Radiative Heating of a Glass Plate

Luc Paquet[*], 59313 Valenciennes France,
Raouf El Cheikh, 59313 Valenciennes France,
Dominique Lochegnies, 59313 Valenciennes France,
Norbert Siedow, 67663 Kaiserslautern Deutchland.

Abstract

This paper aims to prove existence and uniqueness of a solution to the coupling of a nonlinear heat equation with nonlinear boundary conditions with the exact radiative transfer equation, assuming the absorption coefficient $\kappa(\lambda)$ to be piecewise constant and null for small values of the wavelength λ as in the paper of N. Siedow, T. Grosan, D. Lochegnies, E. Romero, "Application of a New Method for Radiative Heat Tranfer to Flat Glass Tempering", *J. Am. Ceram. Soc.,* **88** [8] 2181-2187 (2005). An important observation is that for a fixed value of the wavelength λ, Planck's function is a Lipschitz function with respect to the temperature. Using this fact, we deduce that the solution is at most unique. To prove existence of a solution, we define a fixed point problem related to our initial boundary value problem to which we apply Schauder's theorem in a closed convex subset of the Banach separable space $L^2(0, t_f; C([0, l]))$. We use also Stampacchia's truncation method to derive lower and upper bounds on the solution.

Keywords: elementary pencil of rays, Planck's function, radiative transfer equation, glass plate, nonlinear heat-conduction equation, Stampacchia's truncation method, Schauder's theorem, Vitali's theorem.

Mathematics Subject Classification (2010): 35K20,35K55,35K58,35K90, 35Q20, 35Q60, 35Q80.

1 Introduction and statement of the problem

We consider an infinite plane horizontal glass plate of width l, laid down on its lower face $x_g = 0$, on a black sheet-metal maintained at absolute ambient temperature $T = T_a$. The x-axis is directed upward orthogonally to the glass plate

---

[*] Author to whom all correspondence should be addressed. e-mail: luc.paquet@univ-valenciennes.fr, paquet.luc@wanadoo.fr (complete addresses of the authors are written at the end of the manuscript).

so that the upper (resp. lower) face of the glass-plate has $x_g = l$ (resp. $x_g = 0$) for equation. An infinite plane black sheet metal S, at absolute temperature $T_S(t)$ at time t, placed above the glass plate emits radiation in every direction, whose radiative intensity at wavelength λ in the (dry) air gap between is given by Planck's function:

$$B(T, \lambda) = \frac{2C_1}{\lambda^5 (e^{\frac{C_2}{\lambda T}} - 1)} \text{ with } T = T_S(t). \tag{1}$$

$C_1 = hc_0^2 = 0.595531\ 10^{-16}$ W.m^2/sr and $C_2 = \frac{hc_0}{k_B} = 1.438786\ 10^{-2}$ m.°K ([5], p.98). For a thermal ray, we denote by θ its polar angle with the $x-$axis and by $\mu := \cos(\theta)$. After refraction at the interface $x_g = l$ between air and glass, some part of the radiative energy emitted by the source S, will be absorbed i.e. converted into heat in the glass producing in such a way an increase of the temperature $T(x, t)$ in the glass plate. We assume independency with respect to the coordinates y and z of all the dependent variables. This radiative heating of the glass plate is modelized by the following equations and boundary conditions in the glass plate in the unknowns $I^k(x, t, \mu)$ $(k = 1, \ldots, M)$ and the temperature $T(x, t)$ $(0 \le x \le l,\ 0 \le t \le t_f)$, t_f denoting the final time of the heating process. Let us explain before what are the unknowns $I^k(x, t, \mu)$ $(k = 1, \ldots, M)$. Firstly, given an electromagnetic wave, we denote by λ the wavelength of the wave in vacuum or (dry) air, the corresponding wavelength in glass being $\frac{\lambda}{n_g}$ where n_g means the refractory index of the glass ($n_g \approx 1.46$) (this wavelength is denoted λ_0 in [19] p.8). Like in [24], [12], [3], we assume that the absorption coefficient $\kappa(\lambda)$ is piecewise constant:

$$\kappa(\lambda) = \kappa_k \in \mathbb{R}_+^* \text{ for } \lambda \in [\lambda_k, \lambda_{k+1}[,\ k = 1, \ldots, M, \tag{2}$$

where the M intervals $[\lambda_k, \lambda_{k+1}[,\ k = 1, \ldots, M$ form a partition of the glass semi-tranparent region in the electromagnetic wave spectrum. By $I(x, t, \mu, \lambda)$, we denote the spectral radiative intensity (to be correct, we should say "specific intensity") at any point P of elevation x, time t, wavelength λ in a direction at point P making a polar angle $\theta = Arc\cos(\mu) \in [0, \pi]$ with the parallel to the x-axis at point P. We assume azimuthal symmetry of the spectral radiative intensity with respect to the direction. The spectral radiative intensity is defined in photometry in the following way: considering a small area dA with normal $\vec{n}$ at the point $P \in dA$, the radiative energy dE with wavelength in the interval $[\lambda, \lambda + d\lambda]$ which flows through dA, during the time interval $[t, t+dt]$, in directions confined to a narrow cone of solid angle $d\Omega$ whose mean axis $\vec{s}$ makes an angle θ with $\vec{n}$, is given by the formula ([10], p. 7) ([22], p. 13):

$$dE = I(P, t, \vec{s}, \lambda)\ \cos(\theta) dA\ dt\ d\Omega\ d\lambda, \tag{3}$$

$I(P, t, \vec{s}, \lambda)$ being a coefficient of proportionality called the spectral radiative intensity at point P, time t, in direction $\vec{s}$ belonging to the unit sphere with center at point P, and at wavelength λ. Due to our azimuthal symmetry hypothesis of the spectral radiative intensity with respect to the direction $\vec{s}$, dE

in the left-hand side of formula (3), does not depend on the azimuthal angle of the mean axis of the narrow cone of directions of solid angle $d\Omega$. By $I^k(x,t,\mu)$ $(k=1,\ldots,M)$, we mean:

$$I^k(x,t,\mu) := \int_{\lambda_k}^{\lambda_{k+1}} I(x,t,\mu,\lambda)\ d\lambda.$$

Similarly for a given absolute temperature T, we define:

$$B^k(T) := \int_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)\ d\lambda,\ k=1,\ldots,M,$$

where $B(T,\lambda)$ denotes the Planck function defined by formula (1). Let us note that $B(T,\lambda)$ does not depend on the direction nor directly on x; this radiative intensity of pencil of rays emitted in vacuum, or in dry air, by a blackbody at absolute temperature T depends only on the absolute temperature T and on the wavelength λ. We also denote by $B_g(T,\lambda) := n_g^2 B(T,\lambda)$ and by $B_g^k(T) := n_g^2 B^k(T)$ $(k=1,\ldots,M)$. Integrating both sides of the radiative transfer equation in the glass plate, assuming no scattering in the glass plate ([10], p.343, p.9):

$$\begin{gathered}\mu\tfrac{dI(x,t,\mu,\lambda)}{dx} + \kappa(\lambda)I(x,t,\mu,\lambda) = \kappa(\lambda)B_g(T(x,t),\lambda),\\ (0\le x\le l,\ 0\le t\le t_f,\ -1<\mu<1,\ \lambda>0)\end{gathered} \tag{4}$$

and using (2), we obtain the following system of M equations in the glass plate only coupled by the temperature $T(x,t)$:

$$\begin{gathered}\mu\tfrac{dI^k(x,t,\mu)}{dx} + \kappa_k I^k(x,t,\mu) = \kappa_k B_g^k(T(x,t)),\\ (0\le x\le l,\ 0\le t\le t_f,\ -1<\mu<1),\ k=1,\ldots,M,\end{gathered} \tag{5}$$

one equation for each wavelength band $[\lambda_k,\lambda_{k+1}[,\ k=1,\ldots,M$. Now let us derive the boundary conditions for the equations (5). Considering an elementary pencil of rays ([22], p.18, second paragraph) emitted by the black source S of radiative intensity $B(T_S(t),\lambda)$ in (dry) air, a balance of energy shows that after refraction in the "direction" μ $(-1<\mu<0)$ at the interface $x_g=l$ between air and glass, that its radiative intensity will become

$$(1-\rho_g(\mu))n_g^2 B(T_S(t),\lambda) = (1-\rho_g(\mu))B_g(T_S(t),\lambda), \tag{6}$$

where $\rho_g(\mu)$ denotes the reflectivity coefficient given by Fresnel's relation ([19], formula (2.96) p. 47). The factor n_g^2 appearing in the left-hand side of equation (6) is due to the conservation of the optical outspread of an elementary pencil of rays after refraction (see [10] p.8 or [5] formula (29) p.35) (a particular case of the so called Clausius' relation by A. Maréchal: [16], pp.56-59), that is of the product of the square of the refractive index, times the cross-sectional area normal to the elementary pencil of rays, times its solid angle of directions $d\Omega$. Making once again a balance of energy, in an elementary pencil of rays diverging from a small horizontal area dA ([22], p.18, second paragraph) contained in the interface $x_g = l$, in directions lying in a narrow cone with solid angle $d\Omega$, its mean axis making a polar angle $\theta = Arc\cos(\mu)$ $(-1 < \mu < 0)$ with the x-axis, we obtain the following boundary condition on the surface $x_g = l$ of the glass plate (somewhat similar to [22], p.33-34):

$$I(l,t,\mu,\lambda) = \rho_g(\mu) I(l,t,-\mu,\lambda) + (1-\rho_g(\mu)) B_g(T_S(t),\lambda), \qquad \text{for } -1 < \mu < 0,\ 0 \le t \le t_f\ ,\ \lambda > 0.$$

After integration of both sides of this equation with respect to wavelength λ from λ_k to λ_{k+1}, we obtain the following boundary condition on the surface $x_g = l$ of the glass plate for equation (5):

$$I^k(l,t,\mu) = \rho_g(\mu) I^k(l,t,-\mu) + (1-\rho_g(\mu)) B_g^k(T_S(t)),\ -1 < \mu < 0,\ 0 \le t \le t_f\ , \tag{7}$$

for $k = 1, \ldots, M$. On the lower face $x_g = 0$ of the glass plate in contact with the black-sheet metal maintained at ambient absolute temperature $T = T_a$, we have the simple boundary condition for the "integrated" radiative intensity $I^k(x,t,\mu)$:

$$I^k(0,t,\mu) = B_g^k(T_a),\ 0 < \mu < 1,\ 0 \le t \le t_f\ . \tag{8}$$

The heat source in the heat-conduction equation is given by minus times the divergence of the radiative flux ([27], p.221-222), ([26], p.354-355):

$$q(x,t) = 2\pi \int_0^{+\infty} \int_{-1}^{+1} \mu\ I(x,t,\mu,\lambda)\ d\mu\ d\lambda$$

Using the radiative transfer equation (4) and our hypothesis (2) on the absorption coefficient $\kappa(\lambda)$, we have:

$$\begin{aligned} -\tfrac{\partial q}{\partial x}(x,t) &= -2\pi \int_0^{+\infty} \int_{-1}^{+1} \mu \tfrac{\partial I}{\partial x}(x,t,\mu,\lambda)\ d\mu\ d\lambda \\ &= -4\pi \int_0^{+\infty} \kappa(\lambda) B_g(T(x,t),\lambda)\ d\lambda + 2\pi \int_0^{+\infty} \int_{-1}^{+1} \kappa(\lambda)\ I(x,t,\mu,\lambda)\ d\mu\ d\lambda \\ &= -\sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{k=M} 2\pi\kappa_k \int_{-1}^{+1} I^k(x,t,\mu)\ d\mu. \end{aligned} \tag{9}$$

Thus the quantities of interest are $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu$ $(k = 1, \ldots, M)$ for which we shall give an explicit formula by solving explicitely for each $k \in \{1, \ldots, M\}$

equation (5) with the boundary condition (7) for $-1 < \mu < 0$, respectively (8) for $0 < \mu < 1$. Now by (9), the heat-conduction equation inside the glass plate is the following:

$$c_p m_g \tfrac{\partial T}{\partial t}(x,t) = k_h \tfrac{\partial^2 T}{\partial x^2}(x,t) - \sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T(x,t)) \\ + \sum_{k=1}^{k=M} 2\pi\kappa_k \int_{-1}^{+1} I^k(x,t,\mu)\ d\mu,\ 0 < x < l,\ 0 < t < t_f\ , \tag{10}$$

where c_p, m_g, k_h are assumed to be positive constants named respectively heat capacity, mass density and thermal conductivity of the glass [27], [26]. Let us observe that the terms $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu\ (k = 1,\dots,M)$ which appear in the right-hand side of the heat-conduction equation (10) depend by equations (5) on the distribution of temperature $T(x,t)$ inside the glass plate. Now, what are the boundary conditions for the heat-conduction equation (10). On the lower face $x_g = 0$ of the glass plate which is in contact with a black sheet-metal maintained at absolute ambient temperature $T = T_a$, we have simply the inhomogeneous Dirichlet boundary condition:

$$T(0,t) = T_a,\ \forall t \in]0, t_f[. \tag{11}$$

On the upper face $x_g = l$ of the glass plate, due to radiative emission and absorption very near the boundary for wavelength λ belonging to the glass opaque region $[\lambda_0, +\infty[$ ($\lambda_0 \approx 5\mu m$ for glass, [25], p.70) in the electromagnetic wave spectrum, we have the following nonlinear boundary condition expressing the continuity of the density of heat flux:

$$-k_h \tfrac{\partial T}{\partial x}(l,t) = h_c(T(l,t) - T_a) + \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda [B(T(l,t),\lambda) - B(T_S(t),\lambda)]\ d\lambda, \\ \forall' t \in]0, t_f[. \tag{12}$$

In the boundary condition (12), ϵ_λ is a positive constant called the spectral hemispherical emittance ([19], pp.62-63); like in ([25], p. 70), we have supposed that the spectral hemispherical absorptance is equal to the spectral hemispherical emittance for wavelength λ belonging to the glass opaque region in the electromagnetic wave spectrum, and independent of the temperature. Boundary condition (12) is the same as boundary condition (3) in [3] or (3) in [12]. In boundary condition (12), h_c denotes the convective heat transfer coefficient, the term $h_c(T(l,t) - T_a)$ representing the conducto-convective flux density at the infinite surface $x_g = l$ of the glass plate according to Newton's law ([27], p.16) ([26], p.13-16). Finally, to close our system of equations, we need an initial condition for the temperature:

$$T(x,0) = T_0(x),\ \forall x \in [0,l]. \tag{13}$$

We assume that the initial condition $T_0(\cdot)$ is a continuous strictly positive function on the closed interval $[0,l]$, as an absolute temperature is always positive in classical physics and $T_0(\cdot)$ is a datum. For the same reasons, we also assume that $T_a > 0$ and that $T_S(t) > 0$, $\forall t \in [0,t_f]$. We suppose that the compatibility condition $T_0(0) = T_a$ between the initial condition and the inhomogeneous Dirichlet boundary condition (11) on the lower face $x_g = 0$ of the glass plate is verified.

We want to prove that the equations (5) with the boundary conditions (7) for $-1 < \mu < 0$, respectively (8) for $0 < \mu < 1$ coupled with equation (10) completed by the boundary conditions (11), (12) and the initial condition (13) possess a unique bounded weak solution

$$T \in \{T \in L^2(0,t_f;H^1(]0,l[);\dot{T} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)\}$$

which is also continuous on $[0,l] \times [0,t_f]$. The proof of the existence will be achieved by defining a fixed point problem to which we will apply Schauder's theorem. Using Stampacchia's trunction method [2], we will also prove that the solution T is positive, lower bounded by T_a if $T_S(t) \geq T_a$, $\forall t \in [0,t_f]$ and $T_0(x) \geq T_a$, $\forall x \in [0,l]$ and upper bounded by

$$\max(\|T_0\|_{\infty,[0,l]}, \|T_S\|_{\infty,[0,t_f]}, T_a).$$

Let us conclude by mentioning, that though an absolute temperature is always positive in classical physics, that for mathematical purposes only, we extend the definition of the Planck's function (1) to negative real numbers T by setting $B(T,\lambda) = 0$, if $T \leq 0$; in that way for fixed $\lambda > 0$, the function $T \mapsto B(T,\lambda)$ is defined on the whole real line and is lipschitzian of Lipschitz constant $\frac{2C_1}{C_2\lambda^4}$ (6).

To close this introductory section, let us situate our paper among existing works in the litterature. The existence and uniqueness of the solution of the SP_1-approximation to the radiative heat tranfer equation assuming the grey property of the material, coupled with the heat conduction equation assuming Robin-type boundary conditions has been established by R. Pinnau in [21]. In the present work, we have considered the exact radiative transfer equation, we do not assume the grey hypothesis and we consider the exact nonlinear boundary condition (12) on the upper face $\{x = l\}$ of the glass plate for the heat conduction equation. In [13], the authors assume the grey hypothesis and consider the nonlinearity arising from the well-known Stefan-Boltzmann law, making the resulting heat equation non-monotone but pseudo-monotone. In M. Laitinen's thesis [14], only grey materials i.e. materials whose radiative properties are independent of wavelength are considered. The paper of P.-E. Druet [6] is concerned by proving the existence of a solution to a time-dependent heat equation modelizing the heating of several opaque bodies contained in an inclosure and separated from each other by a transparent medium. He has mathematically implemented the well-known inverse square law in Heat radiation's theory for two boundary points of the opaque bodies in each other's range of vision (see e.g. [19], pp.133-136). In [18], the problem of optimizing the temperature gradient in the gas phase by directly controlling the heat source in the solid phase is considered in a crucible. The problem is described by the stationary heat equation with a nonlocal radiation interface between the solid and the gas phase and a local radiative boundary condition. In particular, these authors show the boundedness of weak solutions of the state equation.

One of the main characteristic of the model of radiative heating of a glass plate studied mathematically here, defined by the equations (5), (10) completed with the boundary conditions (7), (8), (11), (12) and the initial condition (13), is that the short wavelengths of the radiation emitted by the black source S are neglected, which seems reasonable. One possible justification of that attitude of mind is that the emissive power of the black source S for radiations of wavelength $\lambda \in [0, \lambda_{\text{inf}}]$:

$$\pi \int_0^{\lambda_{\text{inf}}} B(T_S, \lambda)\ d\lambda$$

([19], pp. 6-11) may be made as small as desired with respect to the total emissive power of the black source S (equal to σT_S^4, where σ denotes the Stefan-Boltzmann constant equal to $5.670\ 10^{-8}$ W/m^2.°K^4) by choosing λ_{inf} sufficiently small.

2 Computation of the integrals $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu$ $(k = 1, \ldots, M)$

Firstly, we are going to compute the explicit solution of equation (5) with the boundary condition (7) for $-1 < \mu < 0$, respectively (8) for $0 < \mu < 1$, from which we will derive explicit expressions for the integrals $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu$ $(k = 1, \ldots, M)$. From differential equation (5), follows that for $-1 < \mu < 1$:

$$I^k(x,t,\mu) = I^k(0,t,\mu)\ e^{-\kappa_k \frac{x}{\mu}} + \frac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \int_0^x e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \tag{14}$$

or:

$$I^k(x,t,\mu) = I^k(l,t,\mu)\ e^{\kappa_k \frac{l-x}{\mu}} - \frac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \int_x^l e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx'. \tag{15}$$

Suppose $0 < \mu < 1$. By boundary condition (8), we have:

$$I^k(0,t,\mu) = B_g^k(T_a), \text{ for } 0 < \mu < 1. \tag{16}$$

Thus by equation (14) follows that

$$I^k(x,t,\mu) = B_g^k(T_a)\ e^{-\kappa_k \frac{x}{\mu}} + \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \int\limits_0^x e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx', \\ \text{for } 0 < \mu < 1. \tag{17}$$

Suppose now $-1 < \mu < 0$. By boundary condition (7) and equation (14), we obtain:

$$I^k(l,t,\mu) = \rho_g(\mu)\ I^k(0,t,-\mu) e^{\kappa_k \frac{l}{\mu}} - \rho_g(\mu) \tfrac{\kappa_k}{\mu} e^{\kappa_k \frac{l}{\mu}} \int\limits_0^l e^{-\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \\ +(1-\rho_g(\mu)) B_g^k(T_S(t)), \text{ for } -1 < \mu < 0. \tag{18}$$

For $-1 < \mu < 0$: $0 < -\mu < 1$; thus using (16), (18) becomes:

$$I^k(l,t,\mu) = \rho_g(\mu)\ B_g^k(T_a) e^{\kappa_k \frac{l}{\mu}} - \rho_g(\mu) \tfrac{\kappa_k}{\mu} \int\limits_0^l e^{-\kappa_k \frac{x'-l}{\mu}} B_g^k(T(x',t))\ dx' \\ +(1-\rho_g(\mu)) B_g^k(T_S(t)), \text{ for } -1 < \mu < 0. \tag{19}$$

Now using formula (15), we obtain for $-1 < \mu < 0$:

$$I^k(x,t,\mu) = \rho_g(\mu)\ B_g^k(T_a) e^{\kappa_k \frac{2l-x}{\mu}} + (1-\rho_g(\mu)) B_g^k(T_S(t)) e^{\kappa_k \frac{l-x}{\mu}} \\ -\rho_g(\mu) \tfrac{\kappa_k}{\mu} \int\limits_0^l e^{-\kappa_k \frac{x'+x-2l}{\mu}} B_g^k(T(x',t))\ dx' - \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \int\limits_x^l e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx'. \tag{20}$$

Thus the explicit solution of equation (5) with the boundary condition (7) for $-1 < \mu < 0$, respectively (8) for $0 < \mu < 1$, is given by formula (17) for $0 < \mu < 1$ and by formula (20) for $-1 < \mu < 0$. These two formulas allows us now to compute $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu$:

$$\begin{aligned}
&\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu = \int_{0}^{1} I^k(x,t,\mu)\ d\mu + \int_{-1}^{0} I^k(x,t,\mu)\ d\mu \\
&= B_g^k(T_a) \int_0^1 e^{-\kappa_k \frac{x}{\mu}}\ d\mu + \int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_0^x e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right]\ d\mu \\
&+ B_g^k(T_a) \int_{-1}^0 \rho_g(\mu)\ e^{\kappa_k \frac{2l-x}{\mu}}\ d\mu + B_g^k(T_S(t)) \int_{-1}^0 (1-\rho_g(\mu)) e^{\kappa_k \frac{l-x}{\mu}}\ d\mu \\
&- \int_{-1}^0 \rho_g(\mu) \tfrac{\kappa_k}{\mu} e^{\kappa_k \frac{2l-x}{\mu}} \left[\int_0^l e^{-\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right]\ d\mu \\
&- \int_{-1}^0 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_x^l e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right]\ d\mu\ .
\end{aligned} \tag{21}$$

Let us now examine the different terms in the right-hand side of formula (21). For the first term it is rather immediate:

$$B_g^k(T_a) \int_0^1 e^{-\kappa_k \frac{x}{\mu}}\ d\mu = B_g^k(T_a) \int_1^{+\infty} e^{-\kappa_k x \zeta}\ \frac{d\zeta}{\zeta^2} = B_g^k(T_a)\ E_2(\kappa_k x), \tag{22}$$

where $E_2(\cdot)$ denotes the integro-exponential function $E_2(\cdot)$ ([27], p.244-245) ([19], p.779-781). Let us now inspect the second term in the right-hand side of formula (21):

$$\begin{aligned}
&\int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_0^x e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\ dx' \right]\ d\mu \\
&= \int_0^x \left[\int_0^1 \tfrac{\kappa_k}{\mu} e^{-\kappa_k \frac{x-x'}{\mu}}\ d\mu \right]\ B_g^k(T(x',t))\ dx' \\
&= \int_0^x \left[\kappa_k \int_1^{+\infty} e^{-\kappa_k (x-x')\zeta}\ \tfrac{d\zeta}{\zeta} \right]\ B_g^k(T(x',t))\ dx' \\
&= \int_0^x \kappa_k\ E_1(\kappa_k(x-x'))\ B_g^k(T(x',t))\ dx' = \int_0^x G(x,x')\ B_g^k(T(x',t))\ dx',
\end{aligned} \tag{23}$$

where $E_1(\cdot)$ denotes the integro-exponential function $E_1(\cdot)$ ([27], p.244-245) ([19], p.779-781), and where we have set:

$$G(x,x') := \kappa_k E_1(\kappa_k\, |x-x'|),\ \forall\, (x,x') \in [0,l]^2. \tag{24}$$

Similarly for the sixth term in the right-hand side of formula (21):

$$\begin{aligned}
&-\int_{-1}^{0} \frac{\kappa_k}{\mu} e^{-\kappa_k \frac{x}{\mu}} \left[\int_{x}^{l} e^{\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\, dx' \right] d\mu \\
&= \int_{0}^{1} \frac{\kappa_k}{\mu} e^{\kappa_k \frac{x}{\mu}} \left[\int_{x}^{l} e^{-\kappa_k \frac{x'}{\mu}} B_g^k(T(x',t))\, dx' \right] d\mu \\
&= \int_{x}^{l} \left[\int_{0}^{1} \frac{\kappa_k}{\mu} e^{-\kappa_k \frac{x'-x}{\mu}}\, d\mu \right] B_g^k(T(x',t))\, dx' \\
&= \int_{x}^{l} \left[\int_{1}^{+\infty} \kappa_k \; e^{-\kappa_k (x'-x)\zeta} \; \frac{d\zeta}{\zeta} \right] B_g^k(T(x',t))\, dx' \\
&= \int_{x}^{l} \kappa_k \; E_1(\kappa_k (x'-x)) \; B_g^k(T(x',t))\, dx' \\
&= \int_{x}^{l} \kappa_k \; E_1(\kappa_k \,|x-x'|) \; B_g^k(T(x',t))\, dx' = \int_{x}^{l} G(x,x') \; B_g^k(T(x',t))\, dx'.
\end{aligned} \tag{25}$$

These two terms can be gathered in $\int_{0}^{l} G(x,x') \; B_g^k(T(x',t))\, dx'$. To write the third and fourth terms in the right-hand side of formula (21) each containing $\rho_g(\mu)$ under the integral sign in a compact fashion, let us introduce the function

$$\Phi_2 : \mathbb{R}_+^* \to \mathbb{R}_+ : y \mapsto \Phi_2(y) := \int_{0}^{1} \rho_g(\mu) e^{-\frac{y}{\mu}} d\mu \; . \tag{26}$$

Let us note that $\rho_g(\mu) = 1$ for $\mu \in [0, \sqrt{\frac{n_g^2-1}{n_g^2}}]$, as for such grazing incidence "angles", elementary pencils of rays are competely reflected. Thus $\Phi_2(\cdot)$ is somewhat similar to the integro-exponential function $E_2(\cdot)$, but it takes into account the reflectivity coefficient. Using the function $\Phi_2(\cdot)$ the third term $B_g^k(T_a) \int_{-1}^{0} \rho_g(\mu) \; e^{\kappa_k \frac{2l-x}{\mu}} \; d\mu$ in (21) may now be rewritten:

$$\begin{aligned}
B_g^k(T_a) \int_{-1}^{0} \rho_g(\mu) \; e^{\kappa_k \frac{2l-x}{\mu}} \; d\mu &= B_g^k(T_a) \int_{0}^{1} \rho_g(\mu) \; e^{-\kappa_k \frac{2l-x}{\mu}} \; d\mu \\
&= B_g^k(T_a) \; \Phi_2(\kappa_k (2l-x)).
\end{aligned} \tag{27}$$

The fourth term $B_g^k(T_S(t))\int\limits_{-1}^{0}(1-\rho_g(\mu))e^{\kappa_k\frac{l-x}{\mu}}\ d\mu$ in the right-hand side of formula (21) may be rewritten:

$$\begin{aligned}
B_g^k(T_S(t))\int\limits_{-1}^{0}(1-\rho_g(\mu))e^{\kappa_k\frac{l-x}{\mu}}\ d\mu &= B_g^k(T_S(t))\int\limits_{0}^{1}(1-\rho_g(\mu))e^{-\kappa_k\frac{l-x}{\mu}}\ d\mu \\
&= B_g^k(T_S(t))E_2(\kappa_k(l-x)) - B_g^k(T_S(t))\int\limits_{0}^{1}\rho_g(\mu)\ e^{-\kappa_k\frac{l-x}{\mu}}\ d\mu \\
&= B_g^k(T_S(t))[E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))].
\end{aligned} \tag{28}$$

To write the fifth term in the right-hand side of formula (21) containing also $\rho_g(\mu)$ under the integral sign in a compact fashion, let us introduce the integro-exponential function $\Phi_1(\cdot)$ somewhat similar to $E_1(\cdot)$

$$\Phi_1 : \mathbb{R}_+^* \to \mathbb{R}_+ : y \mapsto \Phi_1(y) := \int\limits_{0}^{1}\rho_g(\mu)e^{-\frac{y}{\mu}}\frac{d\mu}{\mu}\ ,\ \forall y > 0. \tag{29}$$

Using the function $\Phi_1(\cdot)$ the fifth term

$$-\int\limits_{-1}^{0}\rho_g(\mu)\frac{\kappa_k}{\mu}e^{\kappa_k\frac{2l-x}{\mu}}\left[\int\limits_{0}^{l}e^{-\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'\right]\ d\mu$$

in (21) may now be rewritten:

$$\begin{aligned}
&-\int\limits_{-1}^{0}\rho_g(\mu)\tfrac{\kappa_k}{\mu}e^{\kappa_k\frac{2l-x}{\mu}}\left[\int\limits_{0}^{l}e^{-\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'\right]\ d\mu \\
&= \int\limits_{0}^{1}\rho_g(\mu)\tfrac{\kappa_k}{\mu}e^{-\kappa_k\frac{2l-x}{\mu}}\left[\int\limits_{0}^{l}e^{\kappa_k\frac{x'}{\mu}}B_g^k(T(x',t))\ dx'\right]\ d\mu \\
&= \kappa_k\int\limits_{0}^{l}\left[\int\limits_{0}^{1}\rho_g(\mu)\ e^{-\kappa_k\frac{2l-x-x'}{\mu}}\ \frac{d\mu}{\mu}\right]B_g^k(T(x',t))\ dx' \\
&= \kappa_k\int\limits_{0}^{l}\Phi_1(\kappa_k(2l-x-x'))B_g^k(T(x',t))\ dx'.
\end{aligned} \tag{30}$$

Thus formula (21) may be rewritten:

$$\begin{aligned}\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu = \int_{0}^{l} G(x,x')\ B_g^k(T(x',t))\ dx' + B_g^k(T_a)\ E_2(\kappa_k x) \\ +B_g^k(T_a)\ \Phi_2(\kappa_k(2l-x)) + B_g^k(T_S(t))[E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))] \\ +\kappa_k \int_{0}^{l} \Phi_1(\kappa_k(2l-x-x^{'}))B_g^k(T(x',t))\ dx'.\end{aligned} \tag{31}$$

In the following, we will write also $\int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu$ instead of $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu$ to specify the dependence of $\int_{-1}^{+1} I^k(x,t,\mu)\ d\mu$ with respect to the temperature $T(\cdot,\cdot)$ as shown by formula (31).

3 Weak formulation of the nonlinear initial boundary value problem (10)-(13).

Let us set:

$$h_T(x,t) := \sum_{k=1}^{k=M} 2\pi\kappa_k \int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \text{ and } \psi(T(x,t)) := -\sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T(x,t)) \tag{32}$$

for $(x,t) \in]0,l[\times]0,t_f[$. Firstly, we want to prove that if $T \in L_+^2(]0,l[\times]0,t_f[)$ and $T_S \in L_+^2(]0,t_f[)$, then h_T and $\psi \circ T$ belong to $L^2(]0,l[\times]0,t_f[)$. We will need several lemmas.

Lemma 1 *$B_g^k(T) \leq cT$ for every $T \in \mathbb{R}_+^*$, where c denotes some positive constant depending on k.*

Proof. $B(T,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)} \leq \frac{2C_1}{\lambda^5(\frac{C_2}{\lambda T})} \leq 2\frac{C_1}{C_2}\ \frac{T}{\lambda^4}$. Thus $B^k(T) = \int_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)\ d\lambda \leq 2\frac{C_1}{C_2}\ T \int_{\lambda_k}^{\lambda_{k+1}} \frac{d\lambda}{\lambda^4} \leq cT$. A fortiori $B_g^k(T) \leq c n_g^2 T$. ■

Remark 2 *To alleviate the notations, in the following, we shall occasionally use the symbol $\lesssim$ to mean that the left-hand side is bounded by a constant times the right-hand side.*

Corollary 3 *If* $T \in L^2(]0,l[\times]0,t_f[)$ *and* $T_S \in L^2_+(]0,t_f[)$*, then* h_T *and* $\psi \circ T$ *belong to* $L^2(]0,l[\times]0,t_f[)$*.*

Proof. Firstly, a little thought shows that we can reduce us to the case $T \in L^2_+(]0,l[\times]0,t_f[)$. Due to the preceding lemma, the function:

$$B_g^k \circ T :]0,l[\times]0,t_f[\to \mathbb{R}_+ : (x,t) \mapsto B_g^k(T(x,t))$$

belongs to $L^2(]0,l[\times]0,t_f[)$. Thus $\psi \circ T$ which is a linear combination of the functions $B_g^k \circ T$ belong to $L^2(]0,l[\times]0,t_f[)$. To prove that h_T belongs to $L^2(]0,l[\times]0,t_f[)$, it suffices to prove that the functions

$$]0,l[\times]0,t_f[\to \mathbb{R}_+ : (x,t) \mapsto \int_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu$$

belong to $L^2(]0,l[\times]0,t_f[)$, $\forall k = 1,\ldots,M$. Firstly, in view of formula (31), we have to prove that the function

$$\begin{aligned}]0,l[\times]0,t_f[\to \mathbb{R}_+ : (x,t) \mapsto & \int_0^l G(x,x')\ B_g^k(T(x',t))\ dx' \\ &= \kappa_k \int_0^l E_1(\kappa_k\,|x-x'|)\ B_g^k(T(x',t))\ dx' \end{aligned} \tag{33}$$

belongs to $L^2(]0,l[\times]0,t_f[)$, $\forall k = 1,\ldots,M$. By lemma 1

$$\int_0^l E_1(\kappa_k\,|x-x'|)\ B_g^k(T(x',t))\ dx' \le c \int_0^l E_1(\kappa_k\,|x-x'|)\ T(x',t)\ dx'$$

which implies using Cauchy-Schwarz inequality

$$\left(\int_0^l E_1(\kappa_k\,|x-x'|)\ B_g^k(T(x',t))\ dx'\right)^2 \le c^2 \int_0^l E_1(\kappa_k\,|x-x'|)^2\ dx' \cdot \int_0^l T(x',t)^2\ dx'$$

Integrating both sides with respect to x from 0 to l and with respect to t from 0 to t_f, we obtain:

$$\begin{aligned} &\int_0^{t_f}\int_0^l \left(\int_0^l E_1(\kappa_k\,|x-x'|)\ B_g^k(T(x',t))\ dx'\right)^2 dx\ \otimes dt \\ &\qquad \le c^2 \int_0^{t_f}\int_0^l \left[\int_0^l E_1(\kappa_k\,|x-x'|)^2\ dx' \cdot \int_0^l T(x',t)^2\ dx'\right] dx \otimes\ dt \\ &\qquad \le c^2 \int_0^l\int_0^l E_1(\kappa_k\,|x-x'|)^2\ dx' \otimes dx \cdot \int_0^{t_f}\int_0^l T(x',t)^2\ dx' \otimes\ dt. \end{aligned}$$

Using the bound $E_1(y)^2 \leq 2 + 2\left(\ln y\ \right)^2$, it is easily seen that

$$\int_0^l \int_0^l E_1(\kappa_k\, |x - x'|)^2\ dx' \otimes dx \text{ is finite.}$$

Thus

$$\int_0^{t_f} \int_0^l \left(\int_0^l E_1(\kappa_k\, |x - x'|)\ B_g^k(T(x', t))\ dx' \right)^2 dx \otimes dt \lesssim \int_0^{t_f} \int_0^l T(x', t)^2\ dx^{'} \otimes dt < +\infty.$$

We have thus proved (33). The proof that the function

$$(x, t) \mapsto \int_0^l \Phi_1(\kappa_k(2l - x - x^{'}))\ B_g^k(T(x', t))\ dx'$$

belongs also to $L^2(]0, l[\times]0, t_f[)$ is similar to the preceding one, the only point we have to check being that

$$\int_0^l \int_0^l \Phi_1(\kappa_k(2l - x - x^{'}))^2\ dx' \otimes dx = \int_0^l \int_0^l \Phi_1(\kappa_k(x + x^{'}))^2\ dx' \otimes dx$$

is finite. But, by the definition (29) of the function Φ_1, $\Phi_1(y) \leq E_1(y)$, $\forall y \in \mathbb{R}_+^*$ and thus $\Phi_1(\kappa_k(x+x^{'}))^2 \leq E_1(\kappa_k(x+x^{'}))^2 \leq 2+4\left(\ln \kappa_k\right)^2+4\left(\ln\left(x+x^{'}\right)\ \right)^2$. Thus

$$\int_0^l \int_0^l \Phi_1(\kappa_k(2l - x - x^{'}))^2\ dx' \otimes dx < +\infty.$$

To conclude that h_T belongs to $L^2(]0, l[\times]0, t_f[)$, in view of the definition of $h_T(.,.)$ and (31), it remains to prove that the three functions of (x, t): $B_g^k(T_a)\ E_2(\kappa_k x)$, $B_g^k(T_a)\ \Phi_2(\kappa_k(2l - x))$, $B_g^k(T_S(t))[1 - \Phi_2(\kappa_k(l - x))]$ belong to $L^2(]0, l[\times]0, t_f[)$. As $E_2(y) \leq 1$, $\forall y \in \mathbb{R}_+^*$, it is clear for the first term. Also from the definition (26) of Φ_2, it follows that $\Phi_2(y) \leq 1$, $\forall y \in \mathbb{R}_+^*$, so that it is also clear that the second term belong to $L^2(]0, l[\times]0, t_f[)$. For the third term, we have also to check that $B_g^k(T_S(\cdot)) \in L^2(]0, t_f[)$. This results immediately from lemma 1 and the fact that $T_S \in L_+^2(]0, t_f[)$ by hypothesis. ■

Corollary 4 *Let us consider the mapping*

$$\Theta : \mathbb{R} \to \mathbb{R} : T \mapsto \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda B(T, \lambda)\ d\lambda.$$

Θ is an increasing function and for every $\hat{T} \in L^2(]0,t_f[)$, the function $\Theta \circ \hat{T}$ also denoted $\Theta(\hat{T})$ belongs to $L^2_+(]0,t_f[)$. In particular if $T_S \in L^2_+(]0,t_f[)$, then

$$\Theta(T_S) : \;]0,t_f[\to \mathbb{R}_+ : t \mapsto \pi \int_{\lambda_0}^{+\infty} \epsilon_\lambda B(T_S(t),\lambda)\; d\lambda \tag{34}$$

belongs to $L^2_+(]0,t_f[)$.

Proof. That Θ is an increasing function is obvious. As we have seen in the proof of lemma 1:

$$B(T,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)} \leq \frac{2C_1}{\lambda^5(\frac{C_2}{\lambda T})} \leq 2\frac{C_1}{C_2}\frac{T}{\lambda^4},\; \forall\lambda > 0,\; \forall T > 0.$$

Let us recall that we have set $B(T,\lambda) = 0$ if $T \leq 0$. Taking $T = \hat{T}(t)$, it follows that

$$\int_{\lambda_0}^{+\infty} \epsilon_\lambda B(\hat{T}(t),\lambda)\; d\lambda \leq 2\frac{C_1}{C_2}\hat{T}_+(t)\int_{\lambda_0}^{+\infty} \epsilon_\lambda \frac{d\lambda}{\lambda^4} \leq 2\frac{C_1}{C_2}\hat{T}_+(t)\int_{\lambda_0}^{+\infty} \frac{d\lambda}{\lambda^4} \leq \frac{2C_1}{3C_2\lambda_0^3}\hat{T}_+(t)$$

as $\forall\lambda > 0$: $\epsilon_\lambda \in [0,1]$. Thus:

$$\int_0^{t_f} \Theta(\hat{T}(t))^2 dt \leq \pi^2 \int_0^{t_f} \left[\int_{\lambda_0}^{+\infty} \epsilon_\lambda B(\hat{T}(t),\lambda)\; d\lambda\right]^2 dt \lesssim \int_0^{t_f} \hat{T}_+(t)^2 dt < +\infty.$$

Thus $\Theta \circ \hat{T}$ belongs to $L^2(]0,t_f[)$. ■

In terms of the functions $h_T(\cdot,\cdot)$, $\psi(T(\cdot,\cdot))$ and $\Theta(T_S)(.)$, the initial boundary value problem (10), (11), (12), (13) can be rewritten:

$$\left\{\begin{array}{c} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \frac{\partial^2 T}{\partial x^2}(x,t) + \psi(T(x,t)) + h_T(x,t),\; \forall'(x,t) \in]0,l[\times]0,t_f[, \\ T(0,t) = T_a,\; \forall' t \in]0,t_f[, \\ -k_h \frac{\partial T}{\partial x}(l,t) = h_c(T(l,t) - T_a) + \Theta(T(l,t)) - \Theta(T_S)(t),\; \forall' t \in]0,t_f[, \\ T(x,0) = T_0(x),\; \forall' x \in [0,l]. \end{array}\right. \tag{35}$$

(35) has sense if we suppose that

$$T \in L^2(0,t_f; H^2(]0,l[) \text{ and } \dot{T} \in L^2(0,t_f; L^2(]0,l[)).$$

Also, a priori, we do not know if the solution (which could even a priori not be unique) is positive (we will prove that this natural property is true; natural because an absolute temperature in classical physics is always positive). Thus to give sense to (35) and also to the equations which will follow, we have set $B(T,\lambda) = 0$ if $T \leq 0$ as we have said at the end of the introduction. We shall now define what is a weak solution to the initial boundary value problem (35).

Let us consider any function $\varphi \in H^1(]0,l[)$ such that $\varphi(0) = 0$. Multiplying both members of equation $(35)_{(i)}$ by $\varphi(x)$ and integrating by parts from 0 to l using the boundary conditions $(35)_{(iii)}$ and $(35)_{(ii)}$, we obtain for $\forall' t \in]0,t_f[$:

$$\begin{aligned} c_p m_g \int_0^l \frac{\partial T}{\partial t}(x,t)\varphi(x)dx = k_h \int_0^l \frac{\partial^2 T}{\partial x^2}(x,t)\varphi(x)dx + \int_0^l \psi(T(x,t))\varphi(x)dx \\ + \int_0^l h_T(x,t)\varphi(x)dx = -h_c(T(l,t) - T_a)\varphi(l) - \Theta(T(l,t)) \cdot \varphi(l) + \Theta(T_S(t)) \\ \cdot \varphi(l) - k_h \int_0^l \frac{\partial T}{\partial x}(x,t)\varphi'(x)dx + \int_0^l \psi(T(x,t))\varphi(x)dx + \int_0^l h_T(x,t)\varphi(x)dx. \end{aligned} \tag{36}$$

To give sense to (36) under the weak assumption that $T \in L^2(0,t_f;H^1(]0,l[))$ and that $\dot{T} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$, we have to replace in the left-hand side of (36) the integration on $]0,l[$ by a duality bracket: $\forall' t \in]0,t_f[$:

$$\begin{aligned} c_p m_g \left\langle \frac{dT}{dt}(\cdot,t),\varphi \right\rangle_{H^1(]0,l[)^*,H^1(]0,l[)} = -k_h \int_0^l \frac{\partial T}{\partial x}(x,t)\varphi'(x)dx + \int_0^l \psi(T(x,t))\varphi(x)dx \\ + \int_0^l h_T(x,t)\varphi(x)dx + [\Theta(T_S(t)) - \Theta(T(l,t))] \cdot \varphi(l) \\ + [h_c(T_a - T(l,t))]\varphi(l), \quad \forall \varphi \in H^1(]0,l[) \text{ such that } \varphi(0) = 0. \end{aligned} \tag{37}$$

As $T \in L^2(0,t_f;H^1(]0,l[))$ and $\dot{T} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$, it follows that $T \in C([0,t_f];L^2(]0,l[))$ ([9], p.40) which gives sense to the initial condition $T(\cdot,0) = T_0(\cdot)$. As $T \in L^2(0,t_f;H^1(]0,l[))$ and $H^1(]0,l[) \hookrightarrow C([0,l])$, it follows that $T(0,\cdot) \in L^2(]0,t_f[)$ which gives sense to the boundary condition $T(0,\cdot) = T_a$. We can now define what is a weak solution of the initial boundary value problem (35):

Definition 5 *We shall say that*

$$T \in L^2(0,t_f;H^1(]0,l[)) \text{ such that } \dot{T} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$$

is a weak solution of the initial boundary value problem (35) iff (37) is satisfied $\forall \varphi \in H^1(]0,l[)$ *such that* $\varphi(0) = 0$, $T(\cdot,0) = T_0(\cdot)$ *and* $T(0,t) = T_a$, $\forall' t \in]0,t_f[$.

We begin by proving uniqueness of the solution of the initial boundary value problem (37). We will need the following lemmas:

Lemma 6 *Let us fix* $\lambda > 0$. *Then the function* $\mathbb{R} \to \mathbb{R} : \hat{T} \longmapsto B(\hat{T},\lambda)$ *is lipschitzian of Lipschitz constant* $\frac{2C_1}{C_2\lambda^4}$.

Proof. Let us recall that Planck's function (1) is defined by $B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)}$.
First case: $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 > 0$ and $\hat{T}_2 \leq 0$ (or $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 \leq 0$ and $\hat{T}_2 > 0$). Then:

$$\left|B(\hat{T}_1,\lambda) - B(\hat{T}_2,\lambda)\right| = B(\hat{T}_1,\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda \hat{T}_1}}-1)} \leq \frac{2C_1}{C_2\lambda^4}\hat{T}_1 \leq \frac{2C_1}{C_2\lambda^4}\left|\hat{T}_1 - \hat{T}_2\right|.$$

Second case: $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 \leq 0$ and $\hat{T}_2 \leq 0$. Then $B(\hat{T}_1,\lambda) = B(\hat{T}_2,\lambda) = 0$ so that the inequality

$$\left|B(\hat{T}_1,\lambda) - B(\hat{T}_2,\lambda)\right| \leq \frac{2C_1}{C_2\lambda^4}\left|\hat{T}_1 - \hat{T}_2\right| \tag{38}$$

is obvious.
Third case: $\hat{T}_1, \hat{T}_2 \in \mathbb{R}$, $\hat{T}_1 > 0$ and $\hat{T}_2 > 0$. Then by Lagrange's theorem on finite increments:

$$B(\hat{T}_1,\lambda) - B(\hat{T}_2,\lambda) = (\hat{T}_1 - \hat{T}_2)\frac{\partial B}{\partial T}(\check{T},\lambda)$$

where $\check{T} \in \mathbb{R}$ is some intermediate point between $\hat{T}_1$ and $\hat{T}_2$.
$\frac{\partial B}{\partial T}(\check{T},\lambda) = 2C_1C_2\frac{e^{C_2/\lambda\check{T}}}{\lambda^6\check{T}^{\ 2}\left(e^{C_2/\lambda\check{T}}-1\right)^2} = \frac{2C_1}{C_2\lambda^4}\frac{\left(\frac{C_2}{\check{\mu}}\right)^2 e^{C_2/\check{\mu}}}{\left(e^{C_2/\check{\mu}}-1\right)^2}$ where we have set $\check{\mu} := \lambda\check{T}$. This formula shows that $\frac{\partial B}{\partial T}(\check{T},\lambda) > 0$.
Also: $\forall s \in \mathbb{R}_+^* : \frac{s^2e^s}{(e^s-1)^2} = \frac{s^2}{\left(e^{\frac{s}{2}}-e^{-\frac{s}{2}}\right)^2} = \frac{s^2}{4\sinh(\frac{s}{2})^2} = \left(\frac{\frac{s}{2}}{\sinh(\frac{s}{2})}\right)^2 \leq 1$. Thus: $\frac{\partial B}{\partial T}(\check{T},\lambda) \leq \frac{2C_1}{C_2\lambda^4}$. So inequality (38) is still true. ■

Corollary 7 *Let* $T_1, T_2 \in L^2(]0,l[)$. *Then:*

$$\left\|B_g^k(T_1(\cdot)) - B_g^k(T_2(\cdot))\right\|_{L^2([0,l])} \leq 2n_g^2\frac{C_1}{C_2}\frac{\lambda_{k+1}^3 - \lambda_k^3}{3\lambda_k^3\ \lambda_{k+1}^3}\left\|T_1(\cdot) - T_2(\cdot)\right\|_{L^2([0,l])}. \tag{39}$$

Proof. Let us recall that $B_g^k(T) := n_g^2\int\limits_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)d\lambda$. By lemma 6, we have:

$$\left|B_g^k(T_1(x)) - B_g^k(T_2(x))\right| \leq \left(n_g^2\frac{2C_1}{C_2}\int\limits_{\lambda_k}^{\lambda_{k+1}}\frac{d\lambda}{\lambda^4}\right)\ \left|T_1(x) - T_2(x)\right|,$$

from which follows (39). ■

Proposition 8 *There is at most one weak solution* T *belonging to*

$$\{T \in L^2(0,t_f;H^1(]0,l[)); \dot{T} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)\} \cap C([0,l]\times[0,t_f])$$

of the initial boundary value problem (35).

Proof. Let $T_1, T_2 \in \{T \in L^2(0,t_f;H^1(]0,l[)); \dot{T} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)\} \cap C([0,l]\times[0,t_f])$ be two weak solutions of the initial boundary value problem (37). We are going to prove that $T_1 = T_2$. $T := T_1 - T_2$ is solution of

$$\begin{gathered}
c_p m_g \left\langle \tfrac{dT}{dt}(\cdot,t), T(\cdot,t)\right\rangle_{H^1(]0,l[)^*,H^1(]0,l[)} = -k_h \int\limits_0^l \left[\tfrac{\partial T}{\partial x}(x,t)\right]^2 dx \\
+\int\limits_0^l \left[\psi(T_1(x,t)) - \psi(T_2(x,t))\right] T(x,t)dx \\
-h_c T(l,t)^2 - \left[\Theta(T_1(l,t)) - \Theta(T_2(l,t))\right]\cdot T(l,t) \\
+\int\limits_0^l \left[h_{T_1}(x,t) - h_{T_2}(x,t)\right] T(x,t)dx,\ \forall' t \in]0,t_f[.
\end{gathered}$$

This late equation may be rewritten:

$$\begin{aligned}
&\tfrac{c_p m_g}{2}\tfrac{d}{dt}\int\limits_0^l T(x,t)^2 dx + k_h \int\limits_0^l \left[\tfrac{\partial T}{\partial x}(x,t)\right]^2 dx + h_c T(l,t)^2 + [\Theta(T_1(l,t)) - \Theta(T_2(l,t)) \\
&]\cdot T(l,t) + \sum_{k=1}^{k=M} 4\pi\kappa_k \int\limits_0^l \left[B_g^k(T_1(x,t)) - B_g^k(T_2(x,t))\right](T_1(x,t)) - T_2(x,t))\ dx \\
&= \sum_{k=1}^{k=M} 2\pi\kappa_k \int\limits_0^l [\int\limits_{-1}^{+1} I_{T_1}^k(x,t,\mu)\ d\mu - \int\limits_{-1}^{+1} I_{T_2}^k(x,t,\mu)\ d\mu](T_1(x,t)) - T_2(x,t))\ dx.
\end{aligned} \tag{40}$$

As for $T,\ \hat{T} \in \mathbb{R}$, $T \leq \hat{T}$ implies $B(T,\lambda) \leq B(\hat{T},\lambda),\ \forall \lambda > 0$ ([19], p.8), the last two terms in the left-hand side of (40) are positive. Thus by (31):

$$\frac{c_p m_g}{2}\frac{d}{dt}\int_0^l T(x,t)^2dx \le \sum_{k=1}^{k=M} 2\pi\kappa_k\cdot$$

$$\begin{aligned}
&\int_0^l [\int_{-1}^{+1} I^k_{T_1}(x,t,\mu)\ d\mu - \int_{-1}^{+1} I^k_{T_2}(x,t,\mu)\ d\mu](T_1(x,t)) - T_2(x,t))\ dx \le \sum_{k=1}^{k=M} 2\pi\kappa_k \\
&\cdot \int_0^l \left[\int_0^l G(x,x')\ (B^k_g(T_1(x',t)) - B^k_g(T_2(x',t))\ dx'\right](T_1(x,t) - T_2(x,t))\ dx \\
&+ \sum_{k=1}^{k=M} 2\pi\kappa_k^2 \int_0^l \left[\int_0^l \Phi_1(\kappa_k(2l - x - x^{'}))\ (B^k_g(T_1(x',t)) - B^k_g(T_2(x',t))\ dx'\right] \\
&\qquad \cdot\ (T_1(x,t)) - T_2(x,t))\ dx \\
&\le \sum_{k=1}^{k=M} 2\pi\kappa_k \left\|G(\cdot,\cdot)\right\|_{L^2([0,l]^2)} \left\|B^k_g(T_1(\cdot,t)) - B^k_g(T_2(\cdot,t))\right\|_{L^2([0,l])} \\
&\cdot \left\|T_1(\cdot,t) - T_2(\cdot,t)\right\|_{L^2([0,l])} + \sum_{k=1}^{k=M} 2\pi\kappa_k^2 \left\|\Phi_1(\kappa_k(2l - \cdot - \cdot))\right\|_{L^2([0,l]^2)} \\
&\cdot\ \left\|B^k_g(T_1(\cdot,t)) - B^k_g(T_2(\cdot,t))\right\|_{L^2([0,l])} \left\|T_1(\cdot,t) - T_2(\cdot,t)\right\|_{L^2([0,l])}.
\end{aligned}$$

Applying corollary 7 to $T_1(\cdot,t)$ and $T_2(\cdot,t)$, it follows from the preceding inequality that for some positive constant C:

$$\frac{d}{dt}\int_0^l T(x,t)^2dx \le C\int_0^l T(x,t)^2dx.$$

Thus $\frac{d}{dt}\left[e^{-Ct}\int_0^l T(x,t)^2dx\right] = e^{-Ct}\left[\frac{d}{dt}\int_0^l T(x,t)^2dx - C\int_0^l T(x,t)^2dx\right] \le 0.$ Thus the function

$$\mathbb{R}_+^* \to \mathbb{R}_+ : t \mapsto e^{-Ct}\int_0^l T(x,t)^2dx$$

is a decreasing positive function and being 0 at time $t = 0$, is identically 0. Thus $T(\cdot,\cdot) = 0$ i.e. $T_1(\cdot,\cdot) = T_2(\cdot,\cdot)$. ■

Now, we are going to define a “fixed point problem” in order to prove the existence of a weak solution to the initial boundary value problem (37): formally, it is the problem: find $\tilde{T} \in \{\tilde{T} \in L^2(0,t_f;H^1(]0,l[)); \frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)\}$

such that

$$\begin{cases} c_p m_g \frac{\partial \tilde{T}}{\partial t}(x,t) = k_h \frac{\partial^2 \tilde{T}}{\partial x^2}(x,t) + \psi(\tilde{T}(x,t)) + h_T(x,t),\ \forall'(x,t) \in]0,l[\times]0,t_f[, \\ \tilde{T}(o,t) = T_a,\ \forall' t \in]0,t_f[, \\ -k_h \frac{\partial \tilde{T}}{\partial x}(l,t) = h_c(\tilde{T}(l,t) - T_a) + \Theta(\tilde{T}(l,t)) - \Theta(T_S(t)),\ \forall' t \in]0,t_f[, \\ \tilde{T}(x,0) = T_0(x),\ \forall' x \in [0,l]. \end{cases} \tag{41}$$

By a weak solution of the initial boundary value problem (41), we mean a function

$$\tilde{T} \in \{\tilde{T} \in L^2(0,t_f;H^1(]0,l[));\frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)\}$$

such that $\forall' t \in]0,t_f[$:

$$\begin{cases} c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot,t), \varphi \right\rangle_{H^1(]0,l[)^*, H^1(]0,l[)} = -k_h \int\limits_0^l \frac{\partial \tilde{T}}{\partial x}(x,t)\varphi'(x)dx + \\ \int\limits_0^l \psi(\tilde{T}(x,t))\varphi(x)dx + h_c\left(T_a - \tilde{T}(l,t)\right)\varphi(l) + \left[\Theta(T_S(t)) - \Theta(\tilde{T}(l,t))\right] \\ \cdot\ \varphi(l) + \int\limits_0^l h_T(x,t)\varphi(x)dx,\ \forall \varphi \in H^1(]0,l[) \text{ such that } \varphi(0) = 0, \\ \tilde{T}(0,t) = T_a, \\ \tilde{T}(x,0) = T_0(x),\ \forall' x \in [0,l]. \end{cases} \tag{42}$$

Let us assume that the function T in the definition of $h_T(\cdot,\cdot)$ which appears in the right-hand side of equation $(41)_{\text{(i)}}$ or (42) is given and belongs to $L^2(]0,l[\times]0,t_f[)$. Corollary 3 tells us that h_T belongs to $L^2(]0,l[\times]0,t_f[)$. Firstly, we want to prove that the initial boundary value problem (41) possesses one and only one weak solution $\tilde{T} \in L^2(0,t_f;H^1(]0,l[)) \cap C([0,l]\times[0,t_f])$ such that $\frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$ i.e. that $\tilde{T}$ verifies (42). To prove this, we will use Theorem 1.40 page 49 of [9] on semilinear parabolic equations, but we have to check the hypotheses of that theorem. We know already that h_T belongs to $L^2(]0,l[\times]0,t_f[)$ and thus a fortiori to $L^r(]0,l[\times]0,t_f[)$ with $r > \frac{3}{2}$. The function $g :]0,t_f[\to \mathbb{R} : t \mapsto \frac{h_c T_a}{k_h} + \frac{\Theta(T_S(t))}{k_h} = \frac{h_c T_a}{k_h} + \frac{\pi}{k_h}\int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda B(T_S(t),\lambda)\,d\lambda$ belongs to $L^{s^*}(]0,t_f[)$ for some $s^* > 2$ if we suppose that $T_S \in L^{s^*}(]0,t_f[)$ as follows from the inequality $B(\hat{T},\lambda) \lesssim \frac{\hat{T}}{\lambda^4},\ \forall \hat{T} \in \mathbb{R}_+^*$. In particular, this will be the case if $T_S \in H^1(]0,t_f[)$. Now, the nonlinear term in equation $(41)_{\text{(i)}}$ is given by $-\psi(\tilde{T}(x,t) := \sum\limits_{k=1}^{k=M} 4\pi\kappa_k B_g^k(\tilde{T}(x,t))$ for $(x,t) \in]0,l[\times]0,t_f[$. We must verify that the nonlinear mapping

$$\mathbb{R} \to \mathbb{R} : \hat{T} \mapsto \sum_{k=1}^{k=M} 4\pi\kappa_k B_g^k(\hat{T}) \tag{43}$$

is monotone increasing and Lipschitz continuous (let us recall that we have set $B_g^k(\hat{T}) = 0$ in case $\hat{T}$ would be negative). That it is monotone increasing is evident. By Corollary 7:

$$\left| B_g^k(\hat{T}_1) - B_g^k(\hat{T}_2) \right| \leq 2n_g^2 \frac{C_1}{C_2} \frac{\lambda_{k+1}^3 - \lambda_k^3}{3\lambda_k^3\ \lambda_{k+1}^3} \left| \hat{T}_1 - \hat{T}_2 \right|.$$

This proves that the nonlinear mapping (43) is Lipschitz continuous. The nonlinear term in the boundary condition on the surface $x_g = l$ of the glass plate is given by $\Theta(\tilde{T}(l,\ t)) = \pi \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda[B(\tilde{T}(l,t),\lambda)]d\lambda$ for $t \in]0, t_f[$. As already said in Corollary 4 the nonlinear mapping

$$\mathbb{R} \to \mathbb{R} : \hat{T} \mapsto \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda[B(\hat{T},\lambda)]d\lambda \tag{44}$$

is obviously monotone increasing. By an immediate adaptation of the proof of Corollary 7:

$$\left| \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda[B(\hat{T}_1,\lambda)]d\lambda - \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda[B(\hat{T}_2,\lambda)]d\lambda \right| \leq \frac{2C_1}{C_2} \frac{1}{3\lambda_0^3} \left| \hat{T}_1 - \hat{T}_2 \right|. \tag{45}$$

Thus the nonlinear mapping Θ is also Lipschitz continuous. All the hypotheses of Theorem 1.40 page 49 of [9] on semilinear parabolic equations being verified we have:

Theorem 9 *Let us assume that the function T occuring in the definition (32)-(31) of $h_T(\cdot,\cdot)$ which appears in the right-hand side of equation $(41)_{(i)}$ or (42) is given and belongs to $L^2(]0,l[\times]0,t_f[)$. We assume that the initial condition $T_0 \in C([0,l])$ and verifies the compatibility condition $T_0(0) = T_a$ with the boundary condition on the surface $x_g = 0$ of the glass plate and that the absolute temperature of the black source $T_S(\cdot)$ belongs to $L^{s^*}(]0,t_f[)$ for some $s^* > 2$. Then, the initial boundary value problem (41) possesses one and only one weak solution $\tilde{T} \in L^2(0,t_f;H^1(]0,l[)) \cap C([0,l]\times[0,t_f])$ such that $\frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)\}$*

i.e. $\tilde{T}$ verifies equation (42). Moreover, we have the following estimate:

$$\begin{aligned}
&\left\|\tilde{T}\right\|_{L^2(0,t_f;H^1(]0,l[))} + \left\|\frac{d\tilde{T}}{dt}\right\|_{L^2(0,t_f;[H^1(]0,l[)]^*)} + \left\|\tilde{T}\right\|_{C([0,l]\times[0,t_f])} \\
&\quad \le C_1\Big(\left\|\sum_{k=1}^{k=M} 2\pi\kappa_k \int_{-1}^{+1} I_T^k(\cdot,\cdot,\mu)\ d\mu\right\|_{L^2([0,l]\times[0,t_f])} + T_a \\
&\qquad + \|\Theta(T_S(\cdot))\|_{L^{s^*}(]0,t_f[)} + \|T_0\|_{C([0,l])}\Big) \\
&\quad \le C_2(\|T(\cdot,\cdot)\|_{L^2([0,l]\times[0,t_f])} + T_a + \|T_S(\cdot)\|_{L^{s^*}(]0,t_f[)} + \|T_0\|_{C([0,l])}).
\end{aligned} \tag{46}$$

Remark 10 *For uniqueness, the requirement $\tilde{T} \in C([0,l]\times[0,t_f])$ is not necessary as may be seen by adapting to this new initial boundary value problem the proof of Proposition 8 which simplifies greatly, the right-hand side of equation (40) being* 0 *in the present case.*

We want now to prove under certain hypotheses on T, T_0, T_S that the solution of the initial boundary value problem (41) $\tilde{T}$ is lower bounded by T_a. We will use Stampacchia's truncation method [2]. Before, we need the following lemmas:

Lemma 11 *1°) If $\varphi \in H^1(]0,l[)$ and $\theta \in H^1(]0,l[)$, then $\varphi\theta \in H^1(]0,l[)$.*
2°) If $\varphi \in H^1(]0,l[)$ and $\psi \in \left[H^1(]0,l[)\right]^*$, then $\varphi\psi$ defined by $\langle\varphi\psi,\theta\rangle := \langle\psi,\varphi\theta\rangle, \forall\theta \in H^1(]0,l[)$ belongs to $\left[H^1(]0,l[)\right]^*$ and

$$\|\varphi\psi\|_{[H^1(]0,l[)]^*} \lesssim \|\varphi\|_{H^1(]0,l[)} \ \|\psi\|_{[H^1(]0,l[)]^*}. \tag{47}$$

Proof. 1°) $\varphi \in H^1(]0,l[)$ and $\theta \in H^1(]0,l[)$ implies $\varphi \in C([0,l])$ and $\theta \in C([0,l])$ as $H^1(]0,l[) \hookrightarrow C([0,l])$ and thus $\varphi\theta \in C([0,l]) \hookrightarrow L^2(]0,l[)$. $(\varphi\theta)^{'} = \theta\varphi' + \varphi\theta' \in L^2(]0,l[)$. Thus $\varphi\theta \in H^1(]0,l[)$ and

$$\|\varphi\theta\|_{H^1(]0,l[)} \lesssim \|\varphi\|_{H^1(]0,l[)} \|\theta\|_{H^1(]0,l[)}.$$

2°) $\varphi\psi$ is defined by $\langle\varphi\psi,\theta\rangle_{[H^1]^*,\ H^1} := \langle\psi,\varphi\theta\rangle_{[H^1]^*,\ H^1}$, $\forall\theta \in H^1(]0,l[)$ (for short, we have denoted $H^1(]0,l[)$ by H^1). By the previous point $\varphi\theta \in H^1(]0,l[)$ and we have the inequality

$$\left|\langle\psi,\varphi\theta\rangle_{[H^1]^*,\ H^1}\right| \lesssim \|\psi\|_{[H^1(]0,l[)]^*} \|\varphi\|_{H^1(]0,l[)} \|\theta\|_{H^1(]0,l[)} \lesssim \|\theta\|_{H^1(]0,l[)},$$

$\forall\theta \in H^1(]0,l[)$. Thus the mapping $\theta \mapsto \langle\psi,\varphi\theta\rangle_{[H^1]^*,\ H^1}$ is a continuous linear form on $H^1(]0,l[)$, i.e. an element of $\left[H^1(]0,l[)\right]^*$ and inequality (47) holds. ∎

Lemma 12 *We have the following equality:*

$$\begin{aligned}
&\int_0^l G(x,x')\ dx' + E_2(\kappa_k x) + \Phi_2(\kappa_k(2l-x)) + [E_2(\kappa_k(l-x)) - \Phi_2(\kappa_k(l-x))] \\
&\qquad + \kappa_k \int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\ dx' = 2.
\end{aligned}$$

Proof. Firstly:

$$\begin{aligned}
&\int_0^l G(x,x')\ dx' = \int_0^x G(x,x')\ dx' + \int_x^l G(x,x')\ dx' = \kappa_k \int_0^x E_1(\kappa_k(x-x'))\ dx' \\
&+\kappa_k \int_x^l E_1(\kappa_k(x'-x))dx' = \kappa_k \int_0^x E_1(\kappa_k y)dy + \kappa_k \int_0^{l-x} E_1(\kappa_k y)dy = \int_0^{\kappa_k x} E_1(z)dz \\
&\quad + \int_0^{\kappa_k(l-x)} E_1(z)\ dz = \left[-E_2(z)\right]_{z=0}^{z=\kappa_k x} + \left[-E_2(z)\right]_{z=0}^{z=\kappa_k(l-x)} \text{ as } E_2' = -E_1 \\
&\quad = 1 - E_2(\kappa_k x) + 1 - E_2(\kappa_k(l-x)) = 2 - E_2(\kappa_k x) - E_2(\kappa_k(l-x)).
\end{aligned}$$

Thus $\int_0^l G(x,x')\ dx' + E_2(\kappa_k x) + E_2(\kappa_k(l-x)) = 2.$

Secondly:

$$\begin{aligned}
&\kappa_k \int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\ dx' = \int_0^l \left[\int_0^1 \frac{\kappa_k}{\mu} \rho_g(\mu) e^{-\kappa_k \frac{2l-x-x^{'}}{\mu}} d\mu \right]\ dx' \\
&= \int_0^1 \rho_g(\mu) e^{-\kappa_k \frac{2l-x}{\mu}} \left[\int_0^l \frac{\kappa_k}{\mu} e^{\frac{\kappa_k x^{'}}{\mu}}\ dx' \right] d\mu = \int_0^1 \rho_g(\mu) e^{-\kappa_k \frac{2l-x}{\mu}} \left[e^{\frac{\kappa_k l}{\mu}} - 1 \right] d\mu \\
&= \int_0^1 \rho_g(\mu) e^{-\kappa_k \frac{l-x}{\mu}} d\mu - \int_0^1 \rho_g(\mu) e^{-\kappa_k \frac{2l-x}{\mu}} d\mu.
\end{aligned}$$

Thus: $\kappa_k \int_0^l \Phi_1(\kappa_k(2l-x-x^{'}))\ dx' - \Phi_2(\kappa_k(l-x)) + \Phi_2(\kappa_k(2l-x)) = 0.$

The result follows from these two points. ■

In the following, to alleviate the notations, we will allow us to write $L^p(X)$ instead of $L^p(0,t_f;X)$, for $1 \leq p \leq +\infty$ and X a Banach space. Also, in the following, to shorten the notations, we will write sometimes H^1 instead of $H^1(]0,l[)$ and $\left(H^1\right)^*$ instead of $H^1(]0,l[)^*$.

Proposition 13 *We keep the hypotheses of theorem 9. Moreover, we assume that* $T(\cdot,\cdot) \geq T_a$ *a.e. on* $]0,l[\times]0,t_f[$*, that* $T_S(\cdot) \geq T_a$ *a.e. on* $]0,t_f[$*, and that the initial condition* $T_0(.) \geq T_a$*. Then the weak solution* $\tilde{T}$ *of the initial boundary value problem (41) satisfies the lower bound* $\tilde{T}(\cdot,\cdot) \geq T_a$ *on* $[0,l] \times [0,t_f]$.

Proof. Let us introduce the function

$$H : \mathbb{R} \to \mathbb{R} : y \mapsto \begin{cases} \frac{y^2}{2} \text{ if } y < 0, \\ 0 \text{ if } y \geq 0, \end{cases} \tag{48}$$

and let us set

$$\tilde{\varphi}(t) := c_p m_g \int_0^l H(\tilde{T}(x,t) - T_a)dx = c_p m_g \left\langle H(\tilde{T}(\cdot,t) - T_a), \mathbf{1}_{]0,l[} \right\rangle, \ \forall t \in [0,t_f].$$

As the mapping from $L^2(]0,l[)$ into $L^2(]0,l[)$ which sends a function onto its negative part is lipschitzian according to Lemma 24 and thus continuous, $\tilde{\varphi} : [0,t_f] \to \mathbb{R}$ is a continuous function. Moreover $\tilde{\varphi}(0) = 0$ because $\tilde{T}(\cdot,0) - T_a = T_0(.) - T_a \geq 0$. Using the density of $\mathcal{D}([0,t_f];H^1(]0,l[)$ in the space ([4], p.571)

$$W(0,t_f;H^1(]0,l[) := \{u \in L^2(0,t_f;H^1(]0,l[)); \dot{u} \in L^2(0,t_f;H^1(]0,l[))\}$$

endowed with its natural norm, one can prove that

$$\frac{d}{dt}H(\tilde{T} - T_a) = -(\tilde{T} - T_a)_- \frac{d\tilde{T}}{dt}, \tag{49}$$

in the sense of distributions. Formula (49) is first proved for regular functions $\tilde{T}_n \in \mathcal{D}([0,t_f];H^1(]0,l[)$ approaching $\tilde{T}$ in the norm of the space $W(0,t_f;H^1(]0,l[)$ as $n \to +\infty$. Proposition 26 implies that $(\tilde{T}_n - T_a)_-$ tends to $(\tilde{T} - T_a)_-$ in $L^2(0,t_f;H^1(]0,l[))$ as $n \to +\infty$. It is then easy to pass to the limit. Let us explain the meaning of the right-hand side of (49). $\frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$ so that $\forall' t \in]0,t_f[$: $\frac{d\tilde{T}}{dt}(\cdot,t) \in \left[H^1(]0,l[)\right]^*$. The function

$$t \mapsto (\tilde{T}(\cdot,t) - T_a)_-$$

being the composition of the function $t \mapsto \tilde{T}(\cdot,t) - T_a$ which belongs to $L^2(0,t_f;H^1(]0,l[))$ and of the function

$$H^1(]0,l[) \to H^1(]0,l[) : \psi \longmapsto \psi_-$$

which is continuous (see Proposition 25 in the appendix) is also measurable, and belongs to $L^2(0,t_f;H^1(]0,l[))$. In particular: $\forall' t \in]0,t_f[$: $(\tilde{T}(\cdot,t) - T_a)_- \in H^1(]0,l[)$. By the second point of lemma 11 follows that $\forall' t \in]0,t_f[$ the product of $\frac{d\tilde{T}}{dt}(\cdot,t) \in \left[H^1(]0,l[)\right]^*$ by $(\tilde{T}(\cdot,t) - T_a)_- \in H^1(]0,l[)$ has sense and belongs to $\left[H^1(]0,l[)\right]^*$. In particular, the right-hand side of (49) has sense at $\forall' t \in]0,t_f[$. Moreover:

$$\left\| (\tilde{T}(\cdot,t) - T_a)_- \frac{d\tilde{T}}{dt}(\cdot,t) \right\|_{(H^1)^*} \lesssim \left\| (\tilde{T}(\cdot,t) - T_a)_- \right\|_{H^1} \left\| \frac{d\tilde{T}}{dt}(\cdot,t) \right\|_{(H^1)^*}.$$

From this late inequality follows that the mapping

$$t \mapsto (\tilde{T}(\cdot,t) - T_a)_- \frac{d\tilde{T}}{dt}(\cdot,t)$$

belongs to $L^1(0,t_f;\left[H^1(]0,l[)\right]^*)$ due to

$$\left\|(\tilde{T}-T_a)_-\frac{d\tilde{T}}{dt}\right\|_{L^1(H^1(]0,l[)^*)} \lesssim \left\|(\tilde{T}-T_a)_-\right\|_{L^2(H^1(]0,l[))}\left\|\frac{d\tilde{T}}{dt}\right\|_{L^2(H^1(]0,l[)^*)}.$$

As $\mathbf{1}_{]0,l[}\in H^1(]0,l[)$, $\left\langle(\tilde{T}-T_a)_-\frac{d\tilde{T}}{dt},\mathbf{1}_{]0,l[}\right\rangle\in L^1(]0,t_f[)$. By the definition of the function $\tilde{\varphi}$ and (49):

$$\frac{d\tilde{\varphi}}{dt}(t)=-c_pm_g\left\langle(\tilde{T}-T_a)_-\frac{d\tilde{T}}{dt},\mathbf{1}_{]0,l[}\right\rangle,\ \forall' t\in]0,t_f[.$$

Thus $\frac{d\tilde{\varphi}}{dt}\in L^1(]0,t_f[)$ which implies that the function $\tilde{\varphi}$ is absolutely continuous on the interval $[0,t_f]$. By the definition of the product of an element of $\left[H^1(]0,l[)\right]^*$ by an element of $H^1(]0,l[)$ stated in lemma 11:

$$\left\langle(\tilde{T}(\cdot,t)-T_a)_-\frac{d\tilde{T}}{dt}(\cdot,t),\mathbf{1}_{]0,l[}\right\rangle=\left\langle\frac{d\tilde{T}}{dt}(\cdot,t),(\tilde{T}(\cdot,t)-T_a)_-\right\rangle,\ \forall' t\in]0,t_f[.$$

$\forall' t\in]0,t_f[$: $(\tilde{T}(\cdot,t)-T_a)_-\in H^1(]0,l[)$ and vanishes at the point $x=0$ of the interval $[0,l]$. Thus by the definition (42) of what is a weak solution of the initial boundary value problem (41): $\forall' t\in]0,t_f[$:

$$\begin{aligned}&\frac{d\tilde{\varphi}}{dt}(t)=-c_pm_g\left\langle(\tilde{T}(\cdot,t)-T_a)_-\frac{d\tilde{T}}{dt}(\cdot,t),\mathbf{1}_{]0,l[}\right\rangle\\ &=-c_pm_g\left\langle\frac{d\tilde{T}}{dt}(\cdot,t),(\tilde{T}(\cdot,t)-T_a)_-\right\rangle_{(H^1)^*,H^1}=-k_h\int_0^l\frac{\partial\tilde{T}}{\partial x}(x,t)^2\mathbf{1}_{\left\{\tilde{T}(\cdot,t)<T_a\right\}}dx\\ &-\int_0^l\psi(\tilde{T}(x,t))(\tilde{T}(x,t)-T_a)_-dx-\int_0^l h_T(x,t)(\tilde{T}(x,t)-T_a)_-dx+h_c\cdot\\ &\left(\tilde{T}(l,t)-T_a\right)(\tilde{T}(l,t)-T_a)_-+\left[\Theta(\tilde{T}(l,t))-\Theta(T_S(t))\right]\cdot(\tilde{T}(l,t)-T_a)_-.\end{aligned}\tag{50}$$

In (50), we have used the fact that

$$\forall' t\in]0,t_f[:\ \frac{\partial}{\partial x}(\tilde{T}(\cdot,t)-T_a)_-=-\frac{\partial\tilde{T}}{\partial x}(\cdot,t)\mathbf{1}_{\left\{\tilde{T}(\cdot,t)<T_a\right\}}$$

by ([11], pp.50-54). Now let us look carefully at each term in the right-hand side of the previous equality to see their respective signs. The first term in the right-hand side of equality (50) is obviously negative. By the explicit expression of $\int_{-1}^{+1}I_T^k(x,t,\mu)\,d\mu$ given by formula (31) and the hypotheses $T(\cdot,\cdot)\geq T_a$ a.e. on $]0,l[\times]0,t_f[$, and $T_S(\cdot)\geq T_a$ a.e. on $]0,t_f[$, it follows by lemma 12 that

$$\int_{-1}^{+1}I_T^k(x,t,\mu)\,d\mu\geq 2B_g^k(T_a),\ \text{a.e. on }]0,l[\times]0,t_f[.$$

Thus $h_T(x,t) := \sum\limits_{k=1}^{k=M} 2\pi\kappa_k \int\limits_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \geq \sum\limits_{k=1}^{k=M} 4\pi\kappa_k B_g^k(T_a)$. But

$$\begin{aligned}
-\int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-T_a)_- dx &= \sum_{k=1}^{k=M} 4\pi\kappa_k \int\limits_0^l B_g^k(\tilde{T}(x,t))(\tilde{T}(x,t)-T_a)_- dx \\
&\leq \sum_{k=1}^{k=M} 4\pi\kappa_k \int\limits_0^l B_g^k(T_a)(\tilde{T}(x,t)-T_a)_- dx \leq \int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-T_a)_- dx
\end{aligned}$$

because if $(\tilde{T}(x,t)-T_a)_- \neq 0$, then $\tilde{T}(x,t) < T_a$ which implies $B_g^k(\tilde{T}(x,t)) < B_g^k(T_a)$. Thus

$$-\int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-T_a)_- dx - \int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-T_a)_- dx \leq 0. \qquad (51)$$

Obviously: $h_c\left(\tilde{T}(l,t)-T_a\right)(\tilde{T}(l,t)-T_a)_- \leq 0$. Also

$$\left[\Theta(\tilde{T}(l,t)) - \Theta(T_S(t))\right] \cdot (\tilde{T}(l,t)-T_a)_-$$

is negative because if $(\tilde{T}(l,t)-T_a)_- \neq 0$, then $\tilde{T}(l,t) < T_a$ implies $\Theta(\tilde{T}(l,t)) \leq \Theta(T_a) \leq \Theta(T_S(t))$ as $T_S(t) \geq T_a$ by hypothesis, and thus $\Theta(\tilde{T}(l,t)) - \Theta(T_S(t)) \leq 0$. Thus

$$\begin{aligned}
&-\int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-T_a)_- dx - \int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-T_a)_- dx \\
&+h_c\left(\tilde{T}(l,t)-T_a\right)(\tilde{T}(l,t)-T_a)_- + \left[\Theta(\tilde{T}(l,t)) - \Theta(T_S(t))\right] \cdot (\tilde{T}(l,t)-T_a)_-
\end{aligned} \qquad (52)$$

is negative. From equation (50) follows that $\frac{d\tilde{\varphi}}{dt}(t) \leq 0,\ \forall' t \in]0,t_f[$. $\tilde{\varphi}$ being an absolutely continuous and positive function, null at $t=0$ due to our hyothesis $T_0(\cdot) \geq T_a$ on the initial condition, it follows that $\tilde{\varphi}(t) = 0,\ \forall t \in [0,t_f[$. Thus $\forall t \in [0,t_f]$: $\tilde{T}(x,t) \geq T_a$, $\forall x \in [0,l]$ as $\tilde{T} \in C([0,l]\times[0,t_f])$. ∎

Now, we want also to prove under certain hypotheses that the solution $\tilde{T}$ of the initial boundary value problem (41) is upper bounded. The proof is more or less similar to the proof of the lower bound (13). Still, we use Stampacchia truncation's method.

Proposition 14 *We keep the hypotheses of theorem (9). Moreover, we assume that* $\bar{T}$ *denotes any positive real number such that* $\bar{T} \geq T_a > 0$, $\bar{T} \geq T_S(t) > 0$, $\forall' t \in]0,t_f[$ *and* $\bar{T} \geq T_0(x) > 0$, $\forall x \in [0,l]$. *We suppose that* $T(\cdot,\cdot) \leq \bar{T}$ *a.e. on* $]0,l[\times]0,t_f[$. *Then the weak solution* $\tilde{T}$ *of the initial boundary value problem (41) satisfies the upper bound* $\tilde{T}(\cdot,\cdot) \leq \bar{T}$ *on* $]0,l[\times]0,t_f[$.

Proof. The proof is very similar to the previous one. Let us introduce this time the function

$$\check{H}:\mathbb{R}\to\mathbb{R}:y\mapsto\left\{\begin{array}{l}\frac{y^2}{2}\text{ if }y>0,\\ 0\text{ if }y\le 0,\end{array}\right.$$

and let us set

$$\tilde{\varphi}(t):=c_p m_g\int_0^l \check{H}(\tilde{T}(x,t)-\bar{T})dx=c_p m_g\left\langle \check{H}(\tilde{T}(\cdot,t)-\bar{T}),\mathbf{1}_{]0,l[}\right\rangle,\ \forall t\in[0,t_f].$$

As the mapping from $L^2(]0,l[)$ into $L^2(]0,l[)$ which sends a function onto its positive part is lipschitzian according to Lemma 24 and thus continuous, $\tilde{\varphi}:[0,t_f]\to\mathbb{R}$ is a continuous function. Moreover $\tilde{\varphi}(0)=0$ because $\tilde{T}(\cdot,0)-\bar{T}=T_0(.)-\ \bar{T}\le 0$. Using the same arguments as in the proof of the preceding proposition, one can see that:

$$\frac{d}{dt}\check{H}(\tilde{T}-\bar{T})=(\tilde{T}-\bar{T})_+\frac{d\tilde{T}}{dt},\tag{53}$$

in the sense of distributions. Let us explain the meaning of the right-hand side. $\frac{d\tilde{T}}{dt}\in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$ so that $\forall' t\in]0,t_f[$: $\frac{d\tilde{T}}{dt}(\cdot,t)\in\left[H^1(]0,l[)\right]^*$. The function

$$t\mapsto(\tilde{T}(\cdot,t)-\bar{T})_+$$

being the composition of the function $t\mapsto\tilde{T}(\cdot,t)-\bar{T}$ which belongs to $L^2(0,t_f;H^1(]0,l[))$ and of the function

$$H^1(]0,l[)\to H^1(]0,l[):\psi\longmapsto\psi_+$$

which is continuous (see Proposition 25 in the appendix) is also measurable, and belongs to $L^2(0,t_f;H^1(]0,l[))$. In particular: $\forall' t\in]0,t_f[$: $(\tilde{T}(\cdot,t)-\bar{T})_+\in H^1(]0,l[)$. By the second point of Lemma 11 follows that $\forall' t\in]0,t_f[$ the product of $\frac{d\tilde{T}}{dt}(\cdot,t)\in\left[H^1(]0,l[)\right]^*$ by $(\tilde{T}(\cdot,t)-\bar{T})_+\in H^1(]0,l[)$ has sense and belongs to $\left[H^1(]0,l[)\right]^*$. In particular, the right-hand side of (53) has sense at $\forall' t\in]0,t_f[$. Moreover:

$$\left\|(\tilde{T}(\cdot,t)-\bar{T})_+\frac{d\tilde{T}}{dt}(\cdot,t)\right\|_{(H^1)^*}\lesssim\left\|(\tilde{T}(\cdot,t)-\bar{T})_+\right\|_{H^1}\left\|\frac{d\tilde{T}}{dt}(\cdot,t)\right\|_{(H^1)^*}.$$

From this late inequality follows that the mapping

$$t\mapsto(\tilde{T}(\cdot,t)-\bar{T})_+\frac{d\tilde{T}}{dt}(\cdot,t)$$

belongs to $L^1(0,t_f;\left[H^1(]0,l[)\right]^*)$ due to

$$\left\|(\tilde{T}-\bar{T})_+\frac{d\tilde{T}}{dt}\right\|_{L^1(H^1(]0,l[)^*)}\lesssim\left\|(\tilde{T}-\bar{T})_+\right\|_{L^2(H^1(]0,l[))}\left\|\frac{d\tilde{T}}{dt}\right\|_{L^2(H^1(]0,l[)^*)}.$$

As $\mathbf{1}_{]0,l[} \in H^1(]0,l[)$, $\left\langle (\tilde{T}-\bar{T})_+ \frac{d\tilde{T}}{dt}, \mathbf{1}_{]0,l[} \right\rangle \in L^1(]0,t_f[)$. By the definition of the function $\tilde{\varphi}$ and (53):

$$\frac{d\tilde{\varphi}}{dt}(t) = c_p m_g \left\langle (\tilde{T}-\bar{T})_+ \frac{d\tilde{T}}{dt}, \mathbf{1}_{]0,l[} \right\rangle, \ \forall' t \in]0,t_f[.$$

Thus $\frac{d\tilde{\varphi}}{dt} \in L^1(]0,t_f[)$ which implies that the function $\tilde{\varphi}$ is absolutely continuous on the interval $[0,t_f]$. By the definition of the product of an element of $\left[H^1(]0,l[)\right]^*$ by an element of $H^1(]0,l[)$ stated in Lemma 11:

$$\left\langle (\tilde{T}(\cdot,t)-\bar{T})_+ \frac{d\tilde{T}}{dt}(\cdot,t), \mathbf{1}_{]0,l[} \right\rangle = \left\langle \frac{d\tilde{T}}{dt}(\cdot,t), (\tilde{T}(\cdot,t)-\bar{T})_+ \right\rangle, \ \forall' t \in]0,t_f[.$$

$\forall' t \in]0,t_f[$: $(\tilde{T}(\cdot,t)-\bar{T})_+ \in H^1(]0,l[) \hookrightarrow C([0,l])$ and vanishes at the point $x=0$ of the interval $[0,l]$. Thus by the definition (42) of what is a weak solution of the initial boundary value problem (41): $\forall' t \in]0,t_f[$:

$$\begin{array}{c} \frac{d\tilde{\varphi}}{dt}(t) = c_p m_g \left\langle (\tilde{T}(\cdot,t)-\bar{T})_+ \frac{d\tilde{T}}{dt}(\cdot,t), \mathbf{1}_{]0,l[} \right\rangle \\ = c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot,t), (\tilde{T}(\cdot,t)-\bar{T})_+ \right\rangle_{(H^1)^*,H^1} = -k_h \int\limits_0^l \frac{\partial\tilde{T}}{\partial x}(x,t)^2 \mathbf{1}_{\left\{\tilde{T}(\cdot,t)>\bar{T}\right\}} dx \\ + \int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-\bar{T})_+ dx + \int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-\bar{T})_+ dx \\ + h_c(T_a - \tilde{T}(l,t))(\tilde{T}(l,t)-\bar{T})_+ + (\Theta(T_S(t)) - \Theta(\tilde{T}(l,t)) \cdot (\tilde{T}(l,t)-\bar{T})_+. \end{array} \tag{54}$$

In (54), we have used the fact that

$$\forall' t \in]0,t_f[: \frac{\partial}{\partial x}(\tilde{T}(\cdot,t)-\bar{T})_+ = \frac{\partial\tilde{T}}{\partial x}(\cdot,t)\mathbf{1}_{\left\{\tilde{T}(\cdot,t)>\bar{T}\right\}}$$

by ([11], pp.50-54). Now let us look carefully at each term in the right-hand side of the previous equality to see their respective signs. The first term in the right-hand side of equality (54) is obviously negative. By the explicit expression of $\int\limits_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu$ given by formula (31) and the hypothesis $T(\cdot,\cdot) \leq \bar{T}$ a.e. on $]0,l[\times]0,t_f[$, it follows by Lemma 12 that

$$\int\limits_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \leq 2B_g^k(\bar{T}), \text{ a.e. on }]0,l[\times]0,t_f[.$$

Thus $h_T(x,t) := \sum\limits_{k=1}^{k=M} 2\pi\kappa_k \int\limits_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \leq \sum\limits_{k=1}^{k=M} 4\pi\kappa_k B_g^k(\bar{T})$. But

$$\begin{aligned}&\int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-\bar{T})_+ dx = -\sum_{k=1}^{k=M} 4\pi\kappa_k \int\limits_0^l B_g^k(\tilde{T}(x,t))(\tilde{T}(x,t)-\bar{T})_+ dx\\ &\leq -\sum_{k=1}^{k=M} 4\pi\kappa_k \int\limits_0^l B_g^k(\bar{T})(\tilde{T}(x,t)-\bar{T})_+\ dx \leq -\int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-\bar{T})_+\ dx\end{aligned}$$

because if $(\tilde{T}(x,t)-\bar{T})_+ \neq 0$, then $\tilde{T}(x,t) > \bar{T}$ which implies $B_g^k(\tilde{T}(x,t)) > B_g^k(\bar{T})$ and $B_g^k(T(x,t)) \leq B_g^k(\bar{T})$. Thus

$$\int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-\bar{T})_+ dx + \int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-\bar{T})_+ dx \leq 0. \tag{55}$$

Obviously: $h_c(T_a - \tilde{T}(l,t))(\tilde{T}(l,t)-\bar{T})_+ \leq 0$ because if $(\tilde{T}(l,t)-\bar{T})_+ \neq 0$, then $\tilde{T}(l,t) > \bar{T} \geq T_a$ which implies $(T_a - \tilde{T}(l,t)) \leq 0$. $(\tilde{T}(l,t)-\bar{T})_+ \neq 0$ implies also $(\Theta(T_S(t)) - \Theta(\tilde{T}(l,t)) \leq 0$ as $T_S(t) \leq \bar{T}$, $\forall' t \in]0, t_f[$ by hypothesis. Thus:

$$(\Theta(T_S(t)) - \Theta(\tilde{T}(l,t)) \cdot (\tilde{T}(l,t)-\bar{T})_+ \leq 0.$$

Consequently:

$$\begin{aligned}&\int\limits_0^l \psi(\tilde{T}(x,t))(\tilde{T}(x,t)-\bar{T})_+ dx + \int\limits_0^l h_T(x,t)(\tilde{T}(x,t)-\bar{T})_+ dx\\ &+h_c(T_a - \tilde{T}(l,t))(\tilde{T}(l,t)-\bar{T})_+ + (\Theta(T_S(t)) - \Theta(\tilde{T}(l,t)) \cdot (\tilde{T}(l,t)-\bar{T})_+ \leq 0.\end{aligned} \tag{56}$$

From (54) and (56) follows that $\frac{d\tilde{\varphi}}{dt}(t) \leq 0$, $\forall' t \in]0, t_f[$. $\tilde{\varphi}$ being an absolutely continuous and positive function null for $t = 0$, it follows that $\tilde{\varphi}(t) = 0$, $\forall t \in [0, t_f]$ i.e. $\forall t \in [0, t_f]$: $\tilde{T}(x,t) \leq \bar{T}$, $\forall x \in [0, l]$ as $\tilde{T} \in C([0,l] \times [0, t_f])$. ■

In the following, we will assume at almost every time $t \in]0, t_f[$, that the absolute temperature $T_S(t)$ of the black source S satisfies

$$T_a \leq T_S(t) \leq \bar{T}. \tag{57}$$

To prove that the initial boundary value problem (35) possesses a solution, we will apply Schauder's fixed point theorem to prove that the initial boundary value problem (41) possesses a fixed point. We will apply the version of Schauder's fixed point theorem stated in A.Friedman's book ([8], p.171):

Theorem 15 *Let S be a closed convex set in a Banach space Y and let Φ be a continuous operator from S into Y such that $\Phi(S)$ is contained in S and such that the closure of $\Phi(S)$ is compact. Then Φ has a fixed point.*

This version of Schauder's fixed point theorem follows from the classical statement of Schauder's theorem ([7], p.502) applied to the closed convex hull of $\Phi(S)$: $\overline{co}(\overline{\Phi(S)})$ which is compact by Mazur's theorem, and to $\Phi_{|\overline{co}(\overline{\Phi(S)})}$.
As a Banach space Y, we choose $L^2(0,t_f;C([0,l]))$. We consider two positive real numbers $0 < T_a < \bar{T}$ such that

$$\forall x \in [0,l] : \ T_a \leq T_0(x) \leq \bar{T} \text{ and } \forall' t \in]0,t_f[: \ T_a \leq T_S(t) \leq \bar{T}.$$

Now, for S we consider:

$$S := \{T \in L^2(0,t_f;C([0,l])); \ \forall' t \in]0,t_f[: \ T_a \leq T(.,t) \leq \bar{T}\}, \tag{58}$$

and for Φ, the mapping which sends $T \in S$ onto $\tilde{T}$ solution of the initial boundary value problem (41) which still belongs to S due to proposition 13 and proposition 14. That S is closed, results from the fact every convergent sequence in the space $L^2(0,t_f;C([0,l]))$ possesses an almost everywhere convergent subsequence (see e.g. [17], p.192 for L^2 spaces of square integrable functions with values in a Banach space). Firstly, we are going to prove that the mapping Φ is continuous in several steps. To shorten the notations, we set $Q :=]0,l[\times]0,t_f[$.

Proposition 16 *The mapping from $L^2(Q)$ into $L^2(Q)$ which sends T onto $B_g^k(T)$ is lipschitzian and thus a fortiori continuous.*

Proof. This is an immediate consequence of Corollary 7. ■
Similarly:

Proposition 17 *The mapping from $L^2(]0,t_f[)$ into $L^2(]0,t_f[)$ which sends $\xi \in L^2(]0,t_f[)$ onto*

$$]0,t_f[\to \mathbb{R} : t \mapsto \int\limits_{\lambda_0}^{+\infty} \epsilon_\lambda B(\xi(t),\lambda) \ d\lambda$$

is lipschitzian and thus a fortiori continuous.

Proof. This follows from inequality (45). ■
We want now to deduce using in particular Proposition 16, that the mapping $L^2(Q) \to L^2(Q) : T \mapsto h_T$ is lipschitzian and thus a fortiori continuous. This amounts to prove by (32), that for every $k \in \{1,\ldots,M\}$, that the mapping

$$L^2(Q) \to L^2(Q) : T \mapsto \int\limits_{-1}^{+1} I_T^k(\cdot,\cdot,\mu) \ d\mu$$

is lipschitzian. In view of the explicit formula (31) for $\int\limits_{-1}^{+1} I_T^k(\cdot,\cdot,\mu) \ d\mu$, we are reduced to the following lemmas. The first one is a variant of a classical result about integral operator of the Hilbert-Schmidt type ([28], pp.197-198):

Lemma 18 *Let* $(x,x')\mapsto K(x,x')$ *be an almost everywhere defined real-valued measurable function on the square* $]0,l[^2$ *such that* $\int\limits_0^l\int\limits_0^l K(x,x')^2dx\otimes dx'<+\infty$. *For every* $\varphi\in L^2(Q)$, *let us set:*

$$\mathcal{K}\varphi(x,t)=\int\limits_0^l K(x,x')\varphi(x',t)dx',\ \forall'(x,t)\in Q. \tag{59}$$

Then, $\mathcal{K}$ *so defined, is a linear continuous operator in* $L^2(Q)$ *whose operator norm satisfies the inequality:*

$$\|\mathcal{K}\|\leq\|K\|_{L^2(]0,l[^2)}=\left(\int\limits_0^l\int\limits_0^l K(x,x')^2dx\otimes dx'\right)^{\frac{1}{2}}. \tag{60}$$

Proof. By Cauchy-Schwarz's inequality and Fubini's theorem:

$$\begin{aligned}\int\limits_{t=0}^{t_f}\int\limits_{x=0}^{l}(\mathcal{K}\varphi(x,t))^2dx\otimes dt&=\int\limits_{t=0}^{t_f}\int\limits_{x=0}^{l}(\int\limits_0^l K(x,x')\varphi(x',t)dx')^2dx\otimes dt\\&\leq\int\limits_{t=0}^{t_f}\int\limits_{x=0}^{l}\left(\int\limits_0^l K(x,x')^2dx'\cdot\int\limits_0^l\varphi(x',t)^2dx'\right)dx\otimes dt\\ \leq\int\limits_0^l\int\limits_0^l K(x,x')^2dx\otimes dx'\int\limits_{t=0}^{t_f}\int\limits_{x'=0}^{l}&\varphi(x',t)^2dx'\otimes dt=\|K\|^2_{L^2(]0,l[^2)}\|\varphi\|^2_{L^2(Q)}.\end{aligned}$$

Thus $\|\mathcal{K}\varphi\|_{L^2(Q)}\leq\|K\|_{L^2(]0,l[^2)}\|\varphi\|_{L^2(Q)}$, $\forall\varphi\in L^2(Q)$, so that:

$$\|\mathcal{K}\|:=\sup_{\|\varphi\|_{L^2(Q)}\leq1}\|\mathcal{K}\varphi\|_{L^2(Q)}\leq\|K\|_{L^2(]0,l[^2)}.$$

■

Corollary 19 *Let* $K(x,x'):=G(x,x'):=\kappa_kE_1(\kappa_k|x-x'|)$ *(24). For every* $\varphi\in L^2(Q)$, *let us set:*

$$\mathcal{G}\varphi(x,t)=\int\limits_0^l G(x,x')\varphi(x',t)dx',\ \forall'(x,t)\in Q. \tag{61}$$

Then, $\mathcal{G}$ *so defined, is a linear continuous operator in* $L^2(Q)$ *whose operator norm satisfies the inequality:*

$$\|\mathcal{G}\|\leq\sqrt{2}\ \kappa_k\left(l^2+\int\limits_0^l\int\limits_0^l\ln(\kappa_k|x-x'|)^2dx\otimes dx'\right)^{\frac{1}{2}}. \tag{62}$$

Proof. In view of lemma 18, it suffices to verify that $\int_0^l\int_0^l G(x,x')^2 dx\otimes dx' < +\infty$ i.e. that

$$\int_0^l\int_0^l E_1(\kappa_k\left|x-x'\right|)^2 dx\otimes dx' < +\infty. \tag{63}$$

But $E_1(y)^2 \leq 2+2\left(\ln y\right)^2$ and thus $E_1(\kappa_k\left|x-x'\right|)^2 \leq 2\left[1+\ln(\kappa_k\left|x-x'\right|)^2\right]$, so that (63) is trivial. (62) follows from (60) and the bound $G(x,x')^2 \leq 2\kappa_k^2\left[1+\ln(\kappa_k\left|x-x'\right|)^2\right]$. ■

Corollary 20 *Let* $K(x,x') := \Phi_1(\kappa_k(2l-x-x'))$*. For every* $\varphi \in L^2(Q)$*, let us set:*

$$\mathcal{U}\varphi(x,t) = \int_0^l K(x,x')\varphi(x',t)dx',\ \forall'(x,t)\in Q. \tag{64}$$

Then, $\mathcal{U}$ *so defined, is a linear continuous operator in* $L^2(Q)$ *whose operator norm*

$$\left\|\mathcal{U}\right\| \leq \sqrt{2}\left(l^2+\int_0^l\int_0^l\left[\ln(\kappa_k(x+x'))\right]^2 dx\otimes dx'\right)^{\frac{1}{2}}. \tag{65}$$

Proof. In view of lemma 18, it suffices to verify that $\int_0^l\int_0^l K(x,x')^2 dx\otimes dx' < +\infty$ i.e. that

$$\int_0^l\int_0^l \Phi_1(\kappa_k(2l-x-x'))^2 dx\otimes dx' < +\infty.$$

But by the definition of Φ_1 (see 29) and due to $0\leq\rho_g(\cdot)\leq 1$: $\Phi_1(\kappa_k(2l-x-x')) \leq E_1(\kappa_k(2l-x-x'))$. Thus

$$\int_0^l\int_0^l \Phi_1(\kappa_k(2l-x-x'))^2 dx\otimes dx' \leq \int_0^l\int_0^l \left[E_1(\kappa_k((l-x)+(l-x')))\right]^2 dx\otimes dx'$$

$$= \int_0^l\int_0^l \left[E_1(\kappa_k\left(y+y'\right))\right]^2 dy\otimes dy' \leq 2l^2+2\int_0^l\int_0^l \left[\ln(\kappa_k\left(y+y'\right))\right]^2 dy\otimes dy' < +\infty.$$

■

Proposition 21 *The nonlinear mapping* $L^2(Q)\to L^2(Q) : T\mapsto h_T$ *is lipschitzian and thus a fortiori continuous.*

Proof. This follows from formulas (31), (32), Proposition 16, Corollary 19 and Corollary 20. ■

We are now in a position to prove the continuity of the mapping Φ:

Theorem 22 *The mapping* Φ *which sends* $T \in S$ *defined by (58) onto* $\tilde{T} \in S$*, the unique weak solution of the initial boundary value problem (41), i.e. the unique* $\tilde{T} \in L^2(0,t_f;H^1(]0,l[))$ *such that* $\frac{d\tilde{T}}{dt} \in L^2(0,t_f;\left[H^1(]0,l[)\right]^*)$ *verifying equation (42), is continuous.*

Proof. So, let us consider a sequence of functions $(T_n)_{n\in\mathbb{N}}$ belonging to S converging to some $T \in S$ in the sense of the norm of $L^2(0,t_f;C([0,l])$. Let $\tilde{T}_n := \Phi(T_n)$, $\forall n \in \mathbb{N}$ and $(\tilde{T}_n)_{n\in\mathbb{N}}$ the corresponding sequence. By the estimate (46) of Theorem 9, the sequence $(\tilde{T}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,t_f;H^1(]0,l[))$ and $(\frac{d\tilde{T}_n}{dt})_{n\in\mathbb{N}}$ is bounded in $L^2(0,t_f;H^1(]0,l[)^*)$. Thus some subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is weakly convergent in $L^2(0,t_f;H^1(]0,l[))$ and $(\frac{d\tilde{T}_{n_k}}{dt})_{k\in\mathbb{N}}$ is weakly convergent in $L^2(0,t_f;H^1(]0,l[)^*)$. The injection from the space $H^1(]0,l[) \hookrightarrow C([0,l])$ being compact [2] and $C([0,l]) \hookrightarrow H^1(]0,l[)^*$, by the compacity Theorem 5.1 p.58 of [15], the continuous embedding from the space $\{\check{T} \in L^2(0,t_f;H^1(]0,l[)); \frac{d\check{T}}{dt} \in L^2(0,t_f;H^1(]0,l[)^*)\}$ endowed with its natural norm into $L^2(0,t_f;C([0,l]))$, is also compact. Thus the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is also strongly convergent in the space $L^2(0,t_f;C([0,l]))$ ([23], p. 199). Let us call $\tilde{T}$ the weak-limit of the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ in the space $L^2(0,t_f;H^1(]0,l[))$ which is also the strong limit in the space $L^2(0,t_f;C([0,l]))$. Using the definition of the weak time derivative ([9], p.39-40), it is easy to see that the weak-limit of the subsequence $(\frac{d\tilde{T}_{n_k}}{dt})_{k\in\mathbb{N}}$ in the space $L^2(0,t_f;H^1(]0,l[)^*)$ is $\frac{d\tilde{T}}{dt}$. We have to prove that $\tilde{T} = \Phi(T)$. For every $k \in \mathbb{N}$: $\tilde{T}_{n_k}(x=0,t) = T_a$, $\forall' t \in]0,t_f[$. Thus we have also $\tilde{T}(x=0,t) = T_a$, $\forall' t \in]0,t_f[$ by the srong convergence of $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ to $\tilde{T}$ in $L^2(0,t_f;C([0,l]))$. Also by Theorem 1.32 p.40 of [9], we have a continuous imbedding from the space

$$\{\check{T} \in L^2(0,t_f;H^1(]0,l[)); \frac{d\check{T}}{dt} \in L^2(0,t_f;H^1(]0,l[)^*)\} \text{ into } C([0,t_f];L^2(]0,l[)).$$

Thus the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is also weakly convergent in the space $C([0,t_f];L^2(]0,l[))$. For every $k \in \mathbb{N}$: $\tilde{T}_{n_k}(\cdot,t=0) = T_0(\cdot)$. Let ψ be an arbitrary function in $L^2(]0,l[)$. The mapping

$$\varphi^* : C([0,t_f];L^2(]0,l[)) \to \mathbb{R} : v \mapsto \int_0^l v(x,t=0)\psi(x)dx$$

is a continuous linear form on the space $C([0,t_f];L^2(]0,l[))$. Thus $\left\langle \varphi^*, \tilde{T}_{n_k} \right\rangle = \int_0^l T_0(x)\psi(x)dx \to \int_0^l \tilde{T}(x,0)\psi(x)dx$, $\forall \psi \in L^2(]0,l[)$. Thus $\tilde{T}(\cdot,0) = T_0(\cdot)$ in $L^2(]0,l[)$. Therefore, we know already that $\tilde{T}$ verifies the adequate initial condition and the adequate boundary condition on the face $\{x_g = 0\}$ of the glass plate. Now, taking an arbitrary fonction $\xi(\cdot) \in L^2(]0,t_f[)$, multiplying both sides of equation (42) by the function ξ and integrating with respect to time from 0 to t_f, we obtain:

$$\left\{\begin{array}{l}
c_p m_g \int\limits_0^{t_f} \left\langle \frac{\partial \tilde{T}_{n_k}}{\partial t}(\cdot,t), \varphi \right\rangle_{H^1(]0,l[)^*, H^1(]0,l[)} \xi(t)\, dt = \\
-k_h \int\limits_0^{t_f}\int\limits_0^{l} \frac{\partial \tilde{T}_{n_k}}{\partial x}(x,t)\varphi'(x)\xi(t)\, dx \otimes dt + \int\limits_0^{t_f}\int\limits_0^{l} \psi(\tilde{T}_{n_k}(x,t))\varphi(x)\xi(t)\, dx \otimes dt \\
-\int\limits_0^{t_f} \Theta(\tilde{T}_{n_k}(l,t))\xi(t)\, dt \cdot \varphi(l) + \int\limits_0^{t_f}\int\limits_0^{l} h_{T_{n_k}}(x,t)\varphi(x)\xi(t)\, dx \otimes dt \\
+\int\limits_0^{t_f} \Theta(T_S(t))\xi(t)\, dt \cdot \varphi(l) + \int\limits_0^{t_f} h_c \left(T_a - \tilde{T}_{n_k}(l,t)\right)\xi(t)\, dt \cdot \varphi(l), \\
\forall \varphi \in H^1(]0,l[) \text{ such that } \varphi(0) = 0, \qquad \forall \xi \in L^2(]0,t_f[).
\end{array}\right. \tag{66}$$

We have seen a few lines above that the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is also strongly convergent to $\tilde{T}$ in the space $L^2(0,t_f;C([0,l]))$, thus a fortiori in the space $L^2(Q)$ (Q denotes $]0,l[\times]0,t_f[$). Thus by (32) and proposition 16, $\psi \circ \tilde{T}_{n_k}$ converges to $\psi \circ \tilde{T}$ in $L^2(Q)$. $\tilde{T}_{n_k}(l,\cdot) \to \tilde{T}(l,\cdot)$ in $L^2(]0,t_f[)$ and thus by proposition 17:

$$\Theta(\tilde{T}_{n_k}(l,\cdot)) \to \Theta(\tilde{T}(l,\cdot)) \text{ in } L^2(]0,t_f[).$$

Using Proposition 21 and all the previous convergence properties to pass to the limit in (66) as $k \to +\infty$, we obtain:

$$\left\{\begin{array}{l}
c_p m_g \int\limits_0^{t_f} \left\langle \frac{\partial \tilde{T}}{\partial t}(\cdot,t), \varphi \right\rangle_{H^1(]0,l[)^*, H^1(]0,l[)} \xi(t)\, dt = \\
-k_h \int\limits_0^{t_f}\int\limits_0^{l} \frac{\partial \tilde{T}}{\partial x}(x,t)\varphi'(x)\xi(t)\, dx \otimes dt + \int\limits_0^{t_f}\int\limits_0^{l} \psi(\tilde{T}(x,t))\varphi(x)\xi(t)\, dx \otimes dt \\
-\int\limits_0^{t_f} \Theta(\tilde{T}(l,t))\xi(t)\, dt \cdot \varphi(l) + \int\limits_0^{t_f}\int\limits_0^{l} h_T(x,t)\varphi(x)\xi(t)\, dx \otimes dt \\
+\int\limits_0^{t_f} \Theta(T_S(t))\xi(t)\, dt \cdot \varphi(l) + \int\limits_0^{t_f} h_c \left(T_a - \tilde{T}(l,t)\right)\xi(t)\, dt \cdot \varphi(l), \\
\forall \varphi \in H^1(]0,l[) \text{ such that } \varphi(0) = 0, \quad \forall \xi \in L^2(]0,t_f[).
\end{array}\right. \tag{67}$$

(67) being true $\forall \xi \in L^2(]0,t_f[)$, we have that $\tilde{T} \in \{\check{T} \in L^2(0,t_f;H^1(]0,l[));\ \frac{d\check{T}}{dt} \in L^2(0,t_f;H^1(]0,l[)^*)\}$ verifies $\forall' t \in]0,t_f[$:

$$\begin{cases} c_p m_g \left\langle \frac{\partial \tilde{T}}{\partial t}(\cdot,t),\varphi \right\rangle_{H^1(]0,l[)^*,H^1(]0,l[)} = -k_h \int\limits_0^l \frac{\partial \tilde{T}}{\partial x}(x,t)\varphi'(x)dx \\ \qquad + \int\limits_0^l \psi(\tilde{T}(x,t))\varphi(x)dx + \int\limits_0^l h_T(x,t)\varphi(x)dx \\ \qquad + \left\{ h_c(T_a - \tilde{T}(l,t)) + \left[\Theta(T_S(t)) - \Theta(\tilde{T}(l,t))\right]\right\}\varphi(l), \\ \qquad \forall \varphi \in H^1(]0,l[) \text{ such that } \varphi(0)=0. \end{cases}$$

In conclusion, $\tilde{T} \in \{\check{T} \in L^2(0,t_f;H^1(]0,l[));\ \frac{d\check{T}}{dt} \in L^2(0,t_f;H^1(]0,l[)^*)\}$ verifies (42). By proposition 13 and proposition 14, $\tilde{T} \in S$. In conclusion $\tilde{T} = \Phi(T)$. We have seen that the subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ is strongly convergent to $\Phi(T)$ in the space $L^2(0,t_f;C([0,l]))$. A standard argument of general topology allows now to conclude that the sequence $(\tilde{T}_n := \Phi(T_n))_{n\in\mathbb{N}}$ itself is strongly convergent to $\Phi(T)$ in the space $L^2(0,t_f;C([0,l]))$. Thus Φ is continuous from S into S. ■

It remains to prove that $\Phi(S)$ is relatively compact in the space $L^2(0,t_f;C([0,l]))$ to be allowed to apply Schauder's Theorem 15.

Proposition 23 *$\Phi(S)$ is relatively compact in the space $L^2(0,t_f;C([0,l]))$.*

Proof. $\|T\|_{L^2(Q)}$ for T running over the closed convex subset S of the space $L^2(0,t_f;C([0,l]))$ is bounded by a constant depending only on the upper bound $\bar{T} \in \mathbb{R}_+^*$ appearing in the definition of S. Thus by the estimate (46) of Theorem 9, the set $\{\Phi(T); T \in S\}$ is bounded in the space

$$\{\check{T} \in L^2(0,t_f;H^1(]0,l[));\ \frac{d\check{T}}{dt} \in L^2(0,t_f;H^1(]0,l[)^*)\}$$

endowed with its natural norm. But, by the compacity Theorem 5.1 p.58 of [15], the continuous embedding from the space $\{\check{T} \in L^2(0,t_f;H^1(]0,l[));\ \frac{d\check{T}}{dt} \in L^2(0,t_f;H^1(]0,l[)^*)\}$ endowed with its natural norm, into $L^2(0,t_f;C([0,l]))$ is compact. Thus, the set $\{\Phi(T); T \in S\}$ is relatively compact in the space $L^2(0,t_f;C([0,l]))$. ■

We are now in a position to apply Schauder's theorem 15 to the mapping $\Phi : S \to S : T \mapsto \tilde{T}$. This mapping has thus at least one fixed point, which gives us the existence of a solution to the initial boundary value problem (35). This proves also that the solution of the initial boundary value problem (35) which we know to be unique by Proposition 8 is lower bounded by T_a and upper bounded by $\bar{T}$. From these bounds on the temperature follows by Lemma 12:

$$2B_g^k(T_a) \le \int\limits_{-1}^{+1} I_T^k(x,t,\mu)\ d\mu \le 2B_g^k(\bar{T}),\ \forall k=1,\ldots,M.$$

Appendix

Lemma 24 *If* $u,\ v \in L^2(]0,l[)$*, then* $\|u_- - v_-\|_{L^2(]0,l[)} \leq \|u-v\|_{L^2(]0,l[)}$*.*

Proof. $\forall x \in]0,l[$: $u_-(x) = \frac{|u(x)|-u(x)}{2}$ and $v_-(x) = \frac{|v(x)|-v(x)}{2}$. Thus:

$$|u_-(x) - v_-(x)| = \left|\frac{|u(x)|-u(x)-|v(x)|+v(x)}{2}\right|$$

$$\leq \frac{1}{2}\left|\ |u(x)|-|v(x)|\ \right| + \frac{1}{2}|v(x)-u(x)| \leq |u(x)-v(x)|\,,\ \forall x \in]0,l[.$$

Therefore:

$$\|u_- - v_-\|_{L^2(]0,l[)} = \sqrt{\int_0^l |u_-(x)-v_-(x)|^2\ dx}$$

$$\leq \sqrt{\int_0^l |u(x)-v(x)|^2\ dx} = \|u-v\|_{L^2(]0,l[)}\,.$$

■

The following result can be deduced from proposition 4 of [1]; we give only here a direct proof based on Vitali's theorem and lemma A.4 p.53 of [11].

Proposition 25 *The nonlinear mapping* $H^1(]0,l[) \to H^1(]0,l[) : \psi \mapsto \psi_-$ *is continuous. Similarly, the nonlinear mapping* $H^1(]0,l[) \to H^1(]0,l[) : \psi \mapsto \psi_+$ *is also continuous.*

Proof. Let $\psi \in H^1(]0,l[)$ and $(\psi_n)_{n\in\mathbb{N}} \subset H^1(]0,l[)$ a sequence tending to ψ in the norm of $H^1(]0,l[)$. A fortiori, $\psi_n \to \psi$ in $L^2(]0,l[)$. Consequently, by the preceding lemma: $\psi_n^- \to \psi_-$ in $L^2(]0,l[)$ (to alleviate the notation, we have denoted the negative part of ψ_n, ψ_n^-). To conclude that $\psi_n^- \to \psi_-$ in $H^1(]0,l[)$, it remains to prove that $\frac{d\psi_n^-}{dx} \to \frac{d\psi_-}{dx}$ in $L^2(]0,l[)$. By ([11], pp.50-54):

$$\frac{d\psi_n^-}{dx} = -\frac{d\psi_n}{dx}\mathbf{1}_{\{\psi_n<0\}} \text{ and } \frac{d\psi_-}{dx} = -\frac{d\psi}{dx}\mathbf{1}_{\{\psi<0\}}. \tag{68}$$

As $H^1(]0,l[) \hookrightarrow C([0,l])$, we have also that $\psi_n \to \psi$ in $C([0,l])$. The sequence $\left(\frac{d\psi_n}{dx}\right)_{n\in\mathbb{N}}$ tends to $\frac{d\psi}{dx}$ in $L^2(]0,l[)$, and therefore there exists a subsequence $\left(\frac{d\psi_{n_k}}{dx}\right)_{k\in\mathbb{N}}$ which converges a.e. on $]0,l[$ to $\frac{d\psi}{dx}$. Let us show that the sequence $\left(\frac{d\psi_{n_k}^-}{dx}\right)_{k\in\mathbb{N}}$ converges a.e. to $\frac{d\psi_-}{dx}$.

First case: $x \in [0,l]$ and $\psi(x) > 0$. For almost every such x, $\frac{d\psi_{n_k}}{dx}(x) \to \frac{d\psi}{dx}(x)$ and let us consider such an x. We have also that $\psi_{n_k}(x) \to \psi(x)$ as $k \to +\infty$, and therefore for k sufficiently large, we will have also $\psi_{n_k}(x) > 0$. Therefore by formula (68), we have evidently that $\frac{d\psi_{n_k}^-}{dx}(x) \to \frac{d\psi_-}{dx}(x)$.

Second case: $x \in [0,l]$ such that $\psi(x) < 0$. For almost every such x, $\frac{d\psi_{n_k}}{dx}(x) \to$

$\frac{d\psi}{dx}(x)$ and let us consider such an x. We have also that $\psi_{n_k}(x) \to \psi(x)$ as $k \to +\infty$, and therefore for k sufficiently large, we will have also $\psi_{n_k}(x) < 0$. Therefore by formula (68), we have that $\frac{d\psi^-_{n_k}}{dx}(x) \to \frac{d\psi_-}{dx}(x)$.
Third case: $x \in [0,l]$ such that $\psi(x) = 0$. Fortunately as $\psi \in H^1(]0,l[)$ by lemma A.4 p.53 of [11], for almost such x, $\frac{d\psi}{dx}(x) = 0$. Thus let us consider $x \in [0,l]$ such that $\psi(x) = 0$ and $\frac{d\psi}{dx}(x) = 0$. For almost every such x, $\frac{d\psi_{n_k}}{dx}(x) \to \frac{d\psi}{dx}(x)$. By formula (68): $\frac{d\psi_-}{dx}(x) = 0$ and $\left|\frac{d\psi^-_{n_k}}{dx}(x)\right| \leq \left|\frac{d\psi_{n_k}}{dx}(x)\right| \to 0$. Thus still $\frac{d\psi^-_{n_k}}{dx}(x) \to \frac{d\psi_-}{dx}(x)$.
We have thus proved that the sequence $\left(\frac{d\psi^-_{n_k}}{dx}\right)_{k\in\mathbb{N}}$ converges a.e. to $\frac{d\psi_-}{dx}$. To prove that the sequence $\left(\frac{d\psi^-_{n_k}}{dx}\right)_{k\in\mathbb{N}}$ converges also to $\frac{d\psi_-}{dx}$ in $L^2(]0,l[)$, we apply Vitali's theorem ([20], p.16). As $\frac{d\psi_{n_k}}{dx} \to \frac{d\psi}{dx}$ a.e. and in $L^2(]0,l[)$, by the necessary part of Vitali's theorem $\int\limits_A \left|\frac{d\psi_{n_k}}{dx}(x)\right|^2 dx \to 0$ uniformly in k as $meas(A) \to 0$, A arbitrary measurable subset of $[0,l]$. But $\left|\frac{d\psi^-_{n_k}}{dx}(x)\right| \leq \left|\frac{d\psi_{n_k}}{dx}(x)\right|$ and thus also $\int\limits_A \left|\frac{d\psi^-_{n_k}}{dx}(x)\right|^2 dx \to 0$ uniformly in k as $meas(A) \to 0$, A arbitrary measurable subset of $[0,l]$. As we have shown that $\left(\frac{d\psi^-_{n_k}}{dx}\right)_{k\in\mathbb{N}}$ converges a.e. to $\frac{d\psi_-}{dx}$, by the sufficient part of Vitali's theorem, $\left(\frac{d\psi^-_{n_k}}{dx}\right)_{k\in\mathbb{N}}$ converges also to $\frac{d\psi_-}{dx}$ in $L^2(]0,l[)$. A standard argument of general topology allows now to conclude that the sequence $\left(\frac{d\psi^-_{n}}{dx}\right)_{n\in\mathbb{N}}$ itself is strongly convergent to $\frac{d\psi_-}{dx}$ in the space $L^2(]0,l[)$. In conclusion, $\psi^-_n \to \psi_-$ in $H^1(]0,l[)$. ■

Proposition 26 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^2(0,t_f;H^1(]0,l[))$ *converging to* $u \in L^2(0,t_f;H^1(]0,l[))$. *Let us set*

$$u^- :]0,t_f[\to H^1(]0,l[) : t \mapsto (u(t))_-$$

and

$$u^+ :]0,t_f[\to H^1(]0,l[) : t \mapsto (u(t))_+.$$

Let us define similarly u^-_n *and* u^+_n, $\forall n \in \mathbb{N}$. *Then the sequence* $(u^-_n)_{n\in\mathbb{N}}$ *(resp.* $(u^+_n)_{n\in\mathbb{N}}$*) converges to* u^- *(resp.* u^+*) in* $L^2(0,t_f;H^1(]0,l[))$.

Proof. We give the proof for $(u^-_n)_{n\in\mathbb{N}}$, the proof for the sequence $(u^+_n)_{n\in\mathbb{N}}$ being similar. By the necessary part of Vitali's theorem ([20], p.16) $\int\limits_A \|u_n(t)\|^2_{H^1(]0,l[)}\, dt$

$\to 0$ uniformly in $n \in \mathbb{N}$ as $meas(A) \to 0$, A arbitrary measurable subset of $]0, t_f[$. Also there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ which converges a.e. on $]0, t_f[$ to u, as vectorvalued functions with values in $H^1(]0, l[)$. By the previous proposition, the subsequence $\left(u^-_{n_k}\right)_{k \in \mathbb{N}}$ converges to u^- a.e. on $]0, t_f[$ .As $\left\|u^-_{n_k}(t)\right\|_{H^1(]0,l[)} \leq \|u_{n_k}(t)\|_{H^1(]0,l[)}$, it follows that a fortiori $\int\limits_A \left\|u^-_{n_k}(t)\right\|^2_{H^1(]0,l[)} dt$ $\to 0$ uniformly in k as $meas(A) \to 0$, A arbitrary measurable subset of $]0, t_f[$. Applying the sufficient part of Vitali's theorem ([20], p.16) to the subsequence $\left(u^-_{n_k}\right)_{k \in \mathbb{N}}$, implies that $\left(u^-_{n_k}\right)_{k \in \mathbb{N}}$ converges to u^- in $L^2(0, t_f; H^1(]0, l[))$. A standard argument of general topology shows that the sequence $(u^-_n)_{n \in \mathbb{N}}$ itself, converges to u^- in $L^2(0, t_f; H^1(]0, l[))$. ■

Ackowledgements: The authors gratefully acknowledge the supports of the institutions.

References

[1] A. Ancona, "Continuité des contractions dans les espaces de Dirichlet", Séminaire de théorie du potentiel, n° 2, p.1-26, Springer-Verlag (1976) Lecture Notes in Mathematics 563. Zbl 0341.31006

[2] H. Brézis, "Analyse fonctionnelle Théorie et applications", Masson (1993). Zbl 1147.46300

[3] D. Clever, J. Lang, "Optimal Control of Radiative Heat Transfer in Glass Cooling with Restrictions on the Temperature Gradient", Optim. Control Appl. Meth. (2011), Published online in Wiley Online Library, DOI:10|1002|oca.974.

[4] R. Dautray, J.-L. Lions, "Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques Volume 8 Evolution: semi-groupe, variationnel", Masson Paris (1988). Zbl 0749.35004

[5] F. Desvignes, "Rayonnements optiques Radiométrie-Photométrie", Masson, Paris (1991).

[6] P.-E. Druet, "Weak solutions to a time-dependent heat equation with nonlocal radiation boundary condition and arbitrary p-summable right-hand side", Applications of Mathematics, vol. 55 (2010), N°2, pp.111-149.

[7] L.C. Evans, "Partial Differential Equations", GSM Volume 19, AMS Providence, Rhode Island (1999). Zbl 0902.35002

[8] A. Friedman, "Partial Differential Equations", Robert E. Krieger Publishing Company (1976). Zbl 0224.35002

[9] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, "Optimization with PDE Constraints", Springer (2009). Zbl 1167.49001

[10] H.C. Hottel, A.F. Sarofim, "Radiative Transfer", McGraw-Hill Book Company (1967).

[11] D. Kinderlehrer, G. Stampacchia, "An Introduction to Variational Inequalities and their Applications", Academic Press (1980). Zbl 0457.35001

[12] J. Lang, "Adaptive computation for boundary control of radiative heat tranfer in glass", Journal of Computational and Applied Mathematics 183 (2005), 312-326. Zbl 1100.65073

[13] M.T. Laitinen, T. Tiihonen, "Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials", Math. Methods Appl. Sci. 21 (1998), no. 5, pp. 375-392. Zbl 0958.80003

[14] M.T. Laitinen, "Mathematical modelling of conductive-radiative heat transfer", Ph.D. Dissertation, University of Jyväskylä, Finland(2000), published by Jyväskylän yliopiston kirjasto/Julkaisuyksikkö.

[15] J.-L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires", Dunod (1969). Zbl 0189.40603

[16] A. Maréchal, "Optique Géométrique Générale", in Encyclopedia of Physics, volume XXIV: Fundamentals of Optics with 761 figures, edited by S. Flügge, Springer-Verlag (1956), pp. 44-170.

[17] C.-M. Marle, "Mesures et probabilités", Hermann (1974).

[18] C. Meyer, P. Philip, F. Tröltzsch, "Optimal control of a semilinear PDE with nonlocal radiation interface conditions", SIAM J. Control Optim. 45, pp.699-721 (2006). Zbl 1109.49026

[19] M.F. Modest, "Radiative Heat Transfer", second edition, Academic Press (2003).

[20] D. Pascali, S. Sburlan, "Nonlinear mappings of monotone type", Editura Academiei, Bucuresti, Romania, Sijthoff & Noordhoff International Publishers (1978).

[21] R. Pinnau, "Analysis of optimal boundary control for radiative heat ransfer modeled by the SP_1-system", Communications in Mathematical Sciences, 5,4:951-969, (2007). Zbl 1145.49015

[22] M. Planck, "The Theory of Heat Radiation", Dover Publications Inc. (1991). Zbl 0127.21701

[23] M. Reed, B. Simon, "Methods of Modern Mathematical Physics I: Functional Analysis", Academic Press (1972). Zbl 0242.46001

[24] N. Siedow, T. Grosan, D. Lochegnies, E. Romero, "Application of a New Method for Radiative Heat Tranfer to Flat Glass Tempering", *J. Am. Ceram. Soc.,* **88** [8] 2181-2187 (2005).

[25] L. Soudre, “Etude numérique et expérimentale du thermoformage d’une plaque de verre”, thèse soutenue le 9 décembre 2008, Nancy-Université Henri Poincaré, laboratoire LEMTA, tel-00382905, version 1-11 May 2009.

[26] J. Taine, E. Iacona, J.-P. Petit, “Transferts Thermiques Introduction aux transferts d’énergie”, 4^e edition Dunod (2008).

[27] J. Taine, J.-P. Petit, “Heat Transfer”, Prentice Hall (1993). Zbl 0818.73001

[28] K. Yosida, “Functional Analysis”, Third Edition, Springer-Verlag (1971). Zbl 0830.46001

Authors’ addresses:
· Corresponding author: Luc Paquet, Université de Valenciennes et du Hainaut-Cambrésis (UVHC), Institut des Sciences et Techniques de Valenciennes (ISTV), LAMAV-EDP FR n°2956, 59313 Valenciennes, France, e-mail: luc.paquet@univ-valenciennes.fr, paquet.luc@wanadoo.fr (at home);
· Raouf El Cheikh, Université de Valenciennes et du Hainaut-Cambrésis (UVHC), Institut des Sciences et Techniques de Valenciennes (ISTV), LAMAV-EDP FR n°2956, 59313 Valenciennes, France, also: FB Mathematik, TU Kaiserslautern, Erwin-Schrödinger-Str., Gebäude 48 & Fraunhofer Institute for Industrial Mathematics, ITWM, 67663 Kaiserslautern, Germany, e-mail: raouf.elcheikh@univ-valenciennes.fr, raouf_cheikh@hotmail.com;
· Dominique Lochegnies, Université de Valenciennes et du Hainaut-Cambrésis (UVHC), Institut des Sciences et Techniques de Valenciennes (ISTV), TEMPO, 59313 Valenciennes, France, e-mail: Dominique.Lochegnies@univ-valenciennes.fr;
· Norbert Siedow, Fraunhofer Institute for Industrial Mathematics, ITWM, 67663 Kaiserslautern, Germany, e-mail: norbert.sideow@itwm.fraunhofer.de.