Journal of Abstract Differential Equations and Applications
Volume 2, Number 1, pp. 1–25 (2011)

POLYNOMIAL DECAY RATE FOR A WAVE EQUATION WITH WEAK DYNAMIC BOUNDARY FEEDBACK LAWS

DENIS MERCIER*
Université de Valenciennes et du Hainaut-Cambrésis,
Laboratoire de Mathématiques et ses Applications de Valenciennes and FR CNRS 2956
Le Mont Houy, 59313 VALENCIENNES Cedex 9, FRANCE

SERGE NICAISE†
Université de Valenciennes et du Hainaut-Cambrésis,
Laboratoire de Mathématiques et ses Applications de Valenciennes and FR CNRS 2956
Le Mont Houy, 59313 VALENCIENNES Cedex 9, FRANCE

(Communicated by Editor-in-Chief)

Abstract. We consider the stabilization of the wave equation by weak dynamic boundary feedback laws at one extremity. This system is not uniformly stable but we prove a polynomial stability. Our method consists in combining an observability inequality for the associated undamped problem obtained via sharp spectral results with regularity results of the solution of the undamped problem with a specific right-hand side. The optimality of the decay is also shown again thanks to precise spectral results of the operator associated with the damped problem.

AMS Subject Classification: 35L05, 93D15.

Keywords: dynamic boundary control, stability.

Received: Month Day, 2010 || Accepted: Month Day, 2010

1 Introduction

We consider the following one-dimensional evolution problem with a dynamical control at one extremity, described as follows:

*e-mail address: denis.mercier@univ-valenciennes.fr
†e-mail address: serge.nicaise@univ-valenciennes.fr

$$
(P_b) \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, & 0 < x < 1, t > 0, \\ y(0,t) = 0, & t > 0, \\ ay_{tt}(1,t) + y_x(1,t) + \eta(t) = 0, & t > 0, \\ \eta_t(t) - y_t(1,t) + b\eta(t) = 0, & t > 0, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & 0 < x < 1, \\ \eta(0) = \eta_0, \end{cases}
$$

where y represents the transverse displacement of the vibrating string and η denotes the dynamical control variable. Here a and b are positive constants. The damping of the system is made via the indirect damping mechanism at the extremety 1 that involves a vibrating mass and the first order differential equation in η. The notion of indirect damping mechanisms has been introduced by Russell in [19], and since this time, it retains the attention of many authors, for instance, let us quote the papers of Alabau [1, 2] for recent general studies on hyperbolic systems with indirect boundary stabilizations. Note nevertheless that the above system does not enter in the framework of these papers.

The case $a = 0$ was considered in [21] where a polynomial decay in $\frac{1}{t}$ was proved for initial data in the domain of the associated operator. In the case $a > 0$ the third equation in (P_b) is a so–called dynamic boundary condition. Dynamic boundary conditions arise in many physical applications, in particular they occur in elastic models. For instance, these conditions appear in modeling dynamic vibrations of linear viscoelastic rods and beams which have attached tip masses at their free ends. See [4, 9, 12, 16, 8, 13] and the references therein for more details. Using the compact perturbation result of Russell [18], the dissipative system (P_b) is not uniformly stable (see below). Hence we are interested in proving a weaker decay of the energy. More precisely we will show that the energy of our system decay polynomially as $\frac{1}{\sqrt{t}}$ for initial data in the domain of the associated operator. Contrary to [21] we here use a technique, inspired from [3, 14], that consists in combining an observability inequality for the associated undamped problem obtained via sharp spectral results with regularity results of the solution of the undamped problem with a specific right-hand side. This allows to obtain a polynomial decay for less regular initial data. Moreover using a careful spectral analysis of the operator associated with (P_b) we show that our obtained decay rate is optimal. In comparison with [21], we can say that, with the same regularity assumption on the initial data, the additional term $ay_{tt}(1,t)$ in the feedback law affects the decay rate.

The paper is organized as follows. The second section deals with the well-posedness of the problem obtained by using semigroup theory. In section 3, we perform the spectral analysis of the operator associated with the conservative system and deduce the strong stability of the dissipative system. Section 4 is devoted to the proof of some regularity results on the conservative system with a special right-hand side. In section 5 we show the polynomial decay via the observability estimate and these regularity results. Finally the optimality of the decay is shown in section 6.

Let us finish this introduction with some notation used in the remainder of the paper: For a bounded domain D, the usual norm and semi-norm of $H^s(D)$ $(s \geq 0)$ are denoted by $\|\cdot\|_{H^s(D)}$ and $|\cdot|_{H^s(D)}$, respectively. Furthermore, the notation $A \lesssim B$ and $A \sim B$ means the existence of positive constants C_1 and C_2, which are independent of A and B such that $A \leq C_2 B$ and $C_1 B \leq A \leq C_2 B$.

2 Well-posedness results

In order to study the system (P_b) we use a reduction order argument and introduce the new unknown $ay_t(1,\cdot)$. Hence if we set

$$
u := (y, y_t, ay_t(1,\cdot), \eta),
$$

as vectorial unknown, the previous problem (P_b) is formally equivalent to

$$u_t = \mathscr{A} u, \; u(0) = u_0, \tag{2.1}$$

where $u_0 = (y_0, y_1, ay_1(1), \eta_0)$ and $\mathscr{A}(y,z,\xi,\eta) = (z, y_{xx}, -y_x(1) - \eta, z(1) - b\eta)$ if $\xi = az(1)$.

In order to justify this formal argument, we need a proper functional setting. Let

$$V = \{y \in H^1(0,1) : y(0) = 0\}$$

and define the energy space

$$\mathscr{H} = V \times L^2(0,1) \times \mathbb{C}^2,$$

endowed with the inner product

$$(u,\tilde{u})_{\mathscr{H}} = \int_0^1 < y_x, \tilde{y}_x > dx + \int_0^1 < z, \tilde{z} > dx + \frac{1}{a} < \xi, \tilde{\xi} > + < \eta, \tilde{\eta} >,$$

where $u = (y,z,\xi,\eta)$, $\tilde{u} = (\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta}) \in \mathscr{H}$, and $< .,. >$ represents the Hermitian product in $\mathbb{C}$. The associated norm will be denoted by $\|\cdot\|_{\mathscr{H}}$. Next, we define the linear operator $\mathscr{A}$ by

$$\begin{array}{l} \mathscr{D}(\mathscr{A}) = \{u = (y,z,\xi,\eta) \in \mathscr{H} : y \in H^2(0,1), z \in V, \xi = az(1)\}, \\ \mathscr{A}u = (z, y_{xx}, -y_x(1) - \eta, z(1) - b\eta), \; \forall u = (y,z,\xi,\eta) \in \mathscr{D}(\mathscr{A}). \end{array}$$

The following proposition concerns the well-posedness of problem (2.1) in $\mathscr{H}$.

Proposition 2.1. *(i) For an initial datum $u_0 \in \mathscr{H}$, there exists a unique solution $u \in C([0,+\infty), \mathscr{H})$ to problem (2.1). Moreover, if $u_0 \in \mathscr{D}(\mathscr{A})$, then*

$$u \in C([0,+\infty), \mathscr{D}(\mathscr{A})) \cap C^1([0,+\infty), \mathscr{H}). \tag{2.2}$$

(ii) For each $u_0 \in \mathscr{D}(\mathscr{A})$, the energy $E(t)$ of the solution u of (2.1), defined by

$$E(t) = \frac{1}{2}\|u(t)\|^2_{\mathscr{H}},$$

satisfies

$$E_t(t) = -b|\eta(t)|^2, \tag{2.3}$$

therefore the energy is non-increasing.

Proof. (i) By Lumer-Phillips' theorem (see [15, 20]), it suffices to show that $\mathscr{A}$ is maximal dissipative.

We first prove that $\mathscr{A}$ is dissipative. Take $u = (y,z,\xi,\eta) \in \mathscr{D}(\mathscr{A})$. Then, integrating by parts and since $u \in \mathscr{D}(\mathscr{A})$, we have

$$\begin{aligned} \Re[(\mathscr{A}u,u)_{\mathscr{H}}] &= \Re[\textstyle\int_0^1 < z_x, y_x > dx + \int_0^1 < y_{xx}, z > dx \\ &- \frac{1}{a} < y_x(1) + \eta, \xi > + < z(1) - b\eta, \eta >] \\ &= \Re[< y_x(1), z(1) > - < y_x(0), z(0) > \\ &- \frac{1}{a} < y_x(1) + \eta, az(1) > + < z(1) - b\eta, \eta >] \\ &= \Re[< y_x(1), z(1) > - < y_x(1) + \eta, z(1) > + < z(1) - b\eta, \eta >] \\ &= \Re[-b < \eta, \eta >] \\ &= -b|\eta|^2. \end{aligned} \tag{2.4}$$

This shows the dissipativeness of $\mathscr{A}$.

Let us now prove that $\mathscr{A}$ is maximal, i.e. that $\lambda I - \mathscr{A}$ is surjective for some $\lambda > 0$.

Let $u_1 = (y_1, z_1, \xi_1, \eta_1) \in \mathscr{H}$. We look for $u = (y, z, \xi, \eta) \in \mathscr{D}(\mathscr{A})$ solution of $(\lambda I - \mathscr{A})u = u_1$, or equivalently

$$\begin{cases} (i) & \lambda y - z = y_1, \\ (ii) & \lambda z - y_{xx} = z_1, \\ (iii) & \lambda \xi + y_x(1) + \eta = \xi_1, \\ (iv) & \lambda \eta - z(1) + b\eta = \eta_1. \end{cases} \tag{2.5}$$

Suppose that we have found u solution of (2.5) with the appropriate regularity. Then from (i) and (ii), y satisfies

$$\lambda^2 y - y_{xx} = z_1 + \lambda y_1, \tag{2.6}$$

where we remark that the function $f_1 = z_1 + \lambda y_1$ is in $L^2(0,1)$. The general solution of (2.6) is

$$y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + \frac{1}{2\lambda}\left(\int_{x-1}^{0} e^{\lambda u} f_1(x-u)du + \int_0^x e^{-\lambda u} f_1(x-u)du\right), \forall x \in [0,1], \tag{2.7}$$

where c_1, c_2 are constants.

Since $y(0) = 0$, we get

$$c_2 = -c_1 - \frac{1}{2\lambda}\int_0^1 e^{-\lambda s} f_1(s)ds. \tag{2.8}$$

Now from (i), $z = \lambda y - y_1$ and recalling that

$$\xi = az(1) = a\lambda y(1) - y_1(1),$$

the two last equations $(iii), (iv)$ of (2.5) lead to the following linear system with unknowns c_1 and η:

$$\begin{cases} \lambda e^{-\lambda}\left(1 + e^{2\lambda} + a\left(-1 + e^{2\lambda}\right)\lambda\right)c_1 + \eta = v_1(u_1, \lambda) \\ \left(e^{-\lambda} - e^{\lambda}\right)\lambda c_1 + (\lambda - b)\eta = v_2(u_1, \lambda), \end{cases}$$

where $v_i(u_1, \lambda), i = 1, 2$, depend only on the datum u_1 and on λ. This system admits a unique solution if and only if its determinant is different from zero. This determinant is actually equal to

$$\lambda e^{-\lambda}\left(-1 - b + e^{2\lambda} - b e^{2\lambda} + \lambda + ab\lambda + e^{2\lambda}\lambda - abe^{2\lambda}\lambda - a\lambda^2 + ae^{2\lambda}\lambda^2\right).$$

Hence it is clear that this determinant does not vanish for λ large enough. For such a $\lambda > 0$ the solution (c_1, η) furnishes a solution $u = (y, z, \xi, \eta)$ through the identities (2.7) and (2.8). Finally we easily verify that $u \in \mathscr{D}(\mathscr{A})$ and consequently $\mathscr{A}$ is maximal.

(ii) For an initial datum in $\mathscr{D}(\mathscr{A})$ from (2.2), we know that u is of class C^1 in time, thus we can derive the energy $E(t)$:

$$E_t(t) = Re[(u_t, u)_{\mathscr{H}}] = Re[(\mathscr{A}u, u)_{\mathscr{H}}] = -b|\eta(t)|^2 \le 0.$$

Hence the energy is non-increasing. □

To end up this section, let us notice that for an initial datum in $\mathscr{D}(\mathscr{A})$, it is a simple exercice to check that the solution $u = (y, z, \xi, \eta)$ of (2.1) having the regularity (2.2), it restores a solution (y, η) of problem (2.1) with the relations $z = y_t$, $\xi = ay_t(1, \cdot)$.

3 The conservative problem

3.1 Formulation

In this section, we introduce another operator A in $\mathscr{H}$, which corresponds to the conservative problem associated with (P_b):

$$(P_0) \qquad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, & 0 \leq x \leq 1, \\ y(0,t) = 0, \\ ay_{tt}(1,t) + y_x(1,t) + \eta(t) = 0, \\ \eta_t(t) - y_t(1,t) = 0. \end{cases}$$

In other words, we have put $b = 0$ in problem (P_b).

The abstract formulation of (P_0) is:

$$(P_0) \qquad u_t = Au,\ u(0) = u_0,\ u_0 \in \mathscr{H},$$

where the linear operator A is defined by:

$$\mathscr{D}(A) = \mathscr{D}(\mathscr{A}),$$
$$Au = (z, y_{xx}, -y_x(1) - \eta, z(1)),\ \forall u = (y, z, \xi, \eta) \in \mathscr{D}(A).$$

A is clearly a skew adjoint operator with a compact resolvent, then there is an orthonormal system of eigenvectors of A which is complete in $\mathscr{H}$. Moreover, from the proof of Proposition 2.1 we see that A is dissipative and monotone. Consequently the well-posedness of problem (P_0) holds.

Proposition 3.1. *For an initial datum $u_0 \in \mathscr{H}$, there exists a unique solution*

$$u \in C([0, +\infty), \mathscr{H})$$

to problem (P_0). Moreover, if $u_0 \in \mathscr{D}(\mathscr{A})$, then $u \in C([0, +\infty), \mathscr{D}(\mathscr{A})) \cap C^1([0, +\infty), \mathscr{H})$.

Before going on let us notice that as A is skew adjoint and $\mathscr{A} - A$ is a compact operator, using the compact perturbation result of Russell [18], the dissipative system (P_b) is not uniformly stable (i.e. exponentially stable).

3.2 Spectral analysis

Here we give some properties of the system of eigenvalues and eigenvectors associated with the problem (P_0), that we need in the sequel.

3.2.1 The characteristic equation

The main result of this section is in the following lemma:

Lemma 3.2. *Denote $\{\lambda_k = i\mu_k\}_{k\in\mathbb{Z}}$ the sequence of eigenvalues of A, the sequence μ_k being strictly monotone, then the family $\{\mu_k\}_{k\in\mathbb{Z}}$ are the roots of the characteristic equation*

$$h(\mu) = \mu \cos \mu + (1 - a\mu^2) \sin \mu = 0. \tag{3.1}$$

The algebraic and geometrical multiplicity of each eigenvalue is one. The associated eigenvectors ϕ_k are :

$$\begin{cases} \phi_0 = (y_0, z_0, \xi_0, \eta_0) = \dfrac{1}{\sqrt{2}}(x, 0, 0, -1), \\ \phi_k = (y_k, z_k, \xi_k, \eta_k) = \dfrac{1}{\sqrt{N(\mu_k)}}(\sin(\mu_k x), i\mu_k \sin(\mu_k x), ia\mu_k \sin(\mu_k), \sin(\mu_k)), \forall k \in \mathbb{Z} \setminus \{0\}, \end{cases} \tag{3.2}$$

where the factor of normalization $N(\mu)$ is:

$$N(\mu) = \mu^2 + (1 + a\mu^2) \sin^2 \mu. \tag{3.3}$$

Moreover the system of eigenvectors $\{\phi_k\}_{k\in\mathbb{Z}}$ is complete in $\mathscr{H}$ and forms an orthonormal basis of $\mathscr{H}$.

Proof. First, it is easy to see that 0 is an eigenvalue of A with multiplicity 1 and the associated eigenvector ϕ_0 (satisfying $\| \phi_0 \|_{\mathscr{H}} = 1$) is given in (3.2).

Now, let $\lambda \neq 0$, and let $\phi = (y, z, \xi, \eta)$ be a non-trivial associated eigenvector. This means that ϕ satisfies:

$$A\phi = (z, y_{xx}, -y_x(1) - \eta, z(1)) = \lambda(y, z, \xi, \eta), \tag{3.4}$$

$$y(0) = z(0) = 0, \tag{3.5}$$

$$\xi = az(1). \tag{3.6}$$

Eliminating z in (3.4) we see that

$$y_{xx} = \lambda^2 y.$$

Thus y is a linear combination of the fundamental basis $(e^{\lambda}, e^{-\lambda})$. From (3.5), we get

$$y(x) = \sinh(\lambda x),$$

(up to a factor) and $z(x) = \lambda \sinh(\lambda x)$.

From (3.4) and (3.6), $y(1) = \eta$ and $-y_x(1) - \eta = (a\lambda^2 + 1)y(1)$. Eliminating η we find

$$-y_x(1) = (a\lambda^2 + 1)y(1),$$

which leads to

$$h_0(\lambda) = (a\lambda^2 + 1) \sinh \lambda + \lambda \cosh \lambda = 0. \tag{3.7}$$

In the previous characteristic equation, we set $\lambda = i\mu, \mu \in \mathbb{R}$, and we conclude that ϕ is a non-trivial solution of the eigenvalue problem if and only if μ satisfies the characteristic equation (3.1).

Remark that our above considerations show that the eigenvector ϕ corresponding to $\lambda = i\mu$ takes the form

$$\phi = (y, z, \xi, \eta) = (\sin(\mu x), i\mu \sin(\mu x), ia\mu \sin \mu, \sin \mu), \tag{3.8}$$

and a simple calculation of its norm gives

$$\|\phi\|^2_{\mathscr{H}} = N(\mu),$$

where $N(\mu)$ is given by (3.3).

It remains to look at the algebraic multiplicity: Since $h'(0) = 2$, the algebraic multiplicity of 0 is one. Suppose that $\mu \in \mathbb{R}^*$ satisfies $h(\mu) = h'(\mu) = 0$, then we have :

$$\begin{cases} (i) & \mu \cos \mu + \left(1 - a\,\mu^2\right) \sin \mu = 0, \\ (ii) & \left(2 - a\,\mu^2\right) \cos \mu - (1 - 2a)\mu \sin \mu = 0. \end{cases}$$

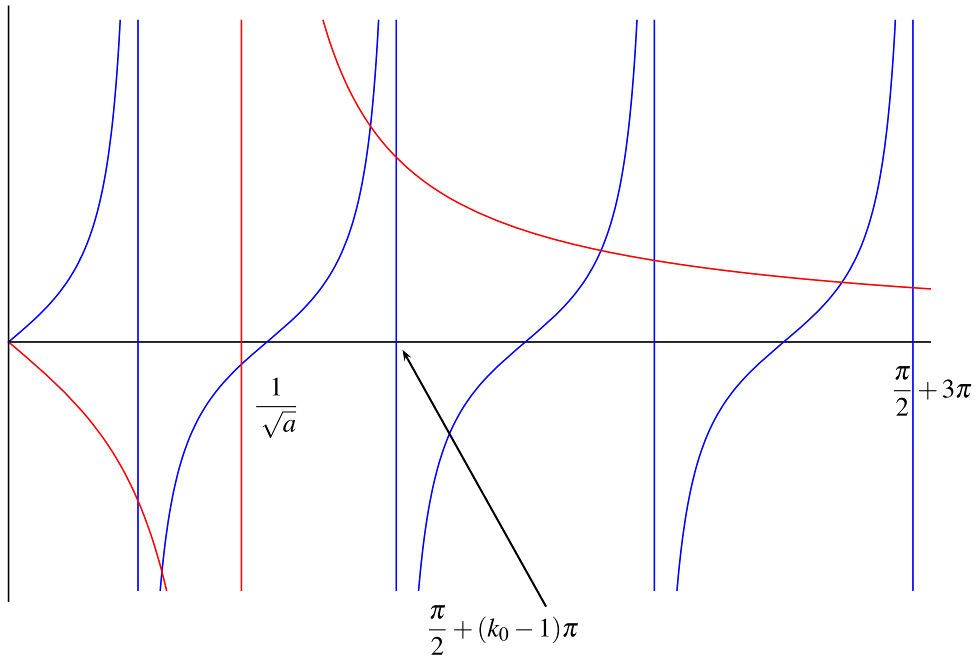

Figure 1. The transcendental equation $\tan x = \dfrac{x}{ax^2-1}$ with $a=\dfrac{1}{8}$, $k_0=2$.

From (i), we eliminate $\cos\mu$ in (ii) to deduce that

$$\frac{\left(2-(-1+a)\,\mu^2+a^2\,\mu^4\right)\,\sin\mu}{\mu}=0.$$

Thanks to (i), we see that $\sin\mu \neq 0$. As we easily check that the polynomial $2-(-1+a)\,\mu^2+a^2\,\mu^4$ has no roots in $\mathbb{R}$, for all $a>0$, we conclude that such a μ cannot exist. □

Remark 3.3. If $a=0$ in problem (P_b), then problem (P_b) becomes (formally) the problem studied in [21]. Moreover we see that $\lim_{a\to 0} h_0(z)$ corresponds to the characteristic equation of the associated conservative problem encountered in [21]. Hence an asympotic analysis as a goes to zero could be of interest.

3.2.2 Asymptotic behavior of the eigenelements

Since for all $k\in\mathbb{Z}, \mu_{-k}=-\mu_k$ and $\phi_{-k}=\overline{\phi_k}$, we only need to study the pair (μ_k,ϕ_k) for all $k\in\mathbb{N}^*$.

Lemma 3.4. *For $k\to+\infty$, we have the following asymptotic behaviors*

$$\mu_{k+1}=k\pi+\frac{1}{ak\pi}+o(\frac{1}{k^2}), \tag{3.9}$$

$$N(\mu_{k+1})=k^2\pi^2+o(k), \tag{3.10}$$

and

$$\eta_{k+1}=(-1)^k\frac{1}{ak^2\pi^2}+o(\frac{1}{k^2}), \tag{3.11}$$

where N is given by (3.3) and η_{k+1} is the last component of ϕ_{k+1} (see Lemma 3.2, (3.2)) .

Proof. If there exists $l\in\mathbb{N}$ which satisfies $a(\dfrac{\pi}{2}+l\pi)^2=1$, then $\mu_{k_0}=\dfrac{\pi}{2}+l\pi=\dfrac{1}{\sqrt{a}}$ is a root of h (this case furnishes at most two roots $\pm\mu_{k_0}$). On the contrary (i.e if $a(\dfrac{\pi}{2}+l\pi)^2\neq 1, \forall l\in\mathbb{N}$),

$h(\mu_k) = 0$ is equivalent with

$$\tan \mu_k = \frac{\mu_k}{a\mu_k^2 - 1}.$$

Let $k_0 \in \mathbb{N}^*$ be the smallest integer satisfying $\frac{1}{\sqrt{a}} \leq \frac{\pi}{2} + (k_0 - 1)\pi$. Since the function $g \mapsto \frac{x}{ax^2 - 1}$ is non-increasing and negative on the interval $]0, \frac{1}{\sqrt{a}}[$, while it is non-increasing and positive on the interval $]\frac{1}{\sqrt{a}}, \frac{\pi}{2} + (k_0 - 1)\pi]$, it is easy to see that h has exactly k_0 roots in $]0, \frac{\pi}{2} + (k_0 - 1)\pi]$. Now, on each interval $]\frac{\pi}{2} + (k-1)\pi; \frac{\pi}{2} + k\pi[$, with $k \geq k_0$, the function g is non-negative, non-increasing, thus the equation $g(x) = \tan x$ admits one and only one solution on this interval (see Fig. 1), or equivalently there is exactly one root of h in this interval, which is μ_{k+1}. Since $\lim_{x\to\infty} g(x) = 0$, we obtain the first asymptotic expansion

$$\mu_{k+1} = k\pi + \varepsilon_k, \varepsilon_k = o(1).$$

Inserting the previous equality in the characteristic equation and using the Taylor series of g at infinity, we get

$$\tan(k\pi + \varepsilon_k) = \tan(\varepsilon_k) = \frac{k\pi + o(1)}{a(k\pi + o(1))^2 - 1} = \frac{1}{ak\pi} + o(\frac{1}{k^2}).$$

Therefore

$$\varepsilon_k = \arctan(\frac{1}{ak\pi} + o(\frac{1}{k^2})) = \frac{1}{ak\pi} + o(\frac{1}{k^2}),$$

thus the asymptotic behavior (3.9) holds. Inserting the expansion (3.9) in (3.3) we obtain (3.10). Analogously, inserting (3.9) and (3.10) in the expression of η_k given in (3.2), we get (3.11). □

3.3 Strong stability of the dissipative system

As a consequence of the previous Lemma, we can prove the following proposition.

Proposition 3.5. *We have*

$$\lim_{t\to+\infty} E(t) = 0 \tag{3.12}$$

for all solution u of (P_b) with u_0 in $\mathscr{H}$.

Proof. First, we show that $\mathscr{A}$ has no eigenvalues on the imaginary axis. If it is not the case, let $i\mu$ be an eigenvalue of $\mathscr{A}$ with $\mu \in \mathbb{R}$ and let $\phi = (y, z, \xi, \eta)$ be its associated eigenvector. Then

$$\Re(\mathscr{A}\phi, \phi)_{\mathscr{H}} = \Re[i\mu(\phi, \phi)_{\mathscr{H}}] = 0,$$

and from (2.4) we deduce that $\eta = 0$. That means $b\eta = 0$ and therefore $\mathscr{A}\phi = A\phi = i\mu\phi$, i.e ϕ is an eigenvector of the operator A. But from (3.2) and (3.1) for such an eigenvector, η is different from 0, this is a contradiction.

Now, we can apply the main theorem of Arendt and Batty [5]: Since $\sigma(\mathscr{A}) \cap i\mathbb{R}$ is empty, we obtain (3.12). □

4 A regularity result and an a priori estimate

In the next section we study the polynomial decay of the energy towards 0. In order to get the result we split up the dissipative system (P_b) into a conservative one with the same initial datum and a remainder one. In this section, we collect some technical results needed for this remaining system.

Lemma 4.1. *The function* $H(x,\lambda) = \dfrac{-\sinh(\lambda x)}{\lambda h_0(\lambda)}$ *admits the following expansion: for all* $x \in [0,1]$ *and for all* $\lambda \in \mathbb{C}, \lambda \neq \lambda_k$, *for all* $k \in \mathbb{Z}$

$$H(x,\lambda) = -\frac{x}{2\lambda} - \sum_{k\in\mathbb{Z}^*} \frac{1}{\lambda-\lambda_k} \frac{\sinh(\lambda_k x)\sinh(\lambda_k)}{N(-i\lambda_k)}, \tag{4.1}$$

where the $\lambda_k{}'s$ *are the roots of the characteristic equation* $h_0(\lambda) = 0$, h_0 *is given by (3.7) and the function N is given by (3.3).*

Proof. First, remark that (4.1) holds for $x = 0$, thus we assume now that $x \neq 0$. We recall that the roots of h_0 are of order 1, isolated, and they are situated on the imaginary axis, then clearly $H(x,\lambda)$ has a singularity of order one in λ_k, $\lambda_k \neq 0$. In the special case $\lambda = 0$, it is easy to see that we also have a singularity of order one. Therefore by Mittag-Leffler's theorem, there exists a function Θ_x, holomorphic on $\mathbb{C}$, such that

$$\forall \lambda \neq \lambda_k, k \in \mathbb{Z}, H(x,\lambda) = \sum_{k\in\mathbb{Z}} \frac{Res_{(\lambda=\lambda_k)}(H(x,\lambda))}{\lambda-\lambda_k} + \Theta_x(\lambda).$$

Now we compute the residues: for $k = 0$ (i.e $\lambda_k = 0$), obviously we have

$$Res_{(\lambda=0)}(H(x,\lambda)) = \frac{-x}{2}. \tag{4.2}$$

On the other hand if $k \neq 0$, then

$$Res_{(\lambda=\lambda_k)}(H(x,\lambda)) = \frac{-\sinh(\lambda_k x)}{\lambda_k h_0'(\lambda_k)}. \tag{4.3}$$

We further need an explicit expression of $h_0'(\lambda_k)$ and, in the sequel, in order to simplify notations, the index k is omitted. From the definition (3.7) of h_0 we have

$$h_0'(\lambda) = \left(2 + a\lambda^2\right)\cosh\lambda + (1+2a)\lambda\sinh\lambda.$$

Setting $\lambda = i\mu$, with $\mu \in \mathbb{R}$, we obtain

$$h_0'(i\mu) = \left(2 - a\mu^2\right)\cos\mu - (1+2a)\mu\sin\mu.$$

The identity (3.1) then leads to

$$\mu h_0'(i\mu)\sin\mu = -\left(2 + \mu^2 - a\mu^2 + a^2\mu^4\right)\sin^2\mu.$$

Again (3.7) yields

$$\mu^2\cos^2\mu = (1 - a\mu^2)^2\sin^2\mu,$$

which gives us

$$\mu^2 = (1 + \mu^2 - 2a\mu^2 + a^2\mu^4)\sin^2\mu,$$

and using this last identity in (3.3), we obtain

$$N(\mu)=(2+\mu^2-a\mu^2+a^2\mu^4)\sin^2\mu=-\mu h_0'(i\mu)\sin\mu. \tag{4.4}$$

Thanks to (4.4) we have found

$$Res_{(\lambda=\lambda_k)}(H(x,\lambda))=-\frac{\sinh(\lambda_k x)\sinh(\lambda_k)}{N(-i\lambda_k)}. \tag{4.5}$$

Using (4.2) and (4.5), we arrive at (4.1) provided $\Theta_x\equiv 0$.

Let $n\in\mathbb{N}^*$ and C_{2n} be the square whose vertices are

$$\pm(2n+\frac{1}{2})\pi\pm i(2n+\frac{1}{2})\pi. \tag{4.6}$$

We first give estimates of $H(x,\lambda)$ on each edge of C_{2n}. Let

$$\lambda=2n\pi+\frac{\pi}{2}+it, t\in[-2n\pi-\frac{\pi}{2},+2n\pi+\frac{\pi}{2}].$$

We have

$$-\sinh((2n\pi+\frac{\pi}{2}+it)x)=-i\cosh(tx)\sin((2n\pi+\frac{\pi}{2})x)-\cos((2n\pi+\frac{\pi}{2})x)\sinh(tx),$$

thus

$$|-\sinh((2n\pi+\frac{\pi}{2}+it)x)|\le 2e^{(2n+\frac{1}{2})\pi x}\le 2e^{(2n+\frac{1}{2})\pi}. \tag{4.7}$$

We have also for n large

$$h_0(2n\pi+\frac{\pi}{2}+it)\sim 2an^2\pi^2e^{2n\pi}. \tag{4.8}$$

We deduce from (4.7)-(4.8) that for n large enough we have:

$$\forall t\in[-2n\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}],\ |H(x,2n\pi+\frac{\pi}{2}+it)|\lesssim\frac{1}{n^3}. \tag{4.9}$$

In the same way, for $\lambda=-2n\pi-\frac{\pi}{2}+it, t\in[-2n\pi-\frac{\pi}{2},+2n\pi+\frac{\pi}{2}]$, the estimate (4.9) still holds. Now let

$$\lambda=t+i(2n\pi+\frac{\pi}{2}), t\in[-2n\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}],$$

then

$$-\sinh(\lambda x)=-\cos((2n\pi+\frac{\pi}{2})x)\sinh(tx)+i\cosh(tx)\sin((2n\pi+\frac{\pi}{2})x)$$

and thus

$$|-\sinh(\lambda x)|\le 4e^{|t|x}\le 4e^{|t|}. \tag{4.10}$$

Now, with

$$\lambda=t+i(2n\pi+\frac{\pi}{2}), t\in[-2n\pi-\frac{\pi}{2},2n\pi+\frac{\pi}{2}],$$

we first remark that $|h_0(t+(2n\pi+\frac{\pi}{2})|=|h_0(-t+(2n\pi+\frac{\pi}{2})|$, thus for the following estimates we will suppose that $t\ge 0$. We have

$$\begin{aligned}\Re(h_0(\lambda)) &= -t((4n+1)\pi\cos(2n\pi)+\sin(2n\pi))\cosh(t)\\ &+ \frac{1}{4}(((4\pi n+\pi)^2-4(t^2+1))\sin(2n\pi)-2(4n+1)\pi\cos(2n\pi))\sinh(t)\\ &= -t(4n+1)\pi\cosh(t)-\frac{1}{2}(4n+1)\pi\sinh(t),\end{aligned}$$

therefore

$$\forall |t| \geq 1, |\Re(h_0(\lambda))| \geq 4n\pi \cosh(t) \geq 2n\pi e^{|t|}. \tag{4.11}$$

As previously, we compute $\Im(h_0(\lambda))$, replace $\cos(2n\pi)$ by 1, $\sin(2n\pi)$ by 0 and find

$$\Im(h_0(\lambda)) = t\sinh(t) - \frac{1}{4}\left((4\pi n+\pi)^2 - 4\left(t^2+1\right)\right)\cosh(t).$$

Thus, for n great enough, we have

$$\forall |t| \leq 1, |\Im(h_0(\lambda))| \geq 4n^2\pi^2 \geq 2n\pi e^{|t|}. \tag{4.12}$$

Therefore, combining (4.11) and (4.12), we find that, for n great enough

$$\forall t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}], |h_0(t+i(2n\pi+\frac{\pi}{2})| \geq 2n\pi e^{|t|}.$$

Hence, using the previous estimate and (4.10) we deduce that for n large enough:

$$\forall t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}], |H(x,t+i(2n\pi+\frac{\pi}{2})| \lesssim \frac{1}{n^2}.$$

The same calculations give the same estimation for $\lambda = t - i(2n\pi + \frac{\pi}{2})$, with $t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}]$.

Finally, we have proved that for n large enough:

$$\forall \lambda \in \partial C_{2n}, |H(x,\lambda)| \lesssim \frac{1}{n^2}. \tag{4.13}$$

For a fixed $\lambda \neq \lambda_k$, for all $k \in \mathbb{Z}$, the general term of the series in the right-hand side of (4.1) can be estimated by

$$\begin{aligned} \left|\frac{1}{\lambda-\lambda_k}\frac{\sinh(\lambda_k)\sinh(\lambda_k x)}{N(-i\lambda_k)}\right| &= \left|\frac{1}{\lambda-i\mu_k}\frac{\sin(\mu_k)\sin(\mu_k x)}{N(\mu_k)}\right| \\ &\lesssim \frac{1}{\mu_k^3}, \end{aligned}$$

and recalling that $\mu_k \sim k\pi$, this shows the absolute convergence of the series.

Let us denote the series in the right-hand side of (4.1) by

$$H_1(x,\lambda) = \sum_{k\in\mathbb{Z}} \frac{1}{\lambda-\lambda_k} c_k(x).$$

By the estimates (3.9) and (3.10) we have $|c_k(x)| \lesssim \frac{1}{k^3}$ (notice that due to (3.9) and the fact that k is an integer, we also have $\sin(\mu_k) \sim 1/k$ as $k \to \infty$).

For $\lambda = 2n\pi + \frac{\pi}{2} + it, t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}]$ we have $|\frac{1}{\lambda - \lambda_k}| \leq \frac{1}{2n\pi}$. Thus,

$$|H_1(x, 2n\pi + \frac{\pi}{2} + it)| \leq \frac{1}{2n\pi}\sum_{k\in\mathbb{Z}} |c_k(x)| \lesssim \frac{1}{2n\pi}. \tag{4.14}$$

The same estimate holds for $\lambda = -2n\pi - \frac{\pi}{2} + it, t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}]$.

Now, let $\lambda = t + i(2n\pi + \frac{\pi}{2}), t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}]$. Since $\lambda_{2n} \sim 2ni\pi$, then

$$\forall k \in \mathbb{Z}, |\lambda - \lambda_k| \geq \min(|i(2n + \frac{\pi}{2}) - \lambda_{2n}|, |i(2n + \frac{\pi}{2}) - \lambda_{2n+1})| \geq \frac{\pi}{4}.$$

Hence,

$$\begin{array}{rcl} |H_1(x,\lambda)| & = & |\sum_{k\in\mathbb{Z}} \frac{1}{\lambda - \lambda_k} c_k(x)| \\ & \leq & \sum_{k\leq n} \frac{1}{|\lambda - \lambda_k|} |c_k(x)| + \frac{\pi}{4} \sum_{k\geq n+1} |c_k(x)| \\ & \leq & \frac{1}{n\pi} \sum_{k\leq n} |c_k(x)| + \frac{\pi}{4} \sum_{k\geq n+1} |c_k(x)| \\ & \lesssim & \frac{1}{n} + \frac{\pi}{4} \sum_{k\geq n+1} \frac{1}{k^3} \\ & \lesssim & \frac{1}{n}. \end{array}$$

The same estimate holds for $\lambda = t - i(2n\pi + \frac{\pi}{2}), t \in [-2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}]$.
Finally, we have shown that:

$$\forall \lambda \in \partial C_{2n}, |H_1(x,\lambda)| \lesssim \frac{1}{n}. \tag{4.15}$$

Therefore estimates (4.13) and (4.15) give :

$$|\Theta_x(\lambda, x)| = |H(\lambda, x) - H_1(\lambda, x)| \lesssim \frac{1}{n}, \forall \lambda \in \partial C_{2n}.$$

By the maximum principle, this estimate obtained on the boundary of C_{2n} is still valid on the whole square. By letting $n \to +\infty$, this implies that $\Theta_x \equiv 0$. □

Now, let us introduce the vector $F_0 = (0,0,0,1) \in \mathscr{H}$. Since $\{\phi_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathscr{H}$, we can write

$$F_0 = \sum_{k\in\mathbb{Z}} \eta_k \phi_k = \sum_{k\in\mathbb{Z}} \eta_k (y_k, z_k, \xi_k, \eta_k), \tag{4.16}$$

where $\phi_k = (y_k, z_k, \xi_k, \eta_k), k \in \mathbb{Z}$, is given by (3.2).

Next, we consider the following problem : Let $T > 0$ be fixed and f a function in $L^2(0,T)$, find $u = (y, z, \xi, \eta)$ satisfying

$$u_t = Au + f(t)F_0,\ u(0) = 0. \tag{4.17}$$

This is equivalent to

$$\left\{ \begin{array}{ll} y_{tt}(x,t) - y_{xx}(x,t) = 0, & \text{in } (0,1)\times(0,T), \\ y(0,t) = 0, & \text{on } (0,T), \\ ay_{tt}(1,t) + y_x(1,t) + \eta(t) = 0, & \text{on } (0,T), \\ \eta_t(t) - y_t(1,t) = f(t), & \text{on } (0,T), \\ y(x,0) = 0, y_t(x,0) = 0, & \text{in } (0,1), \\ \eta(0) = 0. & \end{array} \right. \tag{4.18}$$

Lemma 4.2. *Assume that $f \in L^2(0,T)$. Then problem (4.18) has a unique solution $(y,\eta) \in H^1((0,1)\times(0,T)) \times H^1(0,T)$ which satisfies $y_t \in C([0,T]; H^1(0,1))$ and*

$$\|y\|_{H^1((0,1)\times(0,T))} + \|\eta\|_{H^1(0,T)} \lesssim \|f\|_{L^2(0,T)}. \tag{4.19}$$

Proof. Firstly, remark that the uniqueness is obvious since the difference between two solutions is solution of (P_0) with initial condition $u_0 = 0$, and therefore the difference vanishes. We extend f by 0 on $\mathbb{R}\setminus[0,T]$. Because system (4.18) is reversible in time, the solution $(\tilde{y},\tilde{\eta})$ of (4.18) with $\tilde{f}$ instead of f and replacing $(0,T)$ by $\mathbb{R}$ coincides with (y,η) on $(0,T)$. For shortness, we still denote $(\tilde{y},\tilde{\eta})$ by (y,η). Let $\hat{y}(x,\lambda)$ (resp. $\hat{\eta}(\lambda),\hat{f}(\lambda)$) where $\lambda = \gamma+i\omega, \gamma>0, \omega\in\mathbb{R}$, be the Laplace transform of y with respect to t.

Then $(\hat{y},\hat{\eta})$ satisfies

$$\begin{cases} \lambda^2\hat{y}(x,\lambda)-\hat{y}_{xx}(x,\lambda)=0, & \text{in} \quad (0,1), \\ \hat{y}(0,\lambda)=0, & \\ a\lambda^2\hat{y}(1,\lambda)+\hat{y}_x(1,\lambda)+\hat{\eta}(\lambda)=0, & \\ \lambda(\hat{\eta}(\lambda)-\hat{y}(1,\lambda))=\hat{f}(\lambda), & \end{cases} \tag{4.20}$$

where $\Re\lambda>0$. Consequently, from the previous first and second equations, $\hat{y}(x,\lambda)=\alpha\sinh(\lambda x)$, for some constant α. Hence the third and fourth equations reduce to

$$\begin{cases} a\lambda^2\alpha\sinh(\lambda)+\lambda\alpha\cosh(\lambda)+\hat{\eta}(\lambda)=0, \\ \lambda\hat{\eta}(\lambda)-\lambda\alpha\sinh(\lambda)=\hat{f}(\lambda). \end{cases}$$

Eliminating $\hat{\eta}(\lambda)$ we get

$$\alpha=\frac{-1}{\lambda[\sinh(\lambda)(a\lambda^2+1)+\lambda\cosh(\lambda)]}\hat{f}(\lambda)=\frac{-1}{\lambda h_0(\lambda)}\hat{f}(\lambda),$$

where $h_0(\lambda)$ is the characteristic equation given by (3.7). Remark that the previous equality has a meaning since we know that the roots of h_0 are situated on the imaginary axis.

Therefore, we get

$$\hat{y}(x,\lambda)=\frac{-\sinh(\lambda x)}{\lambda h_0(\lambda)}\hat{f}(\lambda), \tag{4.21}$$

and

$$\hat{\eta}(\lambda)=\frac{a\lambda\sinh(\lambda)+\cosh(\lambda)}{h_0(\lambda)}\hat{f}(\lambda). \tag{4.22}$$

Denoting by $\mathscr{L}^{-1}$ the inverse Laplace transform, the solution of problem (4.18) is given by

$$y(x,t)=\mathscr{L}^{-1}\left(\frac{-\sinh(\lambda x)}{\lambda h_0(\lambda)}\right)\star f,\ \ \eta(t)=\mathscr{L}^{-1}\left(\frac{a\lambda\sinh(\lambda)+\cosh(\lambda)}{h_0(\lambda)}\right)\star f.$$

In order to study the regularity of y and η, we need to estimate the functions $\lambda\mapsto\dfrac{-\sinh(\lambda x)}{\lambda h_0(\lambda)}$ and $\lambda\mapsto\dfrac{a\lambda\sinh(\lambda)+\cosh(\lambda)}{h_0(\lambda)}$ on the line $C_\gamma=\{\lambda:\lambda=\gamma+i\omega,\gamma>0,\omega\in\mathbb{R}\}$.

Since

$$\sinh((\gamma+i\omega)x)=\cos(\omega x)\sinh(\gamma x)+i\cosh(\gamma x)\sin(\omega x),$$

then

$$\forall x\in[0,1],\forall\lambda\in C_\gamma,\ |\sinh(\lambda x)|\le e^\gamma. \tag{4.23}$$

On the other hand h_0 has no roots on the line C_γ and one can show that

$$|h_0(\gamma+i\omega)|\sim\omega^2,\ \text{ if }\ |\omega|\to\infty.$$

This implies that there exists a positive constant $M(\gamma)$ such that $|\dfrac{-\sinh(\lambda x)}{h_0(\lambda)}| \leq M(\gamma)$ on the line C_γ. In the same way,

$$|a\lambda \sinh(\lambda) + \cosh(\lambda)| \lesssim |\omega|, \text{ if } |\omega| \to \infty.$$

Thus there exists a positive constant $M_1(\gamma)$ such that

$$|\lambda|| \frac{(a\lambda \sinh(\lambda) + \cosh(\lambda))}{h_0(\lambda)}| \leq M_1(\gamma) \text{ on the line } C_\gamma. \tag{4.24}$$

Thanks to estimates (4.23) and (4.24), we can directly apply the proof of Lemma 4.2 of [14] and obtain the estimate (4.19).

It remains to check the initial conditions. Let us denote $F(x,t) = \mathscr{L}^{-1}(H(x,\lambda))$; from Lemma 4.1 we have

$$F(x,t) = -\frac{x}{2} + \sum_{k\in\mathbb{Z}^*} e^{i\mu_k t} \frac{\sin(\mu_k x)\sin(\mu_k)}{N(\mu_k)}.$$

We remark that $F \in L^\infty([0,T];L^2(0,1))$, therefore

$$y(x,t) = \int_0^t F(x,t-s)f(s)ds, \ \forall t \in [0,T],$$

and we deduce that

$$y(x,0) = 0.$$

Now, for $t \in [0,T]$,

$$\frac{\partial y}{\partial t}(x,t) = F(x,0)f(t) + \int_0^t \frac{\partial F}{\partial t}(x,t-s)f(s) = \int_0^t \frac{\partial F}{\partial t}(x,t-s)f(s)ds. \tag{4.25}$$

Indeed,

$$\begin{aligned} F(x,0) &= -\frac{x}{2} + \sum_{k\in\mathbb{Z}^*} \frac{\sin(\mu_k x)\sin(\mu_k)}{N(\mu_k)} \\ &= \sum_{k\in\mathbb{Z}} y_k(x)\eta_k, \end{aligned}$$

recalling that y_k, η_k are given in Lemma 3.2. However, we know that $F_0 = (0,0,0,1) = \sum_{k\in\mathbb{Z}} \eta_k \phi_k = \sum_{k\in\mathbb{Z}} \eta_k(y_k,z_k,\xi_k,\eta_k)$ (see (4.16)), thus $F(x,0)$ is the first component of F_0 and consequently $F(x,0) = 0$.

As we have $\dfrac{\partial F}{\partial t}(x,t) = \sum_{k\in\mathbb{Z}^*} i\mu_k e^{i\mu_k t} \dfrac{\sin(\mu_k x)\sin(\mu_k)}{N(\mu_k)}$, then $\dfrac{\partial F}{\partial t}(x,t)$ may be view as the first component of the function

$$\sum_{k\in\mathbb{Z}^*} i\mu_k \frac{\sin(\mu_k)}{\sqrt{N(\mu_k)}} e^{i\mu_k t}\phi_k.$$

Moreover, from Lemma 3.4 we have

$$|i\mu_k \frac{\sin(\mu_k)}{\sqrt{N(\mu_k)}} e^{i\mu_k t}| \sim \frac{1}{k},$$

then this function belongs to $\mathscr{H}$ and we deduce that $\dfrac{\partial F}{\partial t}(.,t) \in H^1(0,1)$, with the following estimate independent of $t \in [0,T]$:

$$\|\frac{\partial F}{\partial t}(.,t)\|_{H^1(0,1)} \lesssim \sum_{k\in\mathbb{Z}^*} \frac{1}{k^2}.$$

Consequently

$$\|\frac{\partial y}{\partial t}(.,t)\|_{H^1(0,1)} \lesssim \int_0^t |f(s)|ds \lesssim \sqrt{t}\|f\|_{L^2(0,T)}.$$

This shows that

$$\frac{\partial y}{\partial t} \in C([0,T];H^1(0,1))$$

and that

$$\frac{\partial y}{\partial t}(x,0) = 0.$$

It remains to compute $\eta(0) = 0$. Since $\eta(t) = -ay_{tt}(1,t) - y_x(1,t)$, we firstly look at $ay_{tt}(1,0)$. From (4.25), we may write:

$$ay_{tt}(1,t) = a\frac{\partial F}{\partial t}(1,0)f(t) + a\int_0^t \frac{\partial^2 F}{\partial t^2}(1,t-s)f(s)ds,$$

and

$$a\frac{\partial F}{\partial t}(1,0) = \sum_{k\in\mathbb{Z}} ia\mu_k \frac{\sin(\mu_k)}{\sqrt{N(\mu_k)}} \frac{\sin(\mu_k)}{\sqrt{N(\mu_k)}} = \sum_{k\in\mathbb{Z}} \eta_k \xi_k = 0,$$

since $\sum_{k\in\mathbb{Z}} \eta_k \xi_k$ is the third component of F_0 (see (4.16)). Consequently

$$ay_{tt}(1,t) = a\int_0^t \frac{\partial^2 F}{\partial t^2}(1,t-s)f(s)ds$$

and hence

$$y_{tt}(1,0) = 0. \tag{4.26}$$

Secondly, we have

$$\frac{\partial y}{\partial x}(x,t) = \int_0^t \frac{\partial F}{\partial x}(x,t-s)f(s)ds, \tag{4.27}$$

and

$$\frac{\partial F}{\partial x}(x,t) = -\frac{1}{2} + \sum_{k\in Z^*} e^{i\mu_k t}\mu_k \cos(\mu_k x)\frac{\sin(\mu_k)}{N(\mu_k)},$$

with the estimate

$$|e^{i\mu_k t}\mu_k \cos(\mu_k x)\frac{\sin(\mu_k)}{N(\mu_k)}| \lesssim \frac{1}{k^2}.$$

This gives a meaning to (4.27) and proves that

$$\frac{\partial y}{\partial x}(x,0) = 0. \tag{4.28}$$

In conclusion, (4.26) and (4.28) lead to $\eta(0) = 0$. □

5 Polynomial stability

5.1 An observability inequality

Lemma 5.1. *Let u be the solution of problem (P_0) with an initial datum $u_0 \in \mathscr{D}(A)$, then there exist a time $T > 0$ and a constant $C > 0$ depending on T such that*

$$\int_0^T |\eta(t)|^2 dt \geq C(|u_0^{(0)}|^2 + \sum_{k\in\mathbb{Z}^*} \frac{|u_0^{(k)}|^2}{k^4}), \tag{5.1}$$

where $u_0 = \sum_{k\in\mathbb{Z}} u_0^{(k)} \phi_k$.

Proof. Since the solution of problem (P_0) admits the following expansion:

$$u(t) = \sum_{k\in\mathbb{Z}} u_0^{(k)} e^{i\mu_k t} \phi_k,$$

then

$$\eta(t) = \sum_{k\in\mathbb{Z}} u_0^{(k)} \eta_k e^{i\mu_k t}.$$

Moreover, from (3.11) we have $\eta_k^2 \sim \frac{1}{a^2 k^4 \pi^2}$, and, from (3.9), the sequence $(\mu_k)_{k\in\mathbb{Z}}$ satisfies the spectral gap condition. The conclusion follows by using the so-called Ingham's inequality (see for instance Theorem 1.5 of [6]). □

5.2 Polynomial decay of the energy

We start with an interpolation inequality.

Lemma 5.2. *For all $u_0 \in \mathscr{D}(A)$, we have:*

$$\|u_0\|_{\mathscr{H}}^3 \leq \|u_0\|_{\mathscr{D}(A^{-2})} \|u_0\|_{\mathscr{D}(A)}^2.$$

Proof. Since for all $s \in \mathbb{R}$, $\|u_0\|_{\mathscr{D}(A^s)}^2 \sim |u_0^{(0)}|^2 + \sum_{k\in\mathbb{Z}^*} k^{2s} |u_0^{(k)}|^2$, where $u_0 = \sum_{k\in\mathbb{Z}} u_0^{(k)} \phi_k$, we apply Hölder's inequality to get the result. □

Now we can state the main result of this paper.

Theorem 5.3. *Let u be the solution of problem (P_b) with an initial datum $u_0 \in \mathscr{D}(\mathscr{A})$, then the energy decays polynomially, i.e,*

$$E(t) \leq \frac{C}{\sqrt{1+t}} \|u_0\|_{\mathscr{D}(\mathscr{A})}^2,$$

for some $C > 0$.

Proof. It follows the arguments of Theorem 2.2 of [3] but adapted to our specific problem. Let $u \in C([0,+\infty), \mathscr{D}(\mathscr{A})) \cap C^1([0,+\infty), \mathscr{H})$ be the solution of problem (P_b) with an initial datum $u_0 \in \mathscr{D}(\mathscr{A})$. Then it can be splitted up in the form

$$u = u^{(1)} + u^{(2)} = (y^{(1)}, z^{(1)}, \xi^{(1)}, \eta^{(1)}) + (y^{(2)}, z^{(2)}, \xi^{(2)}, \eta^{(2)}),$$

where $u^{(1)}$ is the solution of problem (P_0) (with an initial datum u_0) and $u^{(2)}$ is solution of problem (4.18) with $f(t) = -b\eta(t)$. By Lemma 5.1 we have

$$\int_0^T |\eta^{(1)}(t)|^2 dt \geq C\|u_0\|^2_{\mathscr{D}(A^{-2})}, \tag{5.2}$$

if T is great enough. But we may write

$$\|\eta^{(1)}\|_{L^2(0,T)} \leq \|\eta\|_{L^2(0,T)} + \|\eta^{(2)}\|_{L^2(0,T)}, \tag{5.3}$$

and from Lemma 4.2

$$\|\eta^{(2)}\|_{L^2(0,T)} \lesssim \|\eta\|_{L^2(0,T)}. \tag{5.4}$$

Therefore, combining the estimates (5.2)-(5.4) we get

$$\|u_0\|_{\mathscr{D}(A^{-2})} \lesssim \|\eta\|_{L^2(0,T)}. \tag{5.5}$$

Integrating the equality (2.3) of Proposition 2.1 between 0 and T where T is sufficiently large, and using (5.5) we obtain

$$E(T) \leq E(0) - K\|u_0\|^2_{\mathscr{D}(A^{-2})}, \tag{5.6}$$

for some $K > 0$. Thanks to Lemma 5.2, (5.6) becomes

$$E(T) \leq E(0) - K\frac{\|u_0\|^6_{\mathscr{H}}}{\|u_0\|^4_{\mathscr{D}(A)}},$$

which leads to

$$E(T) \leq E(0) - K\frac{E(0)^3}{E_1(0)^2}, \tag{5.7}$$

where we set $E_1(0) = \frac{1}{2}(\|u_0\|^2_{\mathscr{H}} + \|Au_0\|^2_{\mathscr{H}})$.

Since the energy is non-increasing we have also

$$E(T) \leq E(0) - K\frac{E(T)^3}{E_1(0)^2}. \tag{5.8}$$

Following the method used in [3] (see also [14]) we arrive at

$$\varepsilon_{k+1} \leq \varepsilon_k - K_1\varepsilon^3_{k+1}, \tag{5.9}$$

for some $K_1 > 0$, where we have set

$$\varepsilon_k = \frac{E(kT)}{E_1(0)}. \tag{5.10}$$

Using Lemma 5.2 of [3], we deduce from (5.9) that:

$$\varepsilon_k \leq \frac{M}{\sqrt{1+k}}, \forall k > 0,$$

where M is a constant which depends only on K_1.

The result follows from the previous inequality, from the definition (5.10) of the sequence (ε_k), and from the fact that $E(t)$ is non-increasing. □

Remark 5.4. In comparison with the case $a = 0$ treated in [21], we see that the additional term $ay_{tt}(1,t)$ in the feedback law affects the decay rate, since with initial data in $D(\mathscr{A})$, we get a decay in $\frac{1}{\sqrt{t}}$ instead of $\frac{1}{t}$.

Remark 5.5. We could alternatively use the method developed in [21] and derived from [17]. But in this case, the obtained estimate of the energy is only valid for more regular initial data, namely it is in the form

$$E(t) \leq C \frac{\|u_0\|^2_{\mathscr{D}(\mathscr{A}^2)}}{t}. \tag{5.11}$$

This is why we have chosen another method to obtain a decay of the energy for less regular data. On the other hand, using Proposition 3.1 of [7] and Theorem 5.3 we recover the estimate (5.11).

6 Optimal energy decay rate

Since, in the sequel, we need the dependency of the eigenvalues and eigenvectors of $\mathscr{A}$ with respect to the parameter b, we now write $\mathscr{A}_b$ instead of $\mathscr{A}$. Denote further by $\{\lambda_{b,k}\}_{k\in\mathbb{Z}}$ the set of eigenvalues of $\mathscr{A}_b$ enumerated according to their algebraic multiplicity and such that

$$\Im(\lambda_{b,k}) \leq \Im(\lambda_{b,k+1}), \forall k \in \mathbb{Z}.$$

If $u_0 \in \mathscr{D}(A)$ (remark that $\mathscr{D}(\mathscr{A}_b) = \mathscr{D}(A)$), we define the optimal rational decay rate $\omega(u_0)$ by

$$\omega(u_0) = \sup\{\alpha \in \mathbb{R} : E(t) = \frac{1}{2}\|u(t)\|^2_{\mathscr{H}} \leq \frac{c}{t^\alpha}\}. \tag{6.1}$$

From Theorem 5.3, we know that $\omega(u_0) \geq \frac{1}{2}$ for any $u_0 \in \mathscr{D}(\mathscr{A})$. Our goal is to prove that this upper bound is optimal, i.e, for any $\varepsilon > 0$ there exists $u_0^\varepsilon \in \mathscr{D}(\mathscr{A})$ such that $\omega(u_0^\varepsilon) = \frac{1}{2} + \varepsilon$.

Firstly, we recall the following result (see [11], [21]).

Lemma 6.1. *Consider a C^0-semigroup $T(t)$ acting on a (real or complex) Hilbert space $\mathscr{H}$ with infinitesimal generator $\mathscr{A}$. Assume the following*

(i) For $k \in \mathbb{N}^*$, the eigenvalue $\tilde{\lambda}_k$ of $\mathscr{A}$ is of the form $\tilde{\lambda}_k = -\sigma_k + i\tau_k$ with $\sigma_k > \frac{c_1}{k^\delta}$, where $c_1 > 0$ and $\delta > 0$ are independent of k.

(ii) The eigenvectors $\tilde{\phi}_k, k \geq 1$ associated with the eigenvalues $\tilde{\lambda}_k$ form a Riesz basis of $\mathscr{H}$.

(iii) Let $u_0 \in \mathscr{H}$ be such that

$$u_0 = \sum_{k\geq 1} a_k \tilde{\phi}_k,\ |a_k| \leq \frac{c_2}{k^q}, c_2 > 0, q > \frac{1}{2}.$$

Then there exists a constant $c > 0$ depending on u_0 such that $\|T(t)u_0\|_{\mathscr{H}} \leq \frac{c}{t^{(q-1/2)/\delta}}, \forall t > 0$.

Remark 6.2. Note that if $\sigma_k \sim \frac{1}{k^\delta}$ and $|a_k| \sim \frac{1}{k^q}$, then the equivalence

$$\|T(t)u_0\|_{\mathscr{H}} \sim \frac{1}{t^{(q-1/2)/\delta}}, \forall t > 0,$$

holds.

The following Lemma deals with the spectrum of $\mathscr{A}_b$:

Lemma 6.3. *The eigenvalues of $\mathscr{A}_b$ are the roots of h_b given by*

$$h_b(z) = (b+z)\cosh(z) + (1+az(b+z))\sinh(z), \tag{6.2}$$

and consequently satisfy

$$\lambda_{b,k} = \overline{\lambda_{b,k}}.$$

Moreover, their geometrical multiplicity is one.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathscr{A}_b$ and $u = (y, z, \xi, \eta)$ be an associated eigenvector. Then it follows:

$$\begin{cases} y_{xx}(x) = \lambda^2 y(x), & 0 < x < 1, \\ y(0) = 0, \\ a\lambda^2 y(1) + y_x(1) + \eta = 0, \\ \lambda\eta - \lambda y(1) + b\eta = 0. \end{cases} \tag{6.3}$$

From the two first equations of (6.3), we find that a general solution is given by $y(x) = C\sinh(\lambda x)$ for some constant C. Since we easily check that $\lambda = -b$ cannot be an eigenvalue of $\mathscr{A}_b$, then the fourth equation gives $\eta = \dfrac{\lambda}{\lambda+b} y(1)$. Inserting η in the third equation we find that λ is an eigenvalue of $\mathscr{A}_b$ if and only if

$$\frac{\lambda}{b+\lambda}((b+\lambda)\cosh(\lambda) + (1+a\lambda(b+\lambda))\sinh(\lambda)) = 0. \tag{6.4}$$

Notice that $\lambda = 0$ is not an eigenvalue of $\mathscr{A}_b$, then we find that the eigenvalues of $\mathscr{A}_b$ are the roots of the function h_b given by (6.2). Clearly, from the previous considerations, if λ is an eigenvalue of $\mathscr{A}_b$ then $\dim(\mathscr{A}_b - \lambda I) = 1$. □

Lemma 6.4. *There exists an integer n_0 such that, for $n \geq n_0$, the function $h_b(z)$ has $2n+3$ roots (taking into account their multiplicity) in the square*

$$C_n = [-(n+\frac{1}{2})\pi, (n+\frac{1}{2})\pi] \times [-(n+\frac{1}{2})\pi, (n+\frac{1}{2})\pi].$$

The roots of h_b are simple, except for at most four eigenvalues which are also roots of the polynomial

$$P(z) = a^2z^4 + 2a^2bz^3 + (a^2b^2-1)z^2 - 2(1+a)bz + 3 - b^2 - 2ab^2. \tag{6.5}$$

Proof. From the proof of Lemma 3.4, we know that the function h_0 has exactly $2n+3$ roots (which are simple) in the square C_n. So, we will prove that h_b and h_0 has the same number of roots in C_n, for n large enough. By Rouch's theorem, it suffices to prove that

$$|h_b(z) - h_0(z)| < |h_0(z)|, \forall z \in \partial C_n. \tag{6.6}$$

In fact, $h_b(z) - h_0(z) = b(\cosh(z) + az\sinh(z)) = \dfrac{b}{z}(h_0(z) - \sinh(z))$. Thus $|h_b(z) - h_0(z)| \leq \dfrac{b}{|z|}(|h_0(z)| + |\sinh(z)|)$. But some easy calculations allow to verify that $|\sinh(z)| \leq |h_0(z)|$ on ∂C_n for n great enough. Hence (6.6) holds.

Now, we remark that $h_b(z) = 0$ is equivalent to $f(z) = 0$ where f is given by

$$f(z) = e^{2z}(1+b+z+abz+az^2) - (1-b-z+abz+az^2). \tag{6.7}$$

Since $f(z)$ has the form $f(z) = e^{2z}P_1(z) - P_2(z)$, with

$$P_1(z) = 1 + b + z + abz + az^2, P_2(z) = 1 - b - z + abz + az^2, \tag{6.8}$$

we get $f'(z) = e^{2z}(2P_1(z) + P_1'(z)) - P_2'(z)$. Hence if there exists z_0 such that $f(z_0) = f'(z_0) = 0$ then

$$P(z_0) = (P_1'(z_0) + 2P_1(z_0))P_2(z_0) - P_1(z_0)P_2'(z_0) = 0.$$

The conclusion follows since $P(z) = (P_1'(z) + 2P_1(z))P_2(z) - P_1(z)P_2'(z)$ is given by (6.5). □

Remark 6.5. For all $a > 0$ and all $b > 0$, we conjecture that all the roots of h_b are simple. Unfortunately, this is not easy to prove. However this conjecture is confirmed numerically using the Newton-Raphson method.

The following result concerns the algebraic multiplicity of the eigenvalues of $\mathscr{A}_b$:

Lemma 6.6. *For all $k \in \mathbb{Z}$, the algebraic multiplicity of $\lambda_{b,k}$ (as eigenvalue of $\mathscr{A}_b$) is equal to its multiplicity as root of h_b, and is between 1 and 4.*

Moreover we have the following cases:

a) If $\lambda_{b,k}$ is simple (algebraic multiplicity one), then its associated eigenvector is given by

$$\phi_{b,k} = \frac{1}{\lambda_{b,k}}(\sinh(\lambda_{b,k}x), \lambda_{b,k}\sinh(\lambda_{b,k}x), a\lambda_{b,k}\sinh(\lambda_{b,k}), \frac{\lambda_{b,k}}{\lambda_{b,k}+b}\sinh(\lambda_{b,k})) \tag{6.9}$$

and satisfy

$$\|\phi_{b,k}\|_{\mathscr{H}} \sim 1. \tag{6.10}$$

b) If $\lambda_{b,k}$ is of algebraic multiplicity $p_0 \in \{2,3,4\}$, then there exists $k' \in \mathbb{Z}$ such that

$$\lambda_{b,k} = \lambda_{b,k'} = \lambda_{b+k'+1} = \dots = \lambda_{b+k'+p_0-1}$$

and again its associated eigenvector is

$$\phi_{b,k'} = \frac{1}{\lambda_{b,k'}}(\sinh(\lambda_{b,k'}x), \lambda_{b,k'}\sinh(\lambda_{b,k'}x), a\lambda_{b,k'}\sinh(\lambda_{b,k'}), \frac{\lambda_{b,k'}}{\lambda_{b,k'}+b}\sinh(\lambda_{b,k'})), \tag{6.11}$$

while its associated generalized eigenvectors are

$$\forall j = 2, \dots, p_0, \phi_{b,k'+j-1} = \frac{\partial^{j-1}}{\partial\lambda^{j-1}}\phi_{b,k'}|_{\lambda=\lambda_{b,k'}}. \tag{6.12}$$

Proof. First, we start by proving (6.9), (6.10) and (6.11). From the proof of Lemma 6.3, we already know that if λ is an eigenvalue of $\mathscr{A}_b$ then an associated eigenvector u is on the form $u = C(\sinh(\lambda x), \lambda\sinh(\lambda x), a\lambda\sinh(\lambda), \frac{\lambda}{\lambda+b}\sinh(\lambda))$ and λ is a root of the function f given by (6.7), or equivalently, λ satisfies $e^{2\lambda} = \frac{P_2(\lambda)}{P_1(\lambda)}$, with P_1, P_2 given by (6.8). Since $|\lambda|$ goes to infinity, we deduce that $\lim \Re(\lambda) = 0$. Moreover by simple calculations we find that $\|u\|^2_{\mathscr{H}} \sim C^2|\lambda|^2$. Thus, choosing $C = \lambda^{-1}$ we arrive at (6.9), (6.10) and (6.11), replacing λ by $\lambda_{b,k}$.

Now, let λ be a root of h_b or equivalently a root of f, and let p_0 its multiplicity. If $p_0 \geq 5$ then $f^{(p)}(\lambda) = 0, p = 0, .., 4$. If we set

$$f_1 = f' - 2f, \ f_2 = f_1' - 2f_1, \ f_3 = f_2' - 2f_2, \ \text{and} \ f_4 = f_3',$$

then

$$f(\lambda) = f_1(\lambda) = f_2(\lambda) = f_3(\lambda) = f_4(\lambda) = 0.$$

However

$$f_4(\lambda) = 8P_2'(\lambda) - 12P_2^{(2)}(\lambda) + 6P_2^{(3)}(\lambda) = 8(-1 + a(-3 + b + 2\lambda)) = 0,$$

which gives $\lambda = \dfrac{1+3a-ab}{2a}$. Consequently, $\lambda \in \mathbb{R}^-$ and necessarily

$$b > 3 + \frac{1}{a}. \tag{6.13}$$

We also remark that $f_3 = 8P_2 - 12P_2' + 6P_2^{(2)}$ is a polynomial of order 2 and since $f_3(\lambda) = f_3'(\lambda) = 0$, the discriminant Δ of f_3 vanishes. A computation gives

$$\Delta = 1 - 4a + 3a^2 + 2ab + a^2b^2,$$

and implies

$$b = \frac{-a \pm \sqrt{4a^3 - 3a^4}}{a^2}.$$

Since $b > 0$, then

$$0 < a < \frac{4}{3} \text{ and } b = \frac{-a + \sqrt{4a^3 - 3a^4}}{a^2}. \tag{6.14}$$

An easy computation shows that (6.13) and (6.14) are incompatible and therefore $p_0 \leq 4$.

Finally, let λ be an eigenvalue of $\mathscr{A}_b$ and u be its associated eigenvector that, up to a multiplicative factor, takes the form

$$u = (\sinh(\lambda x), \lambda \sinh(\lambda x), a\lambda \sinh(\lambda), \frac{\lambda}{\lambda + b} \sinh(\lambda)) = (y, z, \xi, \eta),$$

and let $v \in \mathscr{D}(\mathscr{A}_b)$ be a solution of

$$(\mathscr{A}_b - \lambda I)v = u. \tag{6.15}$$

A direct computation shows that if $v \in \mathscr{D}(\mathscr{A}_b)$ satisfies the two first equations and the fourth equation of (6.15), then necessarily

$$\begin{aligned} v &= (x\cosh(\lambda x), x\lambda \cosh(\lambda x) + \sinh(\lambda x), a(\lambda \cosh(\lambda) + \sinh(\lambda)), \\ &\quad \frac{(b\lambda + \lambda^2)\cosh(\lambda) + b\sinh(\lambda)}{(\lambda + b)^2}) \\ &= \frac{\partial u}{\partial \lambda}, \end{aligned} \tag{6.16}$$

modulo an element in $\ker(\mathscr{A}_b - \lambda I)$.

It remains to look at the third equation of (6.15) which is:

$$-\tilde{y}_x(1) - \tilde{\eta} - \lambda \tilde{\xi} = \xi,$$

where we have set $v = (\tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta})$.

Since $v = \dfrac{\partial u}{\partial \lambda}$, then the third equation of (6.15) may be rewritten as

$$\frac{\partial}{\partial \lambda}(-y_x(1) - \eta - \lambda \xi) = 0.$$

But, from (6.4), $-y_x(1) - \eta - \lambda\xi = -\dfrac{\lambda+b}{\lambda} h_b(\lambda)$. Consequently, the previous equation is equivalent with $h_b'(\lambda) = 0$ since $h_b(\lambda) = 0$.

That already proves that if λ is simple (as root of h_b) then $\ker(\mathscr{A}_b - \lambda I)^2 = \ker(\mathscr{A}_b - \lambda I)$, thus the algebraic multiplicity of λ is also one. Else $\dim\ker(\mathscr{A}_b - \lambda I)^i = i, i = 1,2$ and $(u, v = \dfrac{\partial u}{\partial \lambda})$ is a basis of $\ker(\mathscr{A}_b - \lambda I)^2$, with $u \in \ker(\mathscr{A}_b - \lambda I)$. Now, in order to determine $\ker(\mathscr{A}_b - \lambda I)^3$ we have to solve

$$w = (\mathscr{A}_b - \lambda I)v. \tag{6.17}$$

For that, we proceed as for the computation of v: Solving the first, second and the fourth equation of (6.17), we find that necessarily $w = \dfrac{\partial v}{\partial \lambda}$. Then the third equation of (6.17) is satisfied if and only if $h_b^{(2)}(\lambda) = 0$ (i.e $p_0 = 3$). In the same way, we will reiterate this argument if $p_0 = 4$. In conclusion we have proved that the algebraic multiplicity of an eigenvalue of $\mathscr{A}_b$ corresponds to its multiplicity as root of h_b and the corresponding system of generalized eigenvectors is given by (6.11-6.12). □

We now look at the asymptotic behavior of the eigenvalues $\{\lambda_{b,k}\}_{k\in\mathbb{Z}}$.

Lemma 6.7. *There exists k_0 large enough such that*

$$\lambda_{b,k+1} = i(k\pi + \frac{1}{ak\pi} - \frac{1}{3a^3k^3\pi^3}) - \frac{b}{a^2k^4\pi^4} + o(\frac{1}{k^4}), \forall |k| \geq k_0. \tag{6.18}$$

Proof. Since the eigenvalues $\lambda_{b,k}$ are the zeroes of f, we have

$$e^{2\lambda_{b,k}} = \frac{1 - b - \lambda_{b,k} + ab\lambda_{b,k} + a\lambda_{b,k}^2}{1 + b + \lambda_{b,k} + ab\lambda_{b,k} + a\lambda_{b,k}^2} = 1 - \frac{2(b+\lambda_{b,k})}{1 + b + \lambda_{b,k} + ab\lambda_{b,k} + a\lambda_{b,k}^2},$$

since $1 + b + \lambda_{b,k} + ab\lambda_{b,k} + a\lambda_{b,k}^2 = 0$ and $f(\lambda_{b,k}) = 0$ imply $\lambda_{b,k} = -b$, that is impossible since $-b$ is not an eigenvalue of $\mathscr{A}_b$. Thus, there exists $k' \in \mathbb{Z}$ such that

$$\lambda_{b,k} = ik'\pi + \frac{1}{2}\ln(1 - \frac{2(b+\lambda_{b,k})}{1 + b + \lambda_{b,k} + ab\lambda_{b,k} + a\lambda_{b,k}^2}). \tag{6.19}$$

Moreover, by Lemma 6.4, if k is great enough then the number k' in (6.19) is $k' = k-1$, if $k > 0$, or $k' = k+1$, if $k < 0$. Since $|\lambda_{b,k}|$ goes to infinity, we have the following asymptotic expansions

$$\lambda_{b,k+1} = ik\pi + o(1), \tag{6.20}$$

and

$$\frac{1}{\lambda_{b,k+1}} = -\frac{i}{k\pi} + o(\frac{1}{k^2}). \tag{6.21}$$

From (6.19) and using the Taylor series of $\ln(1 - \dfrac{2(b+\lambda_{b,k+1})}{1 + b + \lambda_{b,k+1} + ab\lambda_{b,k+1} + a\lambda_{b,k+1}^2})$ in $\dfrac{1}{\lambda_{b,k+1}}$ we find

$$\lambda_{b,k+1} = ik\pi - \frac{1}{a\lambda_{b,k+1}} + \frac{-\frac{1}{3a^3} + \frac{1}{a^2}}{\lambda_{b,k+1}^3} - \frac{b}{a^2\lambda_{b,k+1}^4} + o(\frac{1}{\lambda_{b,k+1}^4}). \tag{6.22}$$

Using (6.21) in (6.22) we successively get:

$$\lambda_{b,k+1} = ik\pi + \frac{i}{ak\pi} + o(\frac{1}{k^2}),$$

$$\frac{1}{\lambda_{b,k+1}} = -\frac{i}{k\pi} + \frac{i}{ak^3\pi^3} + o(\frac{1}{k^4}), \tag{6.23}$$

$$\frac{1}{\lambda_{b,k+1}^3} = \frac{i}{k^3\pi^3} + o(\frac{1}{k^4}), \tag{6.24}$$

$$\frac{1}{\lambda_{b,k+1}^4} = \frac{1}{k^4\pi^4} + o(\frac{1}{k^4}). \tag{6.25}$$

Inserting (6.23), (6.24) and (6.25) in (6.22) we obtain (6.18). □

The previous results lead to the next Riesz basis property.

Lemma 6.8. *The system $\{\phi_{b,k}\}_{k\in\mathbb{Z}}$ of eigenvectors of $\mathscr{A}_b$ forms a Riesz basis of $\mathscr{H}$.*

Proof. Firstly, we prove that the system $\{\phi_{b,k}\}_{k\in\mathbb{Z}}$ is quadratically close to the Riesz basis (here an orthogonal basis) $\{\phi_k\}_{k\in\mathbb{Z}}$ of $\mathscr{H}$, i.e

$$\sum_{k\in\mathbb{Z}} \|\phi_k - \phi_{b,k}\|^2_{\mathscr{H}} < +\infty. \tag{6.26}$$

Indeed by (3.2) we have

$$i\sqrt{N(\mu_k)}\phi_k = (\sinh(\lambda_k x), \lambda_k \sinh(\lambda_k x), a\lambda_k \sinh(\lambda_k), \sinh(\lambda_k)), \tag{6.27}$$

where we recall that $\lambda_k = i\mu_k$.

On the other hand, using estimates (3.9) and (6.18) we have

$$|\sinh(\lambda_k x) - \sinh(\lambda_{b,k} x)| \lesssim |\lambda_k - \lambda_{b,k}| \lesssim \frac{1}{k^2}, \tag{6.28}$$

and

$$|\cosh(\lambda_k x) - \cosh(\lambda_{b,k} x)| \lesssim |\lambda_k - \lambda_{b,k}| \lesssim \frac{1}{k^2}. \tag{6.29}$$

By estimates (3.10) and (6.18) we have also

$$\sqrt{N(\mu_k)} \sim |\lambda_{b,k}| \sim |k|. \tag{6.30}$$

Combining (6.28), (6.29) and (6.30), we get

$$\|\phi_k - \phi_{b,k}\|^2_{\mathscr{H}} \lesssim \frac{1}{k^2}. \tag{6.31}$$

This shows that (6.26) holds.

Using Lemma A.6 of [9], the system $\{\phi_{b,k}\}_{k\in\mathbb{Z}}$ is ω-linearly independent, (i.e, for any sequence $(c_k)_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}} c_k\phi_{b,k} = 0$ in $\mathscr{H}$, then $c_k = 0$, for all $k \in Z$), and since this system is quadratically close to the Riesz basis $\{\phi_k\}_{k\in\mathbb{Z}}$, the conclusion follows applying Bari's theorem (see Theorem 2.3 of [10]). □

We are finally able to prove the optimal energy decay rate.

Theorem 6.9. *For $u_0 \in \mathscr{D}(A)$, let $\omega(u_0)$ defined by (6.1). Then we have:*

$$\inf_{u_0\in\mathscr{D}(A)} \omega(u_0) = \frac{1}{2}. \tag{6.32}$$

Proof. Let $\varepsilon > 0$ and define u_0^ε by

$$u_0^\varepsilon = \sum_{k\in\mathbb{Z}^*} \frac{1}{k^{3/2+\varepsilon}} \phi_{b,k}.$$

Then we have

$$\mathscr{A}_b u_0^\varepsilon = \sum_{k\in\mathbb{Z}^*} \frac{\lambda_{b,k}}{k^{3/2+\varepsilon}} \phi_{b,k}.$$

By estimate (6.18), we deduce that

$$\left|\frac{\lambda_{b,k}}{k^{3/2+\varepsilon}}\right| \sim \frac{1}{k^{1/2+\varepsilon}}.$$

Since $\{\phi_{b,k}\}_{k\in\mathbb{Z}}$ forms a Riesz basis of $\mathscr{H}$, we obtain

$$\|\mathscr{A}_b u_0^\varepsilon\|_{\mathscr{H}}^2 \sim \sum_{k\in\mathbb{Z}^*} \left|\frac{\lambda_{b,k}}{k^{3/2+\varepsilon}}\right|^2 \sim \sum_{k\in\mathbb{Z}^*} \frac{1}{k^{1+2\varepsilon}} < \infty.$$

Consequently $u_0^\varepsilon \in \mathscr{D}(\mathscr{A}_b)$, and thanks to (6.18), we can apply Remark 6.2 with $q = \dfrac{3}{2} + \varepsilon$ and $\delta = 4$, to deduce that

$$E_\varepsilon(t) = \frac{1}{2}\|u^\varepsilon(t)\|_{\mathscr{H}}^2 \sim t^{-\dfrac{1+\varepsilon}{2}}, \ \forall t > 0.$$

It follows that

$$\omega(u_0^\varepsilon) = \frac{1+\varepsilon}{2}.$$

Then, from Theorem 5.3, we have

$$\frac{1}{2} \leq \inf_{u_0\in\mathscr{D}(\mathscr{A})} \omega(u_0) \leq \frac{1+\varepsilon}{2}.$$

Hence, we deduce (6.32) since ε is arbitrarily small. □

Acknowledgments

The authors thank the referee for their careful reading of the manuscript and insightful comments.

References

[1] F. Alabau. Stabilisation frontière indirecte de systèmes faiblement couplés. *C. R. Acad. Sci. Paris Sér. I Math.*, **328** (11), 1015–1020, 1999.

[2] F. Alabau-Boussouira. Indirect boundary stabilization of weakly coupled hyperbolic systems. *SIAM J. Control Optim.*, **41** (2), 511–541, 2002.

[3] K. Ammari and M. Tucsnak, Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, *SIAM J. Control. Optim.*, **39**, 1160–1181, 2000.

[4] K. T. Andrews, K. L. Kuttler and M. Shillor. Second order evolution equations with dynamic boundary conditions. *J. Math. Anal. Appl.*, **197** (3), 781–795, 1996.

[5] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.*, **305**, 837–852, 1988 .

[6] C. Baiocchi, V. Komornik, and P. Loreti, Ingham-Beurling type theorems with weakened gap conditions. *Acta Math. Hungar.*, **97**, 55–95, 2002.

[7] A. Bátkai, K-J. Engel, J. Prüss and R. Schnaubelt, Polynomial stability of operator semigroups, *Math. Nachr.*, **279** (13-14), 1425–1440, 2006.

[8] C. Castro and E. Zuazua. Exact boundary controllability of two Euler-Bernoulli beams connected by a point mass. *Mathematical and Computer Modelling*, **32**, 955–969, 2000.

[9] F. Conrad and Ö. Morgül. On the stabilization of a flexible beam with a tip mass. *SIAM J. Control Optim.*, **36** (6), 1962–1986, 1998.

[10] I. C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, AMS, Providence, B.I., 1969.

[11] W. Littman and L. Markus, Some recent results on control and stabilization of flexible structures, *In Proc. COMCON Workshop*, Montpellier, 1987.

[12] W. Littman and L. Markus. Stabilization of a hybrid system of elasticity by feedback boundary damping. *Ann. Mat. Pura Appl. (4)*, **152**, 281–330, 1988.

[13] D. Mercier and V. Rgnier, Boundary Controllability of a chain of serially connected Euler-Bernoulli beams with interior masses, *Collectanea Mathematica.*, **60**, 2009, 307-334.

[14] S. Nicaise and J. Valein, Stabilization of the wave equation on 1-D networks with a delay term in the nodal feedbacks, *Netw. Heterog. Media*, **2**, 425-479, 2007.

[15] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, New York, 1983.

[16] M. Pellicer and J. Sòla-Morales. Analysis of a viscoelastic spring-mass model. *J. Math. Anal. Appl.*, **294**, 687–698, 2004.

[17] B. Rao, Stabilization of elastic plates with dynamical boundary control, *SIAM J. Control Optim.*, **36** (1), 148–163, 1998.

[18] D. L. Russell. Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods. *J. Diff. Eq.*, **19**, 344–370, 1975.

[19] D. L. Russell. A general framework for the study of indirect damping mechanisms in elastic systems. *J. Math. Anal. Appl.*, **173** (2), 339–358, 1993.

[20] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2009.

[21] A. Wehbe, Rational energy decay rate for a wave equation with dynamical control, *Applied Mathematics Letters*, **16** (3), 357–364, 2003.