{\gamma} - 1}{2\gamma^2} + O(\frac{1}{\mu^2}))}, \\
  h_\lambda(t_4) &= e^{-\frac{1}{\sqrt{\gamma}}(\sqrt{\gamma} \mu^2 + \frac{\sqrt{\gamma} - 1}{2\gamma^2} + O(\frac{1}{\mu^2})).}
\end{align}

Combining (3.7)-(3.16) and (3.5), we obtain
\begin{equation}
  M(\lambda) =
  \begin{pmatrix}
    -\mu e^{i\mu \sqrt{\gamma}} + O(1) & 1 & 1 & 1 \\
    1 & -i\mu e^{i\mu \sqrt{\gamma}} + O(1) & -\frac{1}{\sqrt{\gamma}} + O(1) & -\frac{1}{\sqrt{\gamma}} + O(1) \\
    -\frac{1}{\sqrt{\gamma}} e^\mu e^{i\mu \sqrt{\gamma}} + O(1) & 1 & 1 & 1 \\
    \frac{1}{\gamma} e^{i\mu \sqrt{\gamma}} + O(1) & -\frac{1}{\gamma} e^{-i\mu \sqrt{\gamma}} + O(1) & 1 & 1 \\
    \frac{1}{\gamma} e^{i\mu \sqrt{\gamma}} + O(1) & -\frac{1}{\gamma} e^{-i\mu \sqrt{\gamma}} + O(1) & 1 & 1 \\
    \frac{1}{\gamma} e^{i\mu \sqrt{\gamma}} + O(1) & -\frac{1}{\gamma} e^{-i\mu \sqrt{\gamma}} + O(1) & 1 & 1 \\
  \end{pmatrix}
\end{equation}

where $z_\mu = -\gamma \mu^2 - \frac{id}{\sqrt{\gamma}}$. Then, after some computations, we find the following asymptotic development of $f(\mu)$ the determinant of $M(i\mu)$:
\begin{equation}
  f(\mu) = \mu^5 f_0(\mu) + \mu f_1(\mu) + O(\mu^3)
\end{equation}

where
\begin{equation}
  f_0(\mu) = 4i\gamma^2 \cos(\mu \sqrt{\gamma}) \cosh(\frac{1}{\sqrt{\gamma}}),
\end{equation}

\begin{equation}
  f_1(\mu) = -2i\sqrt{\gamma} \sin(\mu \sqrt{\gamma}) (\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma} \sinh(\frac{1}{\sqrt{\gamma}})).
\end{equation}

Then we set
\begin{equation}
  \tilde{f}(\mu) = \frac{f(\mu)}{\mu^5} = f_0(\mu) + \frac{f_1(\mu)}{\mu} + O(\frac{1}{\mu^2}).
\end{equation}

**Step 2.** We look at the roots of $f_0$ that we denote by $\mu^0_k$.

Solving $f_0(\mu_k) = 0$, we find
\begin{equation}
  \cos(\sqrt{\gamma} \mu_k) = 0.
\end{equation}

This gives
\begin{equation}
  \mu_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}.
\end{equation}

Now with the help of Rouché’s theorem, and for $\mu$ large enough, we show that the roots $\tilde{f}$ are close to those of $f_0$ and :
\begin{equation}
  \mu_k = \frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + o(1) \quad \text{where} \quad k' = k + m.
\end{equation}
Proposition 3.2. The solution \( y \) of the undamped initial value problem (3.3) satisfies the estimates
\[
y(1) = -2\gamma F_1 \frac{F_1}{F_0} \sinh \frac{1}{\sqrt{\gamma}} - 2 \cosh \sqrt{\gamma} + o(1), \quad \|y\|_W \sim O(|\mu_k|^2) \quad \text{and} \quad \|U_k\| \sim O(|\mu_k|^2). \tag{3.24}
\]

**Proof.** For clarity, we divide the proof into several steps.

**Step 1.** There exists a solution \( C(\mu_k) \) of \( \dot{M}(\mu_k)C(\mu_k) = 0 \) which has the form:
\[
C(\mu_k) = C_0 + O\left(\frac{1}{|\mu_k|}\right), \tag{3.25}
\]
where
\[
C_0 = \left(1, 1, -\left(\gamma \frac{F_1}{F_0} + 1\right), (\gamma \frac{F_1}{F_0} - 1)\right).
\]
Let \( c_1 = 1 \), you see in the proof the validation of this choice, using \( \|\| \; \text{asymptotic behavior} \; \text{at} \; \gamma = 1 \), you see in the proof the validation of this choice, using \( \|\| \). we get
\[
\begin{align*}
\alpha_2 &= 2t_1^2 \gamma t_3^3 e^{t_3} - t_3^3 \left(3e^{t_3} + e^{-t_3}\right) - t_1^2 t_3^2 \left(e^{t_3} + e^{-t_3}\right), \\
\alpha_3 &= t_1^3 \left(e^{t_1} + e^{-t_1}\right) - t_1^2 t_3^2 \left(e^{t_1} - e^{-t_1}\right) - 2t_1 t_3^2 e^{t_1}, \\
\alpha_4 &= -t_1^3 \left(e^{t_1} - e^{-t_1}\right) - t_1^2 t_3^2 \left(e^{t_1} + e^{-t_1}\right) + 2t_1 t_3^2 e^{-t_1}, \\
D &= -2t_1^3 e^{t_1} + t_1 t_3^2 \left(e^{t_3} + e^{-t_3}\right) - t_3^2 \left(e^{t_3} - e^{-t_3}\right).
\end{align*}
\]
Substitute (3.3), (3.31) and (3.32) into (3.27), we get
\[
\alpha_2 = -2\sqrt{\gamma} \mu_k^2 e^{\mu_k \sqrt{\gamma}} + O\left(\frac{1}{|\mu_k|}\right). \tag{3.31}
\]
Then, using (3.3) and (3.31), we obtain
\[
\alpha_2 = \mp i 2\sqrt{\gamma} \mu_k^2 + O(|\mu_k|). \tag{3.32}
\]
Similarly, using (3.1), (3.3), (3.31) and (3.32), we get
\[
D = -2i \sqrt{\gamma} \mu_k^2 + O(|\mu_k|). \tag{3.33}
\]
Then using (3.32) and (3.33), we get
\[
c_2 = \frac{\alpha_2}{D} = 1 + O\left(\frac{1}{|\mu_k|}\right). \tag{3.34}
\]
To find \( c_3 \) and \( c_4 \), substitute (3.3) in (3.27) to get
\[
\cos \sqrt{\gamma} \mu_k = -\frac{F_1}{F_0} \sin \sqrt{\gamma} \mu_k \frac{1}{\mu_k} + O\left(\frac{1}{|\mu_k|}\right),
\]
where \( F_0 \) and \( F_1 \) defined in (3.22) and (3.23). Then using (3.27), we obtain
\[
\cos \sqrt{\gamma} \mu_k = -\epsilon_k \frac{F_1}{F_0} \frac{1}{\mu_k} + O\left(\frac{1}{|\mu_k|}\right), \quad \text{where} \; \epsilon_k = \pm 1. \tag{3.35}
\]
Using (3.7)-(3.12) in the determinant \( \alpha_3 \), we obtain

\[
\alpha_3 = t_1^3(e^{t_1} + e^{-t_1}) - t_1^2 t_2(e^{t_1} - e^{-t_1}) - 2 t_1 t_2^2 e^{-t_2}
\]

(3.36)

Combining (3.35) and the (3.36) to obtain the following asymptotic behavior

\[
\alpha_3 = 2i \gamma \sqrt{\gamma} \mu_k (\cos \gamma \mu_k + O(\frac{1}{|\mu_k|})) + 2i \gamma \mu_k \sqrt{\gamma}(\sin \gamma \mu_k + O(\frac{1}{|\mu_k|})).
\]

Finally using (3.37) and (3.33), we get

\[
\alpha_3 = 2i \sqrt{\gamma} \mu_k \epsilon_k (\gamma \frac{F_1}{F_0} + 1) + o(1).
\]

Similarly, we find

\[
c_4 = \gamma \frac{F_1}{F_2} - 1 + o(1).
\]

**Step 2. Estimates of \( y(1) \).** Substitute \( C_0 = (1, 1, -(\frac{F_1}{F_0} + 1), (\frac{F_1}{F_0} - 1)) \) in (3.4) to find:

\[
y(1) = c_1 e^{t_1} + c_2 e^{t_2} + c_3 e^{t_3} + c_4 e^{t_4}
\]

(3.40)

\[
y(1) = 2 \cos \mu \sqrt{\gamma} - 2 \gamma \frac{F_1}{F_0} \sinh \frac{1}{\sqrt{\gamma}} - 2 \cosh \sqrt{\gamma} + o(1).
\]

**Step 3. Estimates of \( ||y||_W \).** Note that

\[
||y||_W^2 = \int_0^1 |y_{xx}|^2 dx.
\]

We start by

\[
\int_0^1 |y_{xx}|^2 dx = \sum_{i,j=1}^4 c_i (\int_0^1 t_1^2 e^{t_1} e_{i1}^2 e_{j1}^2) = C'(\mu_k) G C'(\mu_k)
\]

where

\[
G = (g_{ij}), i,j = 1, 4 \quad \text{with} \quad g_{ij} = \int_0^1 e^{(t_1 + t_2)} dx \quad \text{and} \quad C'(\mu_k) = (t_1^2 e_{i1}).
\]

A direct computation gives

\[
\int_0^1 e^{(t_1 + t_2)} dx = \int_0^1 e^{(t_2 + t_1)} dx = \int_0^1 e^{(t_3 + t_4)} dx = 1.
\]

In addition, for \( t_i + t_j \neq 0 \), we have

\[
\int_0^1 e^{(t_i + t_j)} dx = \frac{e^{t_i + t_j}}{t_i + t_j} = \frac{1}{t_i + t_j}.
\]

Therefore, using (3.41) and (3.42), we find

\[
G = G_0 + O(\frac{1}{\mu_k})
\]

where

\[
G_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{e \sqrt{\gamma} - 1}{2} \sqrt{\gamma} & 1 \\
0 & 0 & 1 & \frac{2}{\sqrt{\gamma} - 1}
\end{pmatrix},
\]

(3.43)
and $O\left(\frac{1}{\mu_k}\right)$ is a matrix with all the entries are $O\left(\frac{1}{\mu_k}\right)$.

Using (5.7)-(5.8), we obtain

$$C'(\mu_k) = C_0'(\mu_k) + O(1).$$

Where

$$C_0'(\mu_k) = (-\gamma\mu_k^2, -\gamma\mu_k^2, 0, 0).$$

Then we deduce that

$$\int_0^1 |y_{xz}|^2 = (C_0'(\mu_k) + O(1))(G_0 + O(1)) (C_0'(\mu_k) + O(1))^T$$

$$= C_0'(\mu_k)G_0C_0'(\mu_k)^T + O(\mu_k^4) \sim O(\mu_k^4).$$

(3.44)

Step 4. Estimates of $\|y\|_V$. Similarly, we prove that

$$\int_0^1 |y^2|dx \sim O(1), \quad \int_0^1 |y_z|^2dx \sim O(\mu_k^2)e^{\mu_kt}.\text{Proposition 3.2}\text{.}$$

Therefore using (5.43), we deduce that

$$\|y\|_V \sim O(\mu_k).$$

Finally using (5.44) and (3.45), we obtain:

$$\|U_k\|_H \sim O(\mu_k^2).$$

(3.47)

**Eigenvectors of $A_0$.** The set of eigenvectors of $A_0$ corresponding to $\mu_k$ is the set $\{U_k = (y_k, z_k) \in D(A_0)\}_k$ where $U_k$ has the following form:

$$U_k = \begin{pmatrix} y_k \\ \mu_k z_k \end{pmatrix}.\text{Proposition 3.2}$$

(3.48)

For the sequel, it is useful to introduce the set $\{\tilde{U}_k\}_{k \in \mathbb{Z}^*}$ of normalized eigenvectors of $A_0$ such that

$$\forall k \in \mathbb{Z}^*, \tilde{U}_k = \frac{1}{\|U_k\|_H} U_k.$$

Remark that if we set $\tilde{U}_k = (y_k, z_k)$, then from Proposition 3.2 and (3.48) we have

$$\|\tilde{y}_k(1)\| = O\left(\frac{1}{\mu_k}\right) = O\left(\frac{1}{|k|}\right), \quad \text{and} \quad |z_k(1)| = O\left(\frac{1}{\mu_k}\right) = O\left(\frac{1}{|k|}\right).$$

(3.49)

4. **Polynomial stability**

We know that the Rayleigh beam equation subject to one boundary control force is strongly but not exponentially stable (see [19]). In this section, our goal is to study the polynomial stability of the energy of the Rayleigh beam equation subject to one boundary control force. Our method uses a methodology introduced by Ammari and Tucsnak in [11], where the polynomial stability for the damped problem is reduced to an observability inequality of the corresponding undamped problem combined to a boundedness property of the transfer function of the associated undamped system. Our main results are the following polynomial-type decay estimation

**Theorem 4.1.** (Polynomial energy decay rate)

There exist a constant $c > 0$, such that, for all $t > 0$ and for all $(y^0, y^1) \in D(A)$ the solution of system (2.16) verifying the following estimate

$$E(y(t)) \leq \frac{c}{(1 + t)^2} \|y^0, y^1\|_{P(A)}^2.$$ \text{Proposition 3.2}

(4.1)

First, we will establish an observability inequality for the undamped problem corresponding to (2.16)

$$\begin{cases} y_t + Ay = 0, \\ y(0) = y_0, \quad y_t(0) = y_1. \end{cases}\text{Proposition 3.2}$$ \text{Proposition 3.2}

(4.2)
Lemma 4.2. (Observability estimate)
There exist $T > 0$ and $C_T > 0$ such that the solution of (4.2) satisfies
\begin{equation}
\int_0^T |B^* y_1(t)|^2 dt \geq C_T \| (y^0, y^1) \|^2_{D(A_0)^'}
\end{equation}
where $D(A_0)'$ is the dual of $D(A_0)$ obtained by means of the inner product in $H$.

Proof. Let $u = (y, y_t)$, the (4.2) is equivalent to following undamped problem
\begin{equation}
\left\{\begin{array}{l}
u_t = A_0 u, \\
u(0) = u_0.
\end{array}\right.
\end{equation}
Since $u_0 = (y^0, y^1) \in D(A_0)$ we can write
$$u(t) = \sum_{k \in \mathbb{Z}^*} u_0^k e^{i\mu_k t} \widetilde{U}_k$$
where $\widetilde{U}_k$ is the normalized eigenvector of the operator $A_0$. Therefore
$$y_t(t) = z(t) = \sum_{k \in \mathbb{Z}^*} e^{i\mu_k t} u_0^k \tilde{z}_k,$$
and
$$B^* y_t = z(1, t) = \sum_{k \in \mathbb{Z}^*} e^{i\mu_k t} u_0^k \tilde{z}_k(1).$$
Since $\exists \gamma > 0$ such that $\mu_{k+1} - \mu_k \geq \gamma$, we can use then Ingham inequality (see [10]) and we obtain : $\exists T > 0$ and $c(T) > 0$ such that
$$\int_0^T |y_t(1, t)|^2 dt \geq c(T) \sum_{k \in \mathbb{Z}^*} |u_0^k|^2 |\tilde{z}_k(1)|^2$$
on the other hand using (4.3), then
$$\int_0^T |y_t(1, t)|^2 dt \geq c(T) \sum_{k \in \mathbb{Z}^*} |u_0^k|^2 \frac{1}{k^2} = \| u_0 \|^2_{D(A_0)^'}.$$
And the proof of theorem is complete.

Next, we will check the boundedness of the following transfer function :
$$H : \mathbb{C}_+ = \{ \lambda \in \mathbb{C} | Re \lambda > 0 \} \rightarrow L(\mathbb{C})$$
\begin{equation}
\lambda \mapsto H(\lambda) = \lambda \beta B^* (\lambda^2 + A)^{-1} B
\end{equation}
Let $\alpha > 0$, we define the set $C_\alpha := \{ \lambda \in \mathbb{C} | Re \lambda = \alpha \}$.

Lemma 4.3. The transfer function $H$ defined in (4.5) is bounded on $C_1$.

Proof. Let $a \in \mathbb{C}$. Using the definition of $B$, we have
$$H(\lambda) = a (\lambda^2 + A)^{-1} B = a (\lambda^2 + A)^{-1} y_0.$$
On the other hand, we can write
\begin{equation}
y_0 = \sum_{k=1}^{+\infty} \gamma_k \tilde{y}_k \text{ with } \sum_{k=1}^{+\infty} |\gamma_k|^2 < +\infty.
\end{equation}
Indeed, since $y_0 \in V$ then $(0, y_0) \in D(A_0)$, hence $(0, y_0) = \sum_{k \in \mathbb{Z}^*} y_0^k \widetilde{U}_k$ with $\sum_{k \in \mathbb{Z}^*} |y_0^k \lambda_k|^2 < +\infty$. Therefore we obtain (4.6) since $U_k = (\tilde{y}_k, i \mu_k \tilde{y}_k), \forall k \in \mathbb{Z}^*$. Consequently we have
\begin{equation}
(\lambda^2 + A)^{-1} B = \sum_{k=1}^{+\infty} \frac{\gamma_k}{\mu_k^2 + \lambda^2} \tilde{y}_k(x).
\end{equation}
Using the definition of $B^*$, we get
\begin{equation}
\frac{H(\lambda)}{\lambda} = \beta B^* (\lambda^2 + A)^{-1} B = \beta \sum_{k=1}^{+\infty} \frac{\gamma_k}{\mu_k^2 + \lambda^2} \tilde{y}_k(1).
\end{equation}
Theorem 5.1. \[ |\frac{1}{\mu^2_k + \lambda^2}| \leq \frac{c_0}{|\lambda|}, \forall k \in \mathbb{N}^*, \lambda \in C_\alpha. \]

For now, assume that there exists a constant \( c_0 > 0 \) such that

(4.8) \[ \frac{1}{\mu^2_k + \lambda^2} \leq \frac{c_0}{|\lambda|}, \forall k \in \mathbb{N}^*, \lambda \in C_\alpha. \]

Substitute (4.8) in (4.7), we obtain

\[ |H(\lambda)| \leq \frac{c_0}{|\lambda|} \sum_{k=1}^{+\infty} |\gamma_k||\tilde{g}_k(1)|. \]

Using Cauchy-Shwartz inequality, we get

\[ |H(\lambda)| \leq \frac{c_0}{|\lambda|} \left( \sum_{k=1}^{+\infty} |\gamma_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{+\infty} |\tilde{g}_k(1)|^2 \right)^{\frac{1}{2}}. \]

Using (4.9), (4.10), we get

\[ |H(\lambda)| \leq +\infty. \]

To complete the proof of the Lemma, we still have (4.8) to prove it. Without loss of generality, we assume that \( \alpha = 1 \). Let \( \lambda = 1 + iy \in C_1 \), then we have

\[ \frac{1}{\mu^2_k + \lambda^2} = \frac{1}{\mu^2_k + (1 + iy)^2} = |g_1(\mu_k) - 2ig_2(\mu_k)|, \]

where

\[ g_1(\mu_k) = \frac{\mu_k^2 + 1 - y^2}{D}, \quad g_2(\mu_k) = \frac{y}{D} \quad \text{and} \quad D = (\mu_k^2 + 1 - y^2)^2 + 4y^2. \]

Now, it’s easy to prove that \( g_1(\mu_k) \) has a maximum value at \( \mu_k = \sqrt{y^2 + 2y - 1} \), then we have

(4.9) \[ |g_1(\mu_k)| \leq \frac{1}{4|y|}. \]

Similarly, we prove that \( g_2(\mu_k) \) has a maximum value at \( \mu_k = \sqrt{y^2 - 1} \), then we have

(4.10) \[ |g_2(\mu_k)| \leq \frac{1}{4|y|}. \]

Then using (4.9), (4.10), we obtain

\[ |\frac{1}{\mu^2_k + \lambda^2}| = |g_1(\mu_k) - 2ig_2(\mu_k)| \leq |g_1(\mu_k)| + 2|g_2(\mu_k)| \leq \frac{3}{4|y|}, \]

Which satisfies (4.8) and the proof is completed. \( \square \)

Proof of theorem 4.1. Let \( Y_1 \times Y_2 = D(A_0); X_1 \times X_2 = D(A_0)' \) then we apply theorem 2.4 in [1].

5. Optimal polynomial decay rate

The aim of this section is to prove the following result

Optimality \[ \text{The energy decay rate given in (4.7) is optimal in the sense that for any } \varepsilon > 0, \text{ we cannot expect the decay rate } \frac{1}{t^{\varepsilon}}, \text{ for all initial data } U_0 \in D(A) \text{ and for all } t > 0. \]

For the optimality, we search the asymptotic behavior of the eigenvalues of \( A \). Let \( \lambda \) be an eigenvalue of \( A \), then \( \exists \ u = (y, z) \in H \) such that \( Au = \lambda u \) or equivalently

(5.1) \[ \begin{cases} y_{xxx} - \gamma \lambda^2 y_{xx} + \lambda^2 y = 0, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) - \gamma \lambda^2 y_x(1) - \beta \lambda y(1) = 0. \end{cases} \]

Similarly, as we do to find the characteristic equation of \( A_0 \), we find that the general solution of (5.1) is

\[ y(x) = \sum_{i=1}^{4} c_i \tilde{e}_i(x), \]

where

\[ \tilde{e}_1(\lambda) = \sqrt{\frac{\gamma \lambda^2 - \lambda \sqrt{\gamma^2 \lambda^2 - 4}}{2}}, \tilde{e}_2(\lambda) = -\tilde{e}_1(\lambda), \tilde{e}_3(\lambda) = \sqrt{\frac{\gamma \lambda^2 + \lambda \sqrt{\gamma^2 \lambda^2 - 4}}{2}}, \tilde{e}_4(\lambda) = -\tilde{e}_3(\lambda). \]
Here and below for simplicity, we denote \( \tilde{t}_i(\lambda) \) by \( \tilde{t}_i \).

Thus the boundary conditions may be written as the following:

\[
N(\lambda)\tilde{C}(\lambda) = 0,
\]

where

\[
N(\lambda) = \begin{pmatrix}
\frac{1}{k_1} & \frac{1}{\tilde{t}_2} & \frac{1}{\tilde{t}_3} & \frac{1}{\tilde{t}_4} \\
\tilde{t}_1 e^{\tilde{t}_1} & \tilde{t}_2 e^{\tilde{t}_2} & \tilde{t}_3 e^{\tilde{t}_3} & \tilde{t}_4 e^{\tilde{t}_4} \\
k_{\lambda}(\tilde{t}_1) & k_{\lambda}(\tilde{t}_2) & k_{\lambda}(\tilde{t}_3) & k_{\lambda}(\tilde{t}_4)
\end{pmatrix},
\]

\( \tilde{C}(\lambda) = \begin{pmatrix}
\tilde{c}_1 \\
\tilde{c}_2 \\
\tilde{c}_3 \\
\tilde{c}_4
\end{pmatrix}, \)

where we have set \( k_{\lambda}(t) = (t^3 - \lambda^2 \gamma t - \beta \lambda) e^{t} \).

**Proposition 5.2.** Let \( \lambda_k \) be an eigenvalues of the operator \( A \). Then there exists \( m \in \mathbb{N} \) such that the following asymptotic behavior holds

\[
\lambda_k = \text{i} \left( \frac{(k + m)\pi}{\sqrt{\gamma}} \right) + \frac{\pi \gamma}{k} + \frac{8(1)^k}{k^2} - \frac{B}{k^2} + O\left( \frac{1}{k^3} \right), \quad k \in \mathbb{N}^*, k \to +\infty
\]

where

\[
A = \frac{2 + 4 \sqrt{\gamma} \tanh(\gamma^{-1/2})}{\pi \gamma^{3/2}} \quad \text{and} \quad B = \frac{\beta(4 \sqrt{\gamma} + 2 \tanh(\gamma^{-1/2}))}{\pi \gamma^{3/2}}.
\]

**Proof.** The proof is decomposed in two steps.

**Step 1.** We start by the expansion of \( \tilde{t}_1 \) and \( \tilde{t}_3 \)

\[
\tilde{t}_1 = \lambda \sqrt{\gamma} - \frac{1}{2 \lambda \gamma \sqrt{\gamma}} + O\left( \frac{1}{\lambda^2} \right),
\]

\[
\tilde{t}_3 = \frac{1}{\sqrt{\gamma}} + O(\lambda).
\]

Using (5.5)–(5.6), we find the asymptotic development of:

\[
\tilde{t}_1 e^{\tilde{t}_1} = e^{\lambda \sqrt{\gamma}} \left( \lambda^2 \sqrt{\gamma} - \frac{\lambda}{2 \sqrt{\gamma}} + \frac{2\gamma - 16 \gamma^2}{16 \gamma^3} \right) + O\left( \frac{1}{\lambda} \right),
\]

\[
\tilde{t}_2 e^{\tilde{t}_2} = e^{-\lambda \sqrt{\gamma}} \left( \lambda^2 \sqrt{\gamma} + \frac{\lambda}{2 \sqrt{\gamma}} + \frac{2\gamma - 16 \gamma^2}{16 \gamma^3} \right) + O\left( \frac{1}{\lambda} \right),
\]

\[
\tilde{t}_3 e^{\tilde{t}_3} = e^{\frac{\gamma}{\sqrt{\gamma}}} + O\left( \frac{1}{\lambda} \right),
\]

\[
\tilde{t}_4 e^{\tilde{t}_4} = e^{\frac{-\gamma}{\sqrt{\gamma}}} + O\left( \frac{1}{\lambda} \right).
\]

This gives

\[
k_{\lambda}(\tilde{t}_1) = e^{\lambda \sqrt{\gamma}} \left( -\frac{4 \sqrt{2} \gamma - 4 \sqrt{2} \beta \gamma^2}{4 \sqrt{2} \gamma^3} + \frac{4 \sqrt{2} \gamma + 4 \sqrt{2} \beta \gamma^2}{8 \sqrt{2} \gamma^3} \right) + O\left( \frac{1}{\lambda} \right),
\]

\[
k_{\lambda}(\tilde{t}_2) = e^{-\lambda \sqrt{\gamma}} \left( \frac{4 \sqrt{2} \gamma - 4 \sqrt{2} \beta \gamma^2}{4 \sqrt{2} \gamma^3} + \frac{4 \sqrt{2} \gamma + 4 \sqrt{2} \beta \gamma^2}{8 \sqrt{2} \gamma^3} \right) + O\left( \frac{1}{\lambda} \right),
\]

\[
\frac{1}{k_{\lambda}(\tilde{t}_3)} = e^{\sqrt{\gamma}} \left( -\lambda^2 \sqrt{\gamma} - \beta \lambda + \frac{(-1 + \sqrt{\gamma})}{2 \gamma^2} \right) + O\left( \frac{1}{\lambda} \right),
\]

\[
k_{\lambda}(\tilde{t}_4) = e^{\frac{-1}{\sqrt{\gamma}}} \left( \lambda^2 \sqrt{\gamma} - \beta \lambda + \frac{-(1 + \sqrt{\gamma})}{2 \gamma^2} \right) + O\left( \frac{1}{\lambda} \right).
\]

Combining (5.5)–(5.14) and (5.2), we obtain an equivalent system for (5.2),

\[
\tilde{N}(\lambda)\tilde{C}(\lambda) = 0,
\]
where $\tilde{N}(\lambda)$ is the matrix obtained when the entries of $N(\lambda)$ are replaced by their asymptotic behavior. After some computations, we find the following asymptotic development of $g(\lambda)$ the determinant of $\tilde{N}(\lambda)$,

$$g(\lambda) = g_0(\lambda) + \frac{g_1(\lambda)}{\lambda} + \frac{g_2(\lambda)}{\lambda^2} + \mathcal{O}(1).$$

Where

$$g_0(\lambda) = 4\gamma^2 \cosh(\lambda\sqrt{\gamma}) \cosh\left(\frac{1}{\sqrt{\gamma}}\right),$$

and

$$g_1(\lambda) = 4\beta\sqrt{\gamma} \cosh(\lambda\sqrt{\gamma}) \sinh\left(\frac{1}{\sqrt{\gamma}}\right) - 2\sqrt{\gamma} \cosh\left(\frac{1}{\sqrt{\gamma}}\right) \sinh(\lambda\sqrt{\gamma}) - 4\gamma \sinh\left(\frac{1}{\sqrt{\gamma}}\right) \sinh(\lambda\sqrt{\gamma}),$$

and

$$g_2(\lambda) = -8 - 8 \cosh\left(\frac{1}{\sqrt{\gamma}}\right) \cosh(\lambda\sqrt{\gamma}) + \frac{\cosh\left(\frac{1}{\sqrt{\gamma}}\right) \cosh(\lambda\sqrt{\gamma})}{2\gamma} + \frac{4 \cosh(\lambda\sqrt{\gamma}) \sinh\left(\frac{1}{\sqrt{\gamma}}\right)}{\sqrt{\gamma}} - 4\beta\sqrt{\gamma} \cosh\left(\frac{1}{\sqrt{\gamma}}\right) \sinh(\lambda\sqrt{\gamma}) - 2\beta \sinh\left(\frac{1}{\sqrt{\gamma}}\right) \sinh(\lambda\sqrt{\gamma}).$$

**Step 2.** We look at the roots of $g_0$ that we denote by $z_k^0$.

Solving $g_0(z_k) = 0$, we find

$$\cosh(\sqrt{\gamma}z_k) = 0,$$

it’s equivalent to

$$z_k^0 = i\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}.$$

Now with the help of Rouche’s theorem, and for $z$ large enough, we show that the roots of $g$ are close to those of $g_0$ and:

$$z_k = i\left(\frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}\right) + o(1), \quad \text{where} \quad k' = k + m$$

**Step 3.** From step 2, we can write

$$z_k = i\left(\frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}\right) + \epsilon_k,$$

where $\epsilon_k = o(1)$, substitute (15.19) in (15.15) in get

$$g_0(z_k) + \frac{g_1(z_k)}{z_k} + \frac{g_2(z_k)}{z_k^2} = 0,$$

where

$$g_0(z_k) = 4i(-1)^k\frac{\gamma}{2} \cosh\left(\frac{1}{\sqrt{\gamma}}\right) \epsilon_k + \mathcal{O}(\epsilon_k^2),$$

$$g_1(z_k) = (-1)^k i(-2\sqrt{\gamma} \cosh\left(\frac{1}{\sqrt{\gamma}}\right) - 4\gamma \sinh\left(\frac{1}{\sqrt{\gamma}}\right) + 4\beta\gamma^2 \sinh\left(\frac{1}{\sqrt{\gamma}}\right) \epsilon_k + \mathcal{O}(\epsilon_k^2)),$$

$$g_2(z_k) = -8 - (-1)^k i(4\beta\sqrt{\gamma} \cosh\left(\frac{1}{\sqrt{\gamma}}\right) + 2\beta \sinh\left(\frac{1}{\sqrt{\gamma}}\right)) + O(\epsilon_k).$$

Therefore,

$$\frac{g_1(z_k)}{z_k} = (-1)^k \frac{\tilde{A}}{\alpha_k} + \mathcal{O}\left(\frac{\epsilon_k}{k}\right),$$

where

$$\tilde{A} = -2\sqrt{\gamma} \cosh\left(\frac{1}{\sqrt{\gamma}}\right) - 4\gamma \sinh\left(\frac{1}{\sqrt{\gamma}}\right),$$

$$\alpha_k = \frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}.$$
Let $\varepsilon > 0$ and set $l = \frac{\varepsilon}{1 + \varepsilon}$. For $k \in \mathbb{Z}$, let $\lambda_k$ be an eigenvalue of the operator $A$ and $U^k$ the associated normalized eigenfunction. Then consider the following sequences

$$\beta_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + \frac{A}{k} + \frac{8(-1)^k}{\pi^2\gamma^{5/2}\cosh(\gamma^{-1/2})k^2},$$

$$U^k \subset D(A).$$

Using (5.4) we get

$$\lim_{k \to \infty} \beta_k^2 - 2\|i\beta_k - A U_k\| = 0.$$

By applying of Borichev theorem (see [24], [13], [21]), we deduce that the trajectory $S(t)u_0$ decays slower than $\frac{1}{1-t}$ on the time $t \to \infty$.

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Univ. Libanaise, Faculté des Sciences, Hadath, Beyrouth, Liban
E-mail address: bassammay@hotmail.com

Univ. de Valenciennes et du Hainaut Cambrésis, LAMAV, FR-CNRS 2956, Institut des sciences et Techniques, 59313 Valenciennes Cedex 9, France
E-mail address: denis.mercier@univ-valenciennes.fr

Univ. Libanaise, Faculté des Sciences, Hadath, Beyrouth, Liban
E-mail address: ali_wehbe@yahoo.fr