STANDARD MONOMIAL THEORY FOR DESINGULARIZED RICHARDSON VARIETIES IN THE FLAG VARIETY $GL(n)/B$

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Abstract. We consider a desingularization Γ of a Richardson variety $X_w^v = X_w \cap X^v$ in the flag variety $F\ell(n) = GL(n)/B$, obtained as a fibre of a projection from a certain Bott-Samelson variety Z. We then construct a basis of the homogeneous coordinate ring of Γ inside Z, indexed by combinatorial objects which we call w_0-*standard tableaux*.

Introduction

Standard Monomial Theory (SMT) originated in the work of Hodge [19], who considered it in the case of the Grassmannian $G_{d,n}$ of d-subspaces of a (complex) vector space of dimension n. The homogeneous coordinate ring $\mathbf{C}[G_{d,n}]$ is the quotient of the polynomial ring in the Plücker coordinates $p_{i_1 \dots i_d}$ by the Plücker relations, and Hodge provided a combinatorial rule to select, among all monomials in the $p_{i_1 \dots i_d}$, a subset that forms a basis of $\mathbf{C}[G_{d,n}]$: these (so-called standard) monomials are parametrized by semi-standard Young tableaux. Moreover, he showed that this basis is compatible with any Schubert variety $X \subset G_{d,n}$, in the sense that those basis elements that remain non-zero when restricted to X can be characterized combinatorially, and still form a basis of $\mathbf{C}[X]$. The aim of SMT is then to generalize Hodge's result to any flag variety G/P (G a connected semi-simple group, P a parabolic subgroup): in a more modern formulation, the problem consists, given a line bundle L on G/P, in producing a "nice" basis of the space of sections $H^0(X, L)$ ($X \subset G/P$ a Schubert variety), parametrized by some combinatorial objects. SMT was developed by Lakshmibai and Seshadri (see [29, 30]) for groups of classical type, and Littelmann extended it to groups of arbitrary type (including in the Kac-Moody setting), using techniques such as the path model in representation theory [32, 33] and Lusztig's Frobenius map for quantum groups at roots of unity [34]. Standard Monomial Theory has numerous applications in the geometry of Schubert varieties: normality, vanishing theorems, ideal theory, singularities, and so on [26].

Richardson varieties, named after [36], are intersections of a Schubert variety and an opposite Schubert variety inside a flag variety G/P. They previously appeared in [20, Ch. XIV, §4] and [38], as well as the corresponding open subvarieties in [10]. They have since played a role in different contexts, such as equivariant K-theory [25], positivity in Grothendieck groups [5], standard monomial theory [7], Poisson geometry [13], positroid varieties [21], and their generalizations [22, 2]. In particular, SMT on G/P is known to be compatible with Richardson varieties [25] (at least for a very ample line bundle on G/P).

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Like Schubert varieties, Richardson varieties may be singular [24, 23, 40, 1]. Desingularizations of Schubert varieties are well known: they are the Bott-Samelson varieties [4, 9, 14], which are also used for example to establish some properties of Schubert polynomials [35], or to give criteria for the smoothness of Schubert varieties [12, 8]. An SMT has been developed for Bott-Samelson varieties of type A_n in [28], and of arbitrary type in [27] using the path model [32, 33].

In the present paper, we shall describe a Standard Monomial Theory for a desingularization of a Richardson variety. To be more precise, we introduce some notations. Let $G = GL(n, k)$ where k is an algebraically closed field of arbitrary characteristic, B the Borel subgroup of upper triangular matrices, and $T \subset B$ the maximal torus of diagonal matrices. The quotient G/B identifies with the variety $F\ell(n)$ of all complete flags in k^n. Let $(e_1, \dots, e_n)$ be the canonical basis of k^n. To each permutation $w \in S_n$, we can associate a T-fixed point e_w in $F\ell(n)$: its ith constituent is the space generated by $e_{w(1)}, \dots, e_{w(i)}$. We denote by F_{can} the T-fixed point corresponding to the identity e of S_n, and $F_{\mathrm{op\,can}}$ the T-fixed point e_{w_0}, where w_0 is the longest element of S_n. The symmetric group S_n is generated by the simple transpositions $s_i = (i, i+1)$, $i = 1, \dots, n$. We denote a permutation $u \in S_n$ with the one-line notation $[u(1)\ u(2) \dots\ u(n)]$. Denote by B^- the subgroup of G of lower triangular matrices and consider the Schubert cells $C_w = B.e_w$ and the opposite Schubert cells $C^v = B^-.e_v$. The Richardson variety $X_w^v \subset F\ell(n)$ is the intersection of the direct Schubert variety $X_w = \overline{C_w}$ with the opposite Schubert variety $X^v = \overline{C^v} = w_0 X_{w_0 v}$. Fix a reduced decomposition $w = s_{i_1} \dots s_{i_d}$ and consider the Bott-Samelson desingularization $Z = Z_{i_1 \dots i_d}(F_{\mathrm{can}}) \to X_w$, and similarly $Z' = Z_{i_r i_{r-1} \dots i_{d+1}}(F_{\mathrm{op\,can}}) \to X^v$ for a reduced decomposition $w_0 v = s_{i_r} s_{i_{r-1}} \dots s_{i_{d+1}}$. Then the fibred product $Z \times_{F\ell(n)} Z'$ has been considered as a desingularization of X_w^v in [6], but for our purposes, it will be more convenient to realize it as the fibre $\Gamma_{\mathbf{i}}$ ($\mathbf{i} = i_1 \dots i_d i_{d+1} \dots i_r$) of the projection $Z_{\mathbf{i}} = Z_{\mathbf{i}}(F_{\mathrm{can}}) \to F\ell(n)$ over $F_{\mathrm{op\,can}}$ (see Section 1 for the precise connection between those two constructions).

In [28, 27], Lakshmibai, Littelmann, and Magyar define a family of line bundles $L_{\mathbf{i},\mathbf{m}}$ ($\mathbf{m} = m_1 \dots m_r \in \mathbf{Z}^r_{\geq 0}$) on $Z_{\mathbf{i}}$ (they are the only globally generated line bundles on $Z_{\mathbf{i}}$, as pointed out in [31]), and give a basis for the space of sections $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$. In [28], the elements p_T of this basis, called standard monomials, are indexed by combinatorial objects T called standard tableaux: the latter's definition involves certain sequences $J_{11} \supset \cdots \supset J_{1m_1} \supset \cdots \supset J_{r1} \supset \cdots \supset J_{rm_r}$ of subwords of $\mathbf{i}$, called liftings of T (see Section 2 for precise definitions—actually, two equivalent definitions of standard tableaux are given in [28], but we will only use the one in terms of liftings). Note also that $L_{\mathbf{i},\mathbf{m}}$ is very ample precisely when $m_j > 0$ for all j (see [31], Theorem 3.1), in which case $\mathbf{m}$ is called regular.

The main result of this paper states that in this case, SMT on $Z_{\mathbf{i}}$ is compatible with $\Gamma_{\mathbf{i}}$.

Theorem 0.1. *Assume that* $\mathbf{m}$ *is regular. With the above notation, the standard monomials* p_T *such that* $(p_T)_{|\Gamma_{\mathbf{i}}} \neq 0$ *still form a basis of* $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$.
Moreover, $(p_T)_{|\Gamma_{\mathbf{i}}} \neq 0$ *if and only if* T *admits a lifting* $J_{11} \supset \cdots \supset J_{rm_r}$ *such that each subword* J_{km} *contains a reduced expression of* w_0.

We prove this theorem in three steps.

(1) Call T (or p_T) w_0-standard if the above condition on (J_{km}) holds. We prove by induction over $M = \sum_{j=1}^r m_j$ that the w_0-standard monomials p_T are linearly independent on $\Gamma_{\mathbf{i}}$. (Here the assumption that $\mathbf{m}$ is regular is not necessary.)
(2) In the regular case, we prove that a standard monomial p_T does not vanish identically on $\Gamma_{\mathbf{i}}$ if and only if it is w_0-standard, using the combinatorics of the Demazure product (see Definition 4.2). It follows that w_0-standard monomials form a basis of the homogeneous coordinate ring of $\Gamma_{\mathbf{i}}$ (when $\Gamma_{\mathbf{i}}$ is embedded in a projective space via the very ample line bundle $L_{\mathbf{i},\mathbf{m}}$).
(3) We use cohomological techniques to prove that the restriction map
$$H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$$
is surjective. More explicitly, we define a family $(Y_{\mathbf{i}}^u)$ of subvarieties of $Z_{\mathbf{i}}$ indexed by S_n, with the property that $Y_{\mathbf{i}}^e = Z_{\mathbf{i}}$ and $Y_{\mathbf{i}}^{w_0} = \Gamma_{\mathbf{i}}$. We construct a sequence in S_n, $e = u_0 < u_1 < \cdots < u_N = w_0$, such that for every t, $Y_{\mathbf{i}}^{u_{t+1}}$ is defined in $Y_{\mathbf{i}}^{u_t}$ by the vanishing of a single Plücker coordinate p_κ, in such a way that each restriction map $H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}})$ can be shown to be surjective using vanishing theorems (Corollary 5.7 and Theorem 5.23). This shows that the w_0-standard monomials span $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$.

Note that alternate bases for certain Bott-Samelson varieties have been constructed in [39], and the fibred products $Z \times_{F\ell(n)} Z'$ have been studied from this point of view in [11].

Sections are organized as follows: in Section 1, we first fix notation and recall information on Bott-Samelson varieties $Z_{\mathbf{i}}$, and then show that the fibre $\Gamma_{\mathbf{i}}$ of $Z_{\mathbf{i}} \to F\ell(n)$ over $F_{\mathrm{op\,can}}$ is a desingularization of the Richardson variety X_w^v; this fact is most certainly known to experts, but has not, to our knowledge, appeared in the literature. In Section 2, we recall the main results about SMT for Bott-Samelson varieties from [28], in particular the definition of standard tableaux. In Section 3, we define w_0-standard monomials and we prove that they are linearly independent in $\Gamma_{\mathbf{i}}$. In Section 4, we prove that when $\mathbf{m}$ is regular, a standard monomial does not vanish identically on $\Gamma_{\mathbf{i}}$ if and only if it is w_0-standard. Finally, we prove in Section 5 that w_0-standard monomials generate the space of sections $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$.

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1. Desingularized Richardson varieties

The notations are as in the Introduction. In addition, if $k, l \in \mathbf{Z}$, then we denote by $[k, l]$ the set $\{k, k+1, \ldots, l\}$, and by $[l]$ the set $[1, l]$.

We first recall a number of results on Bott-Samelson varieties (see *e.g.* [35]).

Definition 1.1. Two flags F, G in $F\ell(n)$ are called *i-adjacent* if they coincide except (possibly) at their components of dimension i, a situation denoted by $F \overset{i}{—} G$.

Notations 1.2. For $i \in [n]$, we denote by $F\ell(\hat{i})$ the variety of partial flags

$$V_1 \subset V_2 \subset \cdots \subset V_{i-1} \subset V_{i+1} \subset \cdots \subset V_n,\ (\dim V_j = j),$$

and by $\psi_{\hat{i}} : F\ell(n) \to F\ell(\hat{i})$ the natural projection.

Then F and G are i-adjacent if and only if they have the same image by $\psi_{\hat{i}}$.

Consider a word $\mathbf{i} = i_1 \dots i_r$ in $[n-1]$, with $w(\mathbf{i}) = s_{i_1} \dots s_{i_r} \in S_n$ not necessarily reduced. A *gallery of type* $\mathbf{i}$ is a sequence of the form

$$F_0 \overset{i_1}{—} F_1 \overset{i_2}{—} \dots \overset{i_r}{—} F_r. \tag{1}$$

For a given flag F_0, the *Bott-Samelson variety* of type $\mathbf{i}$ starting at F_0 is the set of all galleries (1), *i.e.* the fibred product

$$Z_{\mathbf{i}}(F_0) = \{F_0\} \times_{F\ell(\hat{i}_1)} F\ell(n) \times_{F\ell(\hat{i}_2)} \cdots \times_{F\ell(\hat{i}_r)} F\ell(n)$$

(a subvariety of $F\ell(n)^r$). In particular, $Z_{i_1 \dots i_r}(F_0)$ is a $\mathbf{P}^1$-fibration over $Z_{i_1 \dots i_{r-1}}(F_0)$, which shows by induction over r that Bott-Samelson varieties are smooth.

Each subset $J = \{j_1 < \cdots < j_k\} \subset [r]$ defines a subword $\mathbf{i}(J) = (i_{j_1}, \dots, i_{j_k})$ of $\mathbf{i}$. We then write $Z_J(F_0)$ instead of $Z_{\mathbf{i}(J)}(F_0)$, and we view it as the subvariety of $Z_{\mathbf{i}}(F_0)$ consisting of all galleries (1) such that $F_{j-1} = F_j$ whenever $j \notin J$.

We denote by $F_{\text{can}} : \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, e_2, \dots, e_n \rangle$ the flag associated to the canonical basis, and by $F_{\text{op can}} : \langle e_n \rangle \subset \langle e_n, e_{n-1} \rangle \subset \cdots \subset \langle e_n, e_{n-1}, \dots, e_1 \rangle$ the opposite canonical flag. Note that $F_{\text{op can}} = e_{w_0}$.

In the sequel, we shall only need galleries starting at F_{can} or at $F_{\text{op can}}$; in particular, we write $Z_{\mathbf{i}} = Z_{\mathbf{i}}(F_{\text{can}})$.

The (diagonal) B-action on $F\ell(n)^r$ leaves $Z_{\mathbf{i}}$ invariant. In particular, the T-fixed points of $Z_{\mathbf{i}}$ are the galleries of the form

$$F_{\text{can}} \overset{i_1}{—} e_{u_1} \overset{i_2}{—} e_{u_1 u_2} \overset{i_3}{—} \dots \overset{i_r}{—} e_{u_1 \dots u_r},$$

where each $u_j \in S_n$ is either e or s_{i_j}. This gallery will be denoted $e_J \in Z_{\mathbf{i}}$, where $J = \{j \mid u_j = s_{i_j}\} = \{j_1 < \cdots < j_k\}$.

For $j \in [r]$, we denote by $\text{pr}_j : Z_{\mathbf{i}} \to F\ell(n)$ the projection sending the gallery (1) to F_j. Note that $w(\mathbf{i}(J)) = s_{i_{j_1}} \dots s_{i_{j_k}} = u_1 \dots u_r$, so $\text{pr}_r(e_J) = e_{u_1 \dots u_r} = e_{w(\mathbf{i}(J))}$.

When $\mathbf{i}$ is reduced, *i.e.* $w = s_{i_1} \dots s_{i_r}$ is a reduced expression in S_n, a flag F lies in the Schubert variety X_w if and only if there is a gallery of type $\mathbf{i} = i_1 \dots i_r$ from F_{can} to F, hence the last projection pr_r takes $Z_{\mathbf{i}}$ surjectively to X_w. Moreover, this surjection is birational: it restricts to an isomorphism over the Schubert cell C_w: thus, $\text{pr}_r : Z_{\mathbf{i}} \to X_w$ is a desingularization of X_w, and likewise for the last projection $Z_{\mathbf{i}}(F_{\text{op can}}) \to X^{w_0 w}$.

When $\mathbf{i}$ is not necessarily reduced, $\text{pr}_r(Z_{\mathbf{i}})$ may be described as follows. Recall [28, Definition-Lemma 1] that the poset $\{w(\mathbf{i}(J)) \mid J \subset [r]\}$ admits a unique maximal element, denoted by $w_{\max}(\mathbf{i})$ (so $w_{\max}(\mathbf{i}) = w(\mathbf{i})$ if and only if $\mathbf{i}$ is reduced):

Proposition 1.3. *Let* $\mathbf{i}$ *be an arbitrary word. Then* $\text{pr}_r(Z_{\mathbf{i}})$ *is the Schubert variety* X_w*, where* $w = w_{max}(\mathbf{i})$.

Proof. Since $\text{pr}_r(Z_{\mathbf{i}})$ is B-stable, it is a union of Schubert cells. But $Z_{\mathbf{i}}$ is a projective variety, so the morphism pr_r is closed, hence $\text{pr}_r(Z_{\mathbf{i}})$ is a union of Schubert varieties, and therefore a single Schubert variety X_w since $Z_{\mathbf{i}}$ is irreducible. Moreover, the T-fixed points e_J in $Z_{\mathbf{i}}$ project to the T-fixed points $e_{w(\mathbf{i}(J))}$ in X_w, and all T-fixed points of X_w are obtained in this way (indeed, if e_v is such a point, then the fibre $\mathrm{pr}_r^{-1}(e_v)$ is T-stable, so it must contain some e_J by Borel's fixed point theorem). In particular, e_w corresponds to a choice of $J \subset \{1, \ldots, r\}$ such that $w(\mathbf{i}(J))$ is maximal, hence the result. □

We now turn to the description of a desingularization of a Richardson variety $X_w^v = X_w \cap X^v$, $v \leq w \in S_n$. Let $Z = Z_{i_1 \ldots i_d}$ for some reduced decomposition $w = s_{i_1} \ldots s_{i_d}$ and $Z' = Z_{i_r \ldots i_{d+1}}(F_{\mathrm{op\,can}})$ for some reduced decomposition $w_0 v = s_{i_r} s_{i_{r-1}} \ldots s_{i_{d+1}}$. Since Z desingularizes X_w and Z' desingularizes X^v, a natural candidate for a desingularization of X_w^v is the fibred product $Z \times_{F\ell(n)} Z'$. However, we wish to see this variety in a slightly different way: an element of $Z \times Z'$ is a pair of galleries

$$F_{\mathrm{can}} \overset{i_1}{\text{---}} F_1 \overset{i_2}{\text{---}} \ldots \overset{i_d}{\text{---}} F_d,$$

$$F_{\mathrm{op\,can}} \overset{i_r}{\text{---}} G_{r-1} \overset{i_{r-1}}{\text{---}} \ldots \overset{i_{d+1}}{\text{---}} G_d,$$

and it belongs to $Z \times_{F\ell(n)} Z'$ when the end points F_d and G_d coincide: in this case, by reversing the second gallery, they concatenate to form a longer gallery

$$F_{\mathrm{can}} \overset{i_1}{\text{---}} F_1 \overset{i_2}{\text{---}} \ldots \overset{i_d}{\text{---}} F_d \overset{i_{d+1}}{\text{---}} \ldots \overset{i_r}{\text{---}} F_{\mathrm{op\,can}}.$$

Thus, $Z \times_{F\ell(n)} Z'$ identifies with the set of all galleries in $Z_{\mathbf{i}} = Z_{i_1 \ldots i_r}$ that end in $F_{\mathrm{op\,can}}$, *i.e.* with the fibre

$$\Gamma_{\mathbf{i}} = \mathrm{pr}_r^{-1}(F_{\mathrm{op\,can}})$$

of the last projection $\mathrm{pr}_r : Z_{\mathbf{i}} \to F\ell(n)$. By construction, the dth projection pr_d then maps $\Gamma_{\mathbf{i}}$ onto the Richardson variety X_w^v.

Proposition 1.4. *In the above notation, the dth projection $\mathrm{pr}_d : \Gamma_{\mathbf{i}} \to X_w^v$ is a desingularization,* i.e. pr_d *is birational, and the variety $\Gamma_{\mathbf{i}}$ is smooth and irreducible.*

Proof. We first compute the dimension of $\Gamma_{\mathbf{i}}$: since pr_r is surjective, there exists a non-empty open set O in $F\ell(n)$ such that every point $F \in O$ has a fibre of pure dimension $\dim(Z_{\mathbf{i}}) - \dim(F\ell(n))$. Since the flag variety $F\ell(n)$ is irreducible, O meets the open Schubert cell C_{w_0}. Let $F \in O \cap C_{w_0}$. Since pr_r is B-equivariant, the fibres of F and $F_{\mathrm{op\,can}} = e_{w_0}$ are isomorphic. In particular, they have the same dimension, so $\dim(\Gamma_{\mathbf{i}}) = \dim(Z_{\mathbf{i}}) - \dim(F\ell(n))$.

Next we show that $\Gamma_{\mathbf{i}}$ is smooth. Let $\gamma \in \Gamma_{\mathbf{i}}$. We want to prove that the tangent space $T_\gamma(\Gamma_{\mathbf{i}})$ of $\Gamma_{\mathbf{i}}$ at γ and $\Gamma_{\mathbf{i}}$ have the same dimension. Let $\Omega = \mathrm{pr}_r^{-1}(C_{w_0})$. Let U be the maximal unipotent subgroup of B. This subgroup acts simply transitively on the Schubert cell C_{w_0}. Consider the morphism

$$\begin{array}{rrcl} s: & C_{w_0} = U.e_{w_0} & \to & \Omega \\ & u.e_{w_0} & \mapsto & u.\gamma \end{array}$$

Since pr_r is U-equivariant, we have $\mathrm{pr}_r \circ s = \mathrm{id}_{C_{w_0}}$. Differentiating this equality in e_{w_0} gives $\mathrm{d}\,\mathrm{pr}_r(\gamma) \circ \mathrm{d}\,s(e_{w_0}) = \mathrm{id}_{T_{e_{w_0}} F\ell(n)}$. In particular, the linear map $\mathrm{d}\,\mathrm{pr}_r(\gamma) : T_\gamma(Z_{\mathbf{i}}) \to T_{e_{w_0}}(F\ell(n))$ is surjective. Moreover, $T_\gamma(\Gamma_{\mathbf{i}}) \subset \ker(\mathrm{d}\,\mathrm{pr}_r(\gamma))$. From this, we deduce

$$\begin{aligned} \dim(\Gamma_{\mathbf{i}}) \leq \dim T_\gamma(\Gamma_{\mathbf{i}}) &\leq \dim T_\gamma(Z_{\mathbf{i}}) - \dim T_{e_{w_0}}(F\ell(n)) \\ &\leq \dim Z_{\mathbf{i}} - \dim F\ell(n) \text{ (since } Z_{\mathbf{i}} \text{ and } F\ell(n) \text{ are both smooth)} \\ &\leq \dim \Gamma_{\mathbf{i}}, \end{aligned}$$

hence $\Gamma_{\mathbf{i}}$ is smooth.

Now we show that $\Gamma_{\mathbf{i}}$ is irreducible. Let $C_1, \ldots, C_e$ be the irreducible components of $\Gamma_{\mathbf{i}}$. Since $\Gamma_{\mathbf{i}}$ is smooth, the C_j are also the connected components of $\Gamma_{\mathbf{i}}$. The variety Ω is open in $Z_{\mathbf{i}}$. In particular, Ω is irreducible. Since pr_r is B-equivariant,

$$\Omega = \bigcup_{i=1}^{e} \bigcup_{b \in B} bC_i.$$

Let $\Omega_i = \bigcup_{b\in B} bC_i$. The morphism $f : U \times \Gamma_{\mathbf{i}} \to \Omega, (b, \gamma) \mapsto b.\gamma$ is an isomorphism. In particular, $\Omega_i = f(U \times C_i)$ is an irreducible closed set in Ω. So $\Omega = \bigcup_{i=1}^{e} \Omega_i$ is a disjoint decomposition of Ω into irreducibles. Hence $e = 1$, and $\Gamma_{\mathbf{i}}$ is irreducible.

Finally, to show that $\Gamma_{\mathbf{i}} \to X_w^v$ is birational, we consider the projections $\mathrm{pr}_d : Z \to X_w$ and $\mathrm{pr}_{r-d} : Z' \to X^v$. Since they are birational, there exist open subsets $U_w \subset X_w$ and $O \subset Z$ isomorphic under pr_d, and open subsets $U^v \subset X^v$ and $O' \subset Z'$ isomorphic under pr_{r-d}. Then the open set $(O \times O') \cap (Z \times_{F\ell(n)} Z')$ of $Z \times_{F\ell(n)} Z'$ is isomorphic to the open set $U_w \cap U^v$ of X_w^v under $\mathrm{pr}_d : Z \times_{F\ell(n)} Z' \to X_w^v$. Since X_w^v and $Z \times_{F\ell(n)} Z' \cong \Gamma_{\mathbf{i}}$ are irreducible, these open subsets must be dense. The birationality of $\mathrm{pr}_d : \Gamma_{\mathbf{i}} \to X_w^v$ follows. □

Remark 1.5. In characteristic 0, it can be proved more directly that the fibred product $Z \times_{F\ell(n)} Z'$ is smooth using Kleiman's transversality theorem (*cf.* [16], Theorem 10.8). Moreover, this theorem also states that every irreducible component of $Z \times_{F\ell(n)} Z'$ is of dimension $\dim(Z) + \dim(Z') - \dim(F\ell(n))$. To prove the irreducibility of $Z \times_{F\ell(n)} Z'$, consider ∂Z (resp. $\partial Z'$) the union of all Bott-Samelson varieties X with $X \subsetneq Z$ (resp. $X \subsetneq Z'$). By Kleiman's transversality theorem, the dimension of $(\partial Z \times_{F\ell(n)} Z') \cup (Z \times_{F\ell(n)} \partial Z')$ is less than $\dim(Z \times_{F\ell(n)} Z')$. So, on one hand, the fibred-product $O = (Z \setminus \partial Z) \times_{F\ell(n)} (Z' \setminus \partial Z')$ meets each irreducible component of $Z \times_{F\ell(n)} Z'$, hence O is dense. On the other hand, O is isomorphic to the open Richardson variety $C_w^v = C_w \cap C^v$, hence O is irreducible. Therefore, $Z \times_{F\ell(n)} Z'$ is irreducible.

For $\mathbf{i}$ an arbitrary word, we may still consider the variety $\Gamma_{\mathbf{i}}$ of galleries of type $\mathbf{i}$, beginning at F_{can} and ending at $F_{\mathrm{op\,can}}$. In general this variety is no longer birational to a Richardson variety. But we still have

Proposition 1.6. *Let $\mathbf{i} = i_1 \ldots i_r$ be an arbitrary word, and consider the projection $\mathrm{pr}_j : \Gamma_{\mathbf{i}} \to F\ell(n)$. Then $\mathrm{pr}_j(\Gamma_{\mathbf{i}})$ is the Richardson variety X_y^x where $y = w_{max}(i_1 \ldots i_j)$ and $x = w_0 w_{max}(i_{j+1} \ldots i_r)^{-1}$. Moreover, $\Gamma_{\mathbf{i}}$ is smooth and irreducible.*

Proof. The variety $\Gamma_{\mathbf{i}}$ is isomorphic to the fibred product

$$Z_{i_1 \ldots i_j} \times_{F\ell(n)} Z_{i_r \ldots i_{j+1}}(F_{\mathrm{op\,can}}),$$

hence

$$\begin{aligned}
\mathrm{pr}_j(\Gamma_{\mathbf{i}}) &= \mathrm{pr}_j(Z_{i_1 \ldots i_j}) \cap \mathrm{pr}_{r-j}(Z_{i_r \ldots i_{j+1}}(F_{\mathrm{op\,can}})) \\
&= X_{w_{\max}(i_1 \ldots i_j)} \cap w_0 X_{w_{\max}(i_r \ldots i_{j+1})} \\
&= X_y^x.
\end{aligned}$$

Eventually, we may prove that $\Gamma_{\mathbf{i}}$ is smooth and irreducible exactly as in the proof of Proposition 1.4. □

Example 1.7. We consider the Richardson variety $X_w^v \subset F\ell(4)$ with $w = [4231]$ and $v = [2143]$. A flag $F = (F^1 \subset F^2 \subset F^3 \subset F^4 = k^4)$ belongs to the Schubert variety X_w if and only if F^2 meets $\langle e_1, e_2\rangle$.

Since $w = s_1s_2s_3s_2s_1$ is a reduced decomposition, the Bott-Samelson variety Z_{12321} desingularizes X_w. An element of Z_{12321} is a gallery

$$F_{\text{can}} \overset{1}{—} F_1 \overset{2}{—} F_2 \overset{3}{—} F_3 \overset{2}{—} F_4 \overset{1}{—} F_5.$$

A flag G belongs to the opposite Schubert variety X^v if and only if $G^1 \subset \langle e_2, e_3, e_4\rangle$ and $G^3 \supset \langle e_4\rangle$.

Similarly, $w_0v = s_2s_1s_3s_2$ is a reduced decomposition, so the Bott-Samelson variety $Z_{2312}(F_{\text{op can}})$ desingularizes the opposite Schubert variety X^v. An element of $Z_{2132}(F_{\text{op can}})$ is a gallery

$$F_{\text{op can}} \overset{2}{—} G_8 \overset{1}{—} G_7 \overset{3}{—} G_6 \overset{2}{—} G_5.$$

Therefore, an element of the variety $\Gamma_{123212312}$ has the form

$$\gamma = \big(F_{\text{can}} \overset{1}{—} F_1 \overset{2}{—} F_2 \overset{3}{—} F_3 \overset{2}{—} F_4 \overset{1}{—} F_5 \overset{2}{—} G_6 \overset{3}{—} G_7 \overset{1}{—} G_8 \overset{2}{—} F_{\text{op can}}\big).$$

The projection

$$\text{pr}_5 : \gamma \mapsto F_5 = G_5$$

maps $\Gamma_{123212312}$ birationally to X_w^v.

There are only two singular points on X_w^v, namely e_w and e_v. Their fibres $\text{pr}_5^{-1}(e_w)$ and $\text{pr}_5^{-1}(e_v)$ are 1-dimensional. Indeed, given a gallery $\gamma \in \Gamma_{\mathbf{i}}$, let V_j be the i_j-th component of $\text{pr}_j(\gamma)$. Since $\text{pr}_{j-1}(\gamma) \overset{i_j}{—} \text{pr}_j(\gamma)$, we know $\text{pr}_j(\gamma)$ as soon as we know $\text{pr}_{j-1}(\gamma)$ and V_j. Thus, a gallery can be given by the sequence $V_1, \ldots, V_9$. With this description, a gallery in the fibre of e_w is then given by

$$\langle e_2\rangle,\ \langle e_2, e_3\rangle,\ \langle e_2, e_3, e_4\rangle,\ \langle e_2, e_4\rangle,\ \langle e_4\rangle,\ \langle e_4, xe_2 + ye_3\rangle,\ \langle e_2, e_3, e_4\rangle,\ \langle e_4\rangle,\ \langle e_3, e_4\rangle,$$

with $[x : y] \in \mathbf{P}^1$.

Similarly, the fibre of e_v is given by

$$\langle xe_1 + ye_2\rangle,\ \langle e_1, e_2\rangle,\ \langle e_1, e_2, e_3\rangle,\ \langle e_1, e_2\rangle,\ \langle e_2\rangle,\ \langle e_2, e_4\rangle,\ \langle e_2, e_3, e_4\rangle,\ \langle e_4\rangle,\ \langle e_3, e_4\rangle,$$

with $[x : y] \in \mathbf{P}^1$.

2. Background on SMT for Bott-Samelson varieties

In this section, we recall from [28] the main definitions and results about Standard Monomial Theory for Bott-Samelson varieties.

Definitions 2.1. A *tableau* is a sequence $T = t_1 \ldots t_p$ with $t_j \in [n]$. If $T = t_1 \ldots t_p$ and $T' = t'_1 \ldots t'_{p'}$ are two tableaux, then the *concatenation* $T * T'$ is the tableau $t_1 \ldots t_p\, t'_1 \ldots t'_{p'}$. We denote by $\emptyset$ the empty tableau, so that $T * \emptyset = \emptyset * T = T$.

A *column κ of size i* is a tableau $\kappa = t_1 \ldots t_i$ with $1 \le t_1 < \cdots < t_i \le n$. The set of all columns of size i is denoted by $I_{i,n}$. The *Bruhat order* on $I_{i,n}$ is defined by

$$\kappa = t_1 \ldots t_i \le \kappa' = t'_1 \ldots t'_i \iff t_1 \le t'_1,\ \ldots,\ t_i \le t'_i.$$

The symmetric group S_n acts on $I_{i,n}$: if $w \in S_n$ and $\kappa = t_1 \dots t_i \in I_{i,n}$, then $w\kappa$ is the column obtained by rearranging the tableau $w(t_1) \dots w(t_i)$ in an increasing sequence.

For $i \in [n]$, the *fundamental weight column* ϖ_i is the sequence $12 \dots i$.

We shall be interested in a particular type of tableau, called standard.

Definitions 2.2. Let $\mathbf{i} = i_1 \dots i_r$, and $\mathbf{m} = m_1 \dots m_r \in \mathbf{Z}^r_{\geq 0}$. A *tableau of shape* $(\mathbf{i}, \mathbf{m})$ is a tableau of the form

$$\kappa_{11} * \cdots * \kappa_{1m_1} * \kappa_{21} * \cdots * \kappa_{2m_2} * \cdots * \kappa_{r1} * \cdots * \kappa_{rm_r}$$

where κ_{km} is a column of size i_k for every k, m. (If $m_k = 0$, there is no column in the corresponding position of T.)

A *lifting* of T is a sequence of subwords of $\mathbf{i}$

$$J_{11} \supset \cdots \supset J_{1m_1} \supset J_{21} \supset \cdots \supset J_{2m_2} \supset \cdots \supset J_{r1} \supset \cdots \supset J_{rm_r}$$

such that $J_{km} \cap [k]$ is a reduced subword of $\mathbf{i}$ and $w(\mathbf{i}(J_{km} \cap [k]))\varpi_{i_k} = \kappa_{km}$. If such a lifting exists, then the tableau T is said to be *standard*.

Remark 2.3. The last equality in the definition of a lifting may be viewed geometrically as follows. If $J \subset [r]$ and $j \in [r]$, then $\mathrm{pr}_j : Z_{\mathbf{i}} \to F\ell(n)$ maps $Z_J \subset Z_{\mathbf{i}}$ onto a Schubert variety $X_w \subset F\ell(n)$ (*cf.* Proof of Proposition 1.3). In the notations of Section 1, the images of T-fixed points of Z_J under pr_j are of the form $\mathrm{pr}_j(e_K) = e_{u_1 \dots u_j} = e_{w(\mathbf{i}(K \cap [j]))}$ with K running over all subsets of J, hence $w = w_{\max}(\mathbf{i}(J \cap [j]))$. In turn, the image of $\mathrm{pr}_j(Z_J)$ by the projection $F\ell(n) \to G_{i_j,n}$ is equal to the Schubert variety $X_{w\varpi_{i_j}}$: for $J = J_{km}$ in the above lifting, this projection is therefore equal to $X_{\kappa_{km}}$. We shall follow up on this point of view in Remark 4.7.

Notation 2.4. Each column $\kappa \in I_{i,n}$ identifies with a weight of $GL(n)$, in such a way that the fundamental weight column ϖ_i corresponds to the ith fundamental weight of $GL(n)$. Therefore, we also denote by ϖ_i this fundamental weight.

We recall the Plücker embedding: given an i-subspace V of k^n, choose a basis $v_1, \dots, v_i$ of V, and let M be the matrix of the vectors $v_1, \dots, v_i$ written in the basis $(e_1, \dots, e_n)$. We associate to each column $\kappa = t_1 \dots t_i$ the minor $p_\kappa(V)$ of M on rows $t_1, \dots, t_i$. Then the map $p : V \mapsto [p_\kappa(V) \mid \kappa \in I_{i,n}]$ is the Plücker embedding.

Let $\pi_i : F\ell(n) \to G_{i,n}$ be the natural projection. We denote by L_{ϖ_i} the line bundle $(p \circ \pi_i)^* \mathcal{O}(1)$.

Now consider the tensor product $L_{\varpi_{i_1}}^{\otimes m_1} \otimes \cdots \otimes L_{\varpi_{i_r}}^{\otimes m_r}$ on $F\ell(n)^r$, and denote by $L_{\mathbf{i},\mathbf{m}}$ its restriction to $Z_{\mathbf{i}} \subset F\ell(n)^r$.

Definition 2.5. To a tableau $T = \kappa_{11} * \cdots * \kappa_{1m_1} * \cdots * \kappa_{r1} * \cdots * k_{rm_r}$, one associates the section $p_T = p_{\kappa_{11}} \otimes \cdots \otimes p_{\kappa_{1m_1}} \otimes \cdots \otimes p_{\kappa_{r1}} \otimes \cdots \otimes p_{\kappa_{rm_r}}$ of $L_{\mathbf{i},\mathbf{m}}$. If T is standard of shape $(\mathbf{i}, \mathbf{m})$, then p_T is called a *standard monomial of shape* $(\mathbf{i}, \mathbf{m})$.

Theorem 2.6 ([28])**.**

(1) *The standard monomials of shape* $(\mathbf{i}, \mathbf{m})$ *form a basis of the space of sections* $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$.
(2) *For* $i > 0$, $H^i(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) = 0$.
(3) *The variety* $Z_{\mathbf{i}}$ *is projectively normal for any embedding induced by a very ample line bundle* $L_{\mathbf{i},\mathbf{m}}$.

3. Linear Independence

Example 3.1. We want to see on Example 1.7 how one may construct an SMT for the varieties $\Gamma_{\mathbf{i}}$.

Consider the line bundle $L_{\mathbf{i},\mathbf{m}}$ on $Z_{\mathbf{i}}$ where $\mathbf{i} = 123212312$ and $\mathbf{m} = 200010111$. We consider the restriction map $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$, and a natural idea to get a basis of $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ is to take all the standard monomials that do not belong to its kernel.

So let $T = \kappa_{11} * \kappa_{12} * \kappa_{51} * \kappa_{71} * \kappa_{81} * \kappa_{91}$ be a tableau of shape $(\mathbf{i}, \mathbf{m})$. The monomial p_T does not vanish identically on $\Gamma_{\mathbf{i}}$ if and only if $\kappa_{11}, \kappa_{12} \in \{1, 2\}$, $\kappa_{51} \neq 1$, $\kappa_{71} = 234$, $\kappa_{81} = 4$, $\kappa_{91} = 34$.

One may check (by computer) that there are 708 standard tableaux. Among these tableaux, 9 do not vanish identically:

$$T_1 = 2*2*\emptyset*\emptyset*\emptyset*4*\emptyset*234*4*34 \quad T_4 = 2*1*\emptyset*\emptyset*\emptyset*4*\emptyset*234*4*34$$
$$T_2 = 2*2*\emptyset*\emptyset*\emptyset*3*\emptyset*234*4*34 \quad T_5 = 2*1*\emptyset*\emptyset*\emptyset*3*\emptyset*234*4*34$$
$$T_3 = 2*2*\emptyset*\emptyset*\emptyset*2*\emptyset*234*4*34 \quad T_6 = 2*1*\emptyset*\emptyset*\emptyset*2*\emptyset*234*4*34$$

$$T_7 = 1*1*\emptyset*\emptyset*\emptyset*4*\emptyset*234*4*34$$
$$T_8 = 1*1*\emptyset*\emptyset*\emptyset*3*\emptyset*234*4*34$$
$$T_9 = 1*1*\emptyset*\emptyset*\emptyset*2*\emptyset*234*4*34$$

Moreover, the tableaux T_i admit the following liftings (J^i_{km})

$J^1_{11} = \{1,2,3,4,5,6,7,8,9\}$	$J^4_{11} = \{1,2,3,4,5,6,7,8,9\}$	$J^7_{11} = \{2,3,4,5,6,7,8,9\}$
$J^1_{21} = \{1,2,3,4,5,6,7,8,9\}$	$J^4_{21} = \{2,3,4,5,6,7,8,9\}$	$J^7_{21} = \{2,3,4,5,6,7,8,9\}$
$J^1_{51} = \{3,4,5,6,7,8,9\}$	$J^4_{51} = \{2,3,4,5,6,7,8,9\}$	$J^7_{51} = \{2,3,4,5,6,7,8,9\}$
$J^1_{71} = \{3,4,5,6,7,8,9\}$	$J^4_{71} = \{2,3,4,5,6,7,8,9\}$	$J^7_{71} = \{2,3,4,5,6,7,8,9\}$
$J^1_{81} = \{3,4,5,6,7,9\}$	$J^4_{81} = \{2,3,4,5,6,7,9\}$	$J^7_{81} = \{2,3,4,5,6,7,9\}$
$J^1_{91} = \{3,4,5,6,7,9\}$	$J^4_{91} = \{2,3,4,5,6,7\}$	$J^7_{91} = \{2,3,4,5,6,7\}$
$J^2_{11} = \{1,2,3,4,5,6,7,8,9\}$	$J^5_{11} = \{1,2,3,4,5,6,7,8,9\}$	$J^8_{11} = \{2,3,4,5,6,7,8,9\}$
$J^2_{21} = \{1,2,3,4,5,6,7,8,9\}$	$J^5_{21} = \{2,3,4,5,6,7,8,9\}$	$J^8_{21} = \{2,3,4,5,6,7,8,9\}$
$J^2_{51} = \{1,2,3,5,6,7,8,9\}$	$J^5_{51} = \{2,3,5,6,7,8,9\}$	$J^8_{51} = \{2,3,5,6,7,8,9\}$
$J^2_{71} = \{1,2,3,5,6,8,9\}$	$J^5_{71} = \{2,3,5,6,7,8,9\}$	$J^8_{71} = \{2,3,5,6,7,8,9\}$
$J^2_{81} = \{1,2,3,5,6,8,9\}$	$J^5_{81} = \{2,3,5,6,7,8,9\}$	$J^8_{81} = \{2,3,5,6,7,8,9\}$
$J^2_{91} = \{1,2,3,5,6,8\}$	$J^5_{91} = \{2,3,5,6,7,8\}$	$J^8_{91} = \{2,3,5,6,7,8\}$
$J^3_{11} = \{1,2,3,4,5,6,7,8,9\}$	$J^6_{11} = \{1,2,3,4,5,6,7,8,9\}$	$J^9_{11} = \{2,3,4,5,6,7,8,9\}$
$J^3_{21} = \{1,2,3,4,5,6,7,8,9\}$	$J^6_{21} = \{2,3,4,5,6,7,8,9\}$	$J^9_{21} = \{2,3,4,5,6,7,8,9\}$
$J^3_{51} = \{1,2,3,4,6,7,8,9\}$	$J^6_{51} = \{3,5,6,7,8,9\}$	$J^9_{51} = \{3,5,6,7,8,9\}$
$J^3_{71} = \{1,2,3,4,8,9\}$	$J^6_{71} = \{3,5,6,7,8,9\}$	$J^9_{71} = \{3,5,6,7,8,9\}$
$J^3_{81} = \{1,2,3,4,8,9\}$	$J^6_{81} = \{3,5,6,7,8,9\}$	$J^9_{81} = \{3,5,6,7,8,9\}$
$J^3_{91} = \{1,2,3,4,8,9\}$	$J^6_{91} = \{3,5,6,7,8,9\}$	$J^9_{91} = \{3,5,6,7,8,9\}$

These liftings have the following property: $w_{\max}(\mathbf{i}(J^i_{km})) = w_0$ for each k, m. We then say that T_i is w_0-standard. It can be checked that the standard tableaux that are not w_0-standard vanish identically on $\Gamma_{\mathbf{i}}$.

To see that the remaining monomials p_{T_i} are linearly independent, we may work on an open affine set. There exists an open set Ω of $Z_{\mathbf{i}}$ such that $\Gamma_{\mathbf{i}} \cap \Omega$ is isomorphic to the affine space k^3 (see Definition 5.10 and Proposition 5.19). Here, we have

$$\varphi : (x, y, z) \mapsto (V_1, \dots, V_9),$$

for

$$\begin{array}{ll} V_1 = \langle xe_1 + e_2 \rangle & V_2 = \langle xe_1 + e_2, -xye_1 + e_3 \rangle \\ V_3 = \langle xe_1 + e_2, -xye_1 + e_3, e_4 \rangle & V_4 = \langle xe_1 + e_2, -xyze_1 + ze_3 + e_4 \rangle \\ V_5 = \langle yze_2 + ze_3 + e_4 \rangle & V_6 = \langle yze_2 + ze_3 + e_4, ye_2 + e_3 \rangle \\ V_7 = \langle e_2, e_3, e_4 \rangle & V_8 = \langle e_4 \rangle \\ V_9 = \langle e_3, e_4 \rangle & \end{array}$$

We denote again by p_T the polynomial $\varphi^* \left((p_T)_{|\Omega} \right)$. We then have

$$\begin{array}{ccc} p_{T_1} = 1, & p_{T_4} = x, & p_{T_7} = x^2, \\ p_{T_2} = z, & p_{T_5} = xz, & p_{T_8} = x^2 z, \\ p_{T_3} = yz, & p_{T_6} = xyz, & p_{T_9} = x^2 yz. \end{array}$$

It is clear that these monomials are linearly independent in $k[x, y, z]$.

Definitions 3.2 below will generalize the behaviour of the liftings (J^i_{km}) observed in this example.

Definitions 3.2. Let T be a standard tableau of shape $(\mathbf{i}, \mathbf{m})$. We say that T (or the monomial p_T) is *w_0-standard* if there exists a lifting (J_{km}) of T such that each subword J_{km} contains a reduced expression of w_0.

More generally, if $J \subset [r]$ contains a reduced expression for w_0, then $\Gamma_J = Z_J \cap \Gamma_{\mathbf{i}} \neq \emptyset$, and we say that T (or p_T) is *w_0-standard on Γ_J* if there exists a lifting (J_{km}) of T such that for every k, m, $J \supset J_{km}$ and J_{km} contains a reduced expression of w_0.

Similarly, T (or p_T) is said to be *w_0-standard on a union* $\Gamma = \Gamma_{J_1} \cup \dots \cup \Gamma_{J_k}$ if T is w_0-standard on at least one of the components $\Gamma_{J_1}, \dots, \Gamma_{J_k}$. We then denote by $\mathcal{S}(\Gamma)$ the set of all w_0-standard tableaux on Γ.

We need some results about *positroid varieties*. References for these varieties can be found in [21].

Let π_i be the canonical projection $F\ell(n) \to G_{i,n}$. In general, the projection of a Richardson variety $X^v_w \subset F\ell(n)$ is no longer a Richardson variety. But $\pi_i(X^v_w)$ is still defined inside the Grassmannian $G_{i,n}$ by the vanishing of some Plücker coordinates. More precisely, consider the set $\mathcal{M} = \{\kappa \in I_{i,n} \mid e_\kappa \in \pi_i(X^v_w)\}$. Then

$$\Pi = \pi_i(X^v_w) = \{V \in G_{i,n} \mid \kappa \notin \mathcal{M} \implies p_\kappa(V) = 0\}.$$

The poset $\mathcal{M}$ is a *positroid* (see the paragraph following Lemma 3.20 in [21]), and the variety Π is called a *positroid variety.*

Lemma 3.3. *With the notation above,*

$$\mathcal{M} = \{\kappa \in I_{i,n} \mid \exists u \in [v, w],\ u\varpi_i = \kappa\}.$$

Proof. Let $u \in [v,w]$ and $\kappa = u\varpi_i$. Then $e_u \in X_w^v$, so $e_\kappa = \pi_i(e_u) \in \Pi$. Hence $\kappa \in \mathcal{M}$.

Conversely, let $\kappa \in \mathcal{M}$. The fibre $\pi_i^{-1}\{e_\kappa\}$ in X_w^v is a non-empty T-stable variety, hence, by Borel's fixed point theorem, this variety has a T-fixed point e_u, $u \in S_n$. It follows that $u \in [v,w]$ and $u\varpi_i = \kappa$. □

Theorem 3.4. *For every subword $J_1, \dots, J_k$ containing a reduced expression of w_0, the w_0-standard monomials on the union $\Gamma = \Gamma_{J_1} \cup \cdots \cup \Gamma_{J_k}$ are linearly independent.*

Proof. We imitate the proof of the corresponding proposition for Bott-Samelson varieties appearing in [28, Section 3.2]. Let $\mathcal{T}$ be a non-empty subset of $\mathcal{S}(\Gamma)$, and assume that we are given a linear relation among monomials p_T for T in $\mathcal{T}$:

$$(*) \qquad \sum_{T \in \mathcal{T}} a_T\, p_T(\gamma) = 0 \quad \forall \gamma \in \Gamma.$$

Moreover, we may assume that the coefficients appearing in this relation are all non-zero. We shall proceed by induction on the length of tableaux, that is, on $M = \sum_{i=1}^r m_i$.

If $M = 1$, then $\mathbf{m}$ has the form $0 \dots 1 \dots 0$, that is, we have $m_e = 1$ for some e, and $m_i = 0$ for all $i \neq e$. The tableaux T that appear in relation $(*)$ are of the form $T = \kappa_e$, where $\kappa_e \in I_{i_e,n}$. If $\gamma = (F_{\text{can}}, F_1, \dots, F_r = F_{\text{op can}}) \in \Gamma$ then $p_T(\gamma) = p_{\kappa_e}(F_e)$. Thus, we have a linear relation of Plücker coordinates in a union of Richardson varieties in $F\ell(n)$, hence a linear relation on one of these Richardson varieties. But Standard Monomial Theory for Richardson varieties (*cf.* [25], Theorem 32) shows that such a relation cannot exist.

Now assume that $M > 1$, and $\mathbf{m} = 0 \dots 0\, m_e \dots m_r$ with $m_e > 0$. Here, we denote by κ_{km}^T the columns of a tableau T. Consider an element κ minimal among the first columns of the tableaux of $\mathcal{T}$, that is,

$$\kappa \in \min\{\kappa_{e1}^T \mid T \in \mathcal{T}\}.$$

We consider the set $\mathcal{T}(\kappa)$ of tableaux T in $\mathcal{T}$ with $\kappa_{e1}^T = \kappa$. For every $T \in \mathcal{T}(\kappa)$, fix a maximal lifting $J_{e1}^T \supset \cdots \supset J_{rm_r}^T$ containing a reduced expression of w_0 and with J_{e1}^T contained in one of the subwords $J_1, \dots, J_k$, so that $\Gamma \supset \Gamma_{J_{e1}^T} \neq \emptyset$. Thus, we can restrict the relation $(*)$ on

$$\Gamma(\kappa) = \bigcup_{T \in \mathcal{T}(\kappa)} \Gamma_{J_{e1}^T}.$$

If $T \in \mathcal{T}(\kappa)$, then $T = \kappa * T'$, and T' is a w_0-standard tableau on $\Gamma(\kappa)$ of shape $(\mathbf{i}, 0 \dots 0\, m_e - 1 \dots m_r)$.

If $T \notin S(\kappa)$, then $\kappa_{e1}^T \not\leq \kappa$, so $p_{\kappa_{e1}^T}$ vanishes identically on the Schubert variety $X_\kappa \subset G_{i_e,n}$, hence on each Schubert variety $X_{w_{\max}(\mathbf{i}(J_{e1}^S))}$ for $S \in \mathcal{T}(\kappa)$. In particular, $p_{\kappa_{e1}^T}$ vanishes on $\Gamma(\kappa)$, and p_T as well.

Restrict relation $(*)$ to $\Gamma(\kappa)$:

$$p_\kappa(\gamma) \sum_{T \in \mathcal{T}(\kappa)} a_T\, p_{T'}(\gamma) = 0 \quad \forall \gamma \in \Gamma(\kappa).$$

This product vanishes on each irreducible $\Gamma_{J_{e1}^T}$ $(T \in \mathcal{T}(\kappa))$. Now, p_κ does not vanish identically on $\Gamma(J_{e1}^T)$. Indeed, we know by Proposition 1.6 that $\text{pr}_e(\Gamma(J_{e1}^T))$ is the Richardson variety X_y^x with $y = w(\mathbf{i}(J_{e1}^T)) \geq x$. Since $\kappa = y\varpi_{i_e}$, by Lemma 3.3, p_κ does not vanish identically on X_y^x, hence does not vanish identically on $\Gamma(J_{e1}^T)$.

So we may simplify by p_κ on the irreducible $\Gamma_{J_{e1}^T}$, hence a linear relation between w_0-standard monomials on $\Gamma(\kappa)$ of shape $(\mathbf{i}, 0 \ldots 0\, m_e - 1 \ldots m_r)$. By induction over M, $a_T = 0$ for all $T \in \mathcal{T}(\kappa)$: a contradiction. □

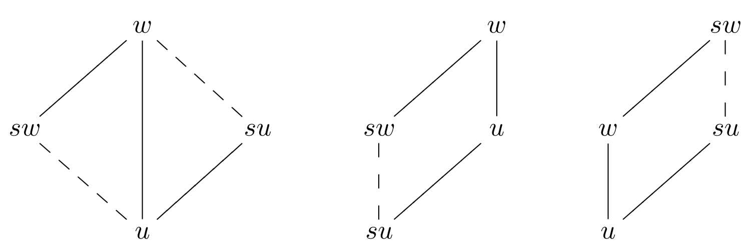
4. Standard monomials that do not vanish on $\Gamma_{\mathbf{i}}$ are w_0-standard

In this section, we shall prove that the standard monomials that do not vanish identically on $\Gamma_{\mathbf{i}}$ are w_0-standard, provided certain assumptions over $\mathbf{m}$, which cover the regular case (*i.e.* $m_i > 0$ for every i).

Lemma 4.1 (Lifting Property [3, Proposition 2.2.7])**.** *Let s be a simple transposition, and $u < w$ in S_n.*

- *If $u < su$ and $w > sw$, then $u \leq sw$ and $su \leq w$.*
- *If $u > su$ and $w > sw$, then $su \leq sw$.*
- *If $u < su$ and $w < sw$, then $su \leq sw$.* □

We may represent these situations by the pictures below



Definition 4.2. Let $x, y \in S_n$. The Demazure product $x * y$ is the unique maximal element of the poset $\mathcal{D}(x, y) = \{uv \mid u \leq x, v \leq y\}$.

Proposition 4.3 ([37, 18])**.** *Let $x, y \in S_n$. The double coset $B(x*y)B$ is the unique $B \times B$-double coset that is open in $BxByB$. In particular, $*$ is associative.* □

Lemma 4.4. *Let s be a simple transposition, and $x \in S_n$. Then $x * s = \max(x, xs)$. Similarly, $s * x = \max(x, sx)$.*

Proof. We shall prove that $x * s = \max(x, xs)$, the proof of $s * x = \max(x, sx)$ being similar.

- Case 1: $x > xs$. Let $u \leq x$. If $us < u$, then $us \leq x$. If $us > u$, then by Lemma 4.1, we have $us \leq x$. Hence every element of $\mathcal{D}(x, s)$ is less than or equal to x, so $x * s = x = \max(x, xs)$.
- Case 2: $x < xs$. Let $u \leq x$. If $us < u$, then $us \leq xs$. If $us > u$, then by Lemma 4.1, $us \leq xs$. Thus, every element of $\mathcal{D}(x, s)$ is less than or equal to xs, so $x * s = xs = \max(x, xs)$. □

Lemma 4.5. *Let J be a subword of $\mathbf{i}$. For every $k \in [r]$,*

$$w_{max}\big(\mathbf{i}(J)\big) = w_{max}\big(\mathbf{i}(J \cap [k])\big) * w_{max}\big(\mathbf{i}(J \cap [k+1, r])\big).$$

Proof. Let

$$\begin{aligned} w =& w_{\max}\big(\mathbf{i}(J)\big) \\ x =& w_{\max}\big(\mathbf{i}(J\cap[k])\big) \\ y =& w_{\max}\big(\mathbf{i}(J\cap[k+1,r])\big) \end{aligned}$$

Each element uv of $\mathcal{D}(x,y)$ has a decomposition of the form $w\big(\mathbf{i}(K_1)\big)w\big(\mathbf{i}(K_2)\big)$ with $K_1\subset J\cap[k]$ and $K_2\subset J\cap[k+1,r]$. Hence,

$$uv = w\big(\mathbf{i}(K_1\cup K_2)\big)\le w,$$

so $x*y\le w$.

Conversely, let $K'\subset J$ be such that $w\big(\mathbf{i}(K')\big)=w$ is a reduced decomposition. Since

$$w = w\big(\mathbf{i}(K'\cap[k])\big)w\big(\mathbf{i}(K'\cap[k+1,r])\big),$$

we have $w\in D(x,y)$, hence $w\le x*y$. □

Lemma 4.6 ([17, 2.2.(4)])**.** *If $x'\le x$ and $y'\le y$, then $x'*y'\le x*y$.*

Proof. By Proposition 4.3, we have

$$B(x'*y')B\subset \overline{BxB}\;\overline{ByB}\subset\overline{BxByB}=\overline{B(x*y)B},$$

hence $x'*y'\le x*y$. □

Let T be a standard tableau of shape $(\mathbf{i},\mathbf{m})$, and e be the least integer such that $m_e\neq 0$, so $\mathbf{m}=0\dots0\,m_e\dots m_r$. We give the construction of a particular type of liftings of T (called optimal), in light of the following

Remark 4.7. Let (K_{km}) be an arbitrary lifting of T and set

$$w_{km}=w\big(\mathbf{i}(K_{km}\cap[k])\big),$$

so that $w_{km}\varpi_k=\kappa_{km}$. By Remark 2.3, $\mathrm{pr}_k(Z_{K_{km}})=X_{w_{km}}$, with the following consequences.

- For each k, $K_{k1}\supset\cdots\supset K_{km_k}$ yields $w_{k1}\ge\cdots\ge w_{km_k}$.
- Let l be the least integer such that $l>k$ and $m_l\neq0$. Then $K_{km_k}\supset K_{l,1}$ yields $\mathrm{pr}_l(Z_{K_{l,1}})\subset\mathrm{pr}_l(Z_{K_{km_k}})$, hence

$$w\big(\mathbf{i}(K_{l,1}\cap[l])\big)\le w_{\max}\big(\mathbf{i}(K_{km_k}\cap[l])\big).$$

  By Lemma 4.5,

$$w_{\max}\big(\mathbf{i}(K_{km_k}\cap[l])\big)=w\big(\mathbf{i}(K_{k,m_k}\cap[k])\big)*w_{\max}\big(\mathbf{i}(K_{k,m_k}\cap[k+1,l])\big).$$

  So

$$w_{l,1}\le w_{k,m_k}*w_{\max}\big(\mathbf{i}(K_{k,m_k}\cap[k+1,l])\big).$$

We shall also need a result due to V. Deodhar:

Notation 4.8. Let $\kappa\in I_{i,n}$ and $w\in S_n$. We set

$$\mathcal{E}(w,\kappa)=\{v\in S_n\mid v\le w,\ v\varpi_i=\kappa\}.$$

Lemma 4.9 ([27, Lemma 11])**.** *Let $\kappa\in I_{i,n}$, and $w\in S_n$. If $\mathcal{E}(w,\kappa)\neq\emptyset$, then it admits a unique maximal element.* □

Remark 4.10. The above lemma admits the following geometric interpretation. Let q be the restriction to X_w of the canonical projection $F\ell(n) \to G_{i,n}$. Since q is B-equivariant, $q^{-1}(X_\kappa)$ is a union of Schubert varieties, namely

$$q^{-1}(X_\kappa) = \bigcup_{v \in \mathcal{E}(x,\kappa)} X_v.$$

By Lemma 4.9, we conclude that $q^{-1}(X_\kappa)$ is a single Schubert variety.

There exists a dual version of the lemma: if $w\varpi_i \geq \kappa$, then the set

$$\{v \in S_n \mid v \geq w,\ v\varpi_i = \kappa\}$$

admits a unique minimal element. Hence, $q^{-1}(X^\kappa)$ is a single opposite Schubert variety.

We now construct elements $v_{km} \in S_n$ inductively, as follows. At the first step, consider the set

$$\mathcal{E}(w_{\max}(i_1 \dots i_e), \kappa_{e1}).$$

Since it contains w_{e1}, it is nonempty, so it has a maximal element v_{e1}, which is unique thanks to Lemma 4.9. Now assume that $v_{km} \geq w_{km}$ has already been constructed. We then proceed in the same way to construct $v_{k,m+1}$ (if $m < m_k$) or $v_{l,1}$ (if $m = m_k$, and $l > k$ is the least integer such that $m_l \neq 0$):

- If $m < m_k$, then the set $\mathcal{E}(v_{k,m}, \kappa_{k,m+1})$ is nonempty (since it contains $w_{k,m+1}$), so let $v_{k,m+1}$ be its unique maximal element.
- If $m = m_k$, then let $v'_{k,m} = v_{k,m} * w_{\max}\big(i_{k+1} \dots i_l\big)$. By Lemma 4.6,

  $$w_{k,m} * w_{\max}\big(\mathbf{i}(K_{k,m} \cap [k+1, l])\big) \leq v'_{k,m}.$$

  Thus, by Remark 4.7 the set $\mathcal{E}(v'_{k,m}, \kappa_{l,1})$ contains $w_{l,1}$, so it is non-empty. Let $v_{l,1}$ be its unique maximal element.

Remark 4.11. Although the *existence* of the v_{km} depends on that of the w_{km} (*i.e.* on the existence of a lifting of the tableau T), the *values* of the v_{km} only depend on the tableau T itself.

Next, we construct subsets $E_{km} \subset [k]$, again inductively. Since

$$v_{e1} \leq w_{\max}(i_1 \dots i_e),$$

choose $E_{e1} \subset \{i_1 \dots i_e\}$ such that v_{e1} admits a reduced expression of the form $\mathbf{i}(E_{e1})$. If $E_{k,m}$ such that $v_{k,m} = w\big(\mathbf{i}(E_{k,m})\big)$ has already been constructed, then define $E_{k,m+1}$ (if $m < m_k$) or $E_{l,1}$ (if $m = m_k$) as follows:

- If $m < m_k$, then $v_{k,m+1} \leq v_{k,m} = w\big(\mathbf{i}(E_{k,m})\big)$, so choose $E_{k,m+1} \subset E_{k,m}$ such that $v_{k,m+1}$ admits a reduced expression of the form $\mathbf{i}(E_{k,m+1})$.
- If $m = m_k$, then by Lemma 4.5

  $$v_{l,1} \leq v'_{k,m_k} = w_{\max}\big(\mathbf{i}(E_{km_k} \cup \{k+1, \dots, l\})\big),$$

  so choose $E_{l,1} \subset E_{km_k} \cup \{k+1, \dots, l\}$ such that $v_{l,1}$ admits a reduced expression of the form $\mathbf{i}(E_{l,1})$.

Definition 4.12. With the above notation, set $J_{km} = E_{km} \cup [k+1, r]$ for each k, m. We will call (J_{km}) an *optimal lifting* of the tableau T.

Remark 4.13. The optimal lifting is not unique. However, while it depends on the choice of reduced expressions for the v_{km}, it is still independent on the choice of the initial lifting (K_{km}).

Example 4.14. Consider the tableau $T = 123 * 13 * 3 * 134 * 24 * 124$ of shape $(3213233213, 1111110000)$. This tableau is standard, and we shall construct an optimal lifting of T.

- $v_{11} = \max \mathcal{E}(s_3, 123) = e$.
- $v_{21} = \max \mathcal{E}(e * s_2, 13) = s_2$.
- $v_{31} = \max \mathcal{E}(s_2 * s_1, 3) = s_2 s_1$.
- $v_{41} = \max \mathcal{E}(s_2 s_1 * s_3, 134) = s_2 s_1 s_3$.
- $v_{51} = \max \mathcal{E}(s_2 s_1 s_3 * s_2, 24) = s_1 s_3 s_2$.
- $v_{61} = \max \mathcal{E}(s_1 s_3 s_2 * s_3, 124) = s_1 s_3$.

Hence

$$\begin{aligned} J_{11} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{21} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{31} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{41} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{51} &= \{3,4,5,6,7,8,9,10\} \\ J_{61} &= \{3,4,7,8,9,10\} \end{aligned}$$

is an optimal lifting of T. Another optimal lifting of T is

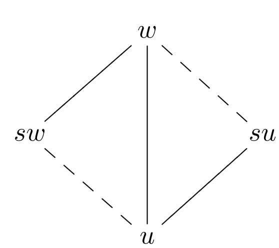
$$\begin{aligned} J'_{11} &= \{2,3,4,5,6,7,8,9,10\} \\ J'_{21} &= \{2,3,4,5,6,7,8,9,10\} \\ J'_{31} &= \{2,3,4,5,6,7,8,9,10\} \\ J'_{41} &= \{2,3,4,5,6,7,8,9,10\} \\ J'_{51} &= \{3,4,5,6,7,8,9,10\} \\ J'_{61} &= \{3,6,7,8,9,10\} \end{aligned}$$

Lemma 4.15. *Let $w \in S_n$ and $\kappa \in I_{i,n}$ be such that $\mathcal{E}(w, \kappa) \neq \emptyset$. Consider a simple transposition s such that $sw < w$.*

(1) If $s\kappa > \kappa$, then $\max \mathcal{E}(w, \kappa) = \max \mathcal{E}(sw, \kappa)$.
(2) If $s\kappa \leq \kappa$, then $\max \mathcal{E}(w, \kappa) = s * \max \mathcal{E}(sw, s\kappa)$.

Proof. Let $u = \max \mathcal{E}(w, \kappa)$.

- Case 1: assume that $su > u$. Then, by Lemma 4.1,

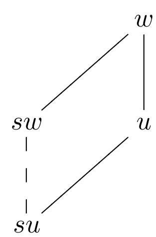


we have $u \leq sw$ and $su \leq w$. Hence $s\kappa \geq \kappa$, but by maximality of u, $su \notin \mathcal{E}(w, \kappa)$, hence $s\kappa > \kappa$. Since $u \leq sw$, $u \in \mathcal{E}(sw, \kappa)$, so

$$u \leq \max \mathcal{E}(sw, \kappa) \leq \max \mathcal{E}(w, \kappa) = u.$$

This proves the part (1) of the lemma.

- Case 2: $su < u$. Then $s\kappa \leq \kappa$, and by Lemma 4.1,



we have $su \leq sw$, so $su \in \mathcal{E}(sw, s\kappa)$, hence $su \leq v = \max \mathcal{E}(sw, s\kappa)$. We distinguish two subcases:
  - Subcase 1: $s\kappa < \kappa$. Then $sv > v$. Since we also have $su < u$, it follows from Lemma 4.1 that $v \leq su$. Similarly, $sv > v$, together with $sw < w$ imply that $sv \leq w$, so $sv \in \mathcal{E}(w, \kappa)$, hence $sv \leq u$. By Lemma 4.1, we have $v \leq su$. So $v = su$, or equivalently
$$u = sv = \max(v, sv) = s * v.$$
  - Subcase 2: $s\kappa = \kappa$.
    * If $u \leq sw$, then $u \in \mathcal{E}(sw, \kappa)$, so $u \leq v$. But $v \leq u$, so $u = v$.
    * If $u \not\leq sw$, then $su \in \mathcal{E}(sw, \kappa)$, so $su \leq v \leq u$. In other words, $v \in \{u, su\}$.

    In each of these two situations, we have $u = v$ or $u = sv$. But, if $sv > v$ then $u \neq v$ (since $su < u$), so $u = sv = \max(v, sv) = s * v$. If $sv < v$, then $u \neq sv$, so
$$u = v = s * v. \square$$

Let $w = s_{i_1} \dots s_{i_j}$ be a reduced expression. The lemma above gives an algorithm to find a reduced expression of $u = \max \mathcal{E}(w, \kappa)$, say $u = w\big(\mathbf{i}(J)\big)$, with $J \subset [j]$: let $s = s_{i_1}$, and compare $s\kappa$ with κ.

- If $s\kappa > \kappa$, then $u = \max \mathcal{E}(sw, \kappa)$.
- If $s\kappa \leq \kappa$, then $u = s * \max \mathcal{E}(sw, s\kappa)$.

We then compute $\max \mathcal{E}(sw, s\kappa)$ or $\max \mathcal{E}(sw, \kappa)$ in the same way, using the decomposition $sw = s_{i_2} \dots s_{i_j}$.

Example 4.16. In S_4, take $w = [4231] = s_1 s_2 s_3 s_2 s_1$ and $\kappa = 13$. We shall compute $u = \max \mathcal{E}(w, \kappa)$ with the previous algorithm. Note that $\kappa \leq 24 = w\varpi_2$, hence $\mathcal{E}(w, \kappa) \neq \emptyset$.

- $s_1\kappa = 23 > \kappa$, so $u = \max \mathcal{E}(s_2 s_3 s_2 s_1, \kappa)$,
- $s_2\kappa = 12 \leq k$, so $u = s_2 * \max \mathcal{E}(s_3 s_2 s_1, 12)$,
- $s_3(12) = 12$, so $u = s_2 * s_3 * \max \mathcal{E}(s_2 s_1, 12)$,
- $s_2(12) = 13 > 12$, so $u = s_2 * s_3 * \max \mathcal{E}(s_1, 12)$,
- $s_1(12) = 12$, so $u = s_2 * s_3 * s_1 * \max \mathcal{E}(e, 12)$.

Now, $\max(e, 12) = e$, so $u = s_2 * s_3 * s_1 = s_2s_3s_1 = [3142]$.

Lemma 4.17 ([3, Proposition 2.4.4])**.** *Let $\kappa \in I_{i,n}$. The set $\{v \in S_n \mid v\varpi_i = \kappa\}$ admits a unique minimal element u. Moreover, if $v \in S_n$ satisfies $v\varpi_i = \kappa$, then v admits a unique factorization $v = uv'$ with $v'\varpi_i = \varpi_i$. This factorization is length-additive, in the sense that $l(v) = l(u) + l(v')$.* □

Remark 4.18. Lemmas 4.1, 4.4, 4.6, Definition 4.2 and Proposition 4.3 are true for every Weyl group. Lemmas 4.9, 4.17, and Remark 4.10 are also true if we replace S_n by a Weyl group W, $I_{i,n}$ by the set W^P of minimal representatives of the quotient W/W_P, W_P a parabolic subgroup of W, and $v\varpi_i = \kappa$ by $v \equiv \kappa \mod W_P$.

Lemma 4.19. *Denote by u_d the minimal permutation such that $u_d\varpi_d = w_0\varpi_d$. Let $w \geq u$, and κ a column of arbitrary size $i \leq n$ such that $\mathcal{E}(w, \kappa) \neq \emptyset$. Assume that $x = \max \mathcal{E}(w, \kappa) \geq u$. Then*

$$\forall v \geq u,\ \mathcal{E}(v, \kappa) \neq \emptyset \implies \max \mathcal{E}(v, \kappa) \geq u.$$

Proof. Since $v \geq u$, we have $v\varpi_d = w_0\varpi_d$, hence by Lemma 4.17, $v = uv'$ with v' in the stabilizer of ϖ_d. Moreover, this decomposition is length-additive, so if $u = s_{i_1} \dots s_{i_j}$ and $v' = s_{i_{j+1}} \dots s_{i_k}$ are reduced expressions, then $s_{i_1} \dots s_{i_j} s_{i_{j+1}} \dots s_{i_k}$ is a reduced expression of v. Similarly, we decompose $x = ux'$ with $x'\varpi_d = \varpi_d$. We then obtain

$$x > s_{i_1}x > \cdots > s_{i_j} \dots s_{i_1}x = x',$$

hence

$$\kappa \geq s_{i_1}\kappa \geq \cdots \geq s_{i_j} \dots s_{i_1}\kappa.$$

Now, we apply the procedure described after Lemma 4.15 for the decomposition $v = s_{i_1} \dots s_{i_j} s_{i_{j+1}} \dots s_{i_k}$. The above inequalities show that $\max \mathcal{E}(v, \kappa)$ is of the form $s_{i_1} * \cdots * s_{i_j} * z$. But, by Lemma 4.6, we have

$$\begin{aligned} s_{i_1} * \cdots * s_{i_j} * z &\geq s_{i_1} * \cdots * s_{i_j} \\ &\geq s_{i_1} \dots s_{i_j} \\ &\geq u. \ \square \end{aligned}$$

Notation 4.20. For $k \in [n-1]$, let j_k be the greatest integer such that $i_{j_k} = k$.

Theorem 4.21. *Assume that for every k, $m_{j_k} > 0$. Then the standard monomials p_T of shape $(\mathbf{i}, \mathbf{m})$ that do not vanish identically on $\Gamma_{\mathbf{i}}$ are w_0-standard.*

Proof. Consider an optimal lifting (J_{km}) of T. Let $(F_{\text{can}}, F_1, \dots, F_r) \in \Gamma_{\mathbf{i}}$ be a gallery such that $p_T(F_{\text{can}}, F_1, \dots, F_r) \neq 0$. By definition of j_k, the flags F_{j_k} and $F_{\text{op can}}$ share the same k-subspace, which then is the T-fixed point $\langle e_n, \dots, e_{n-k+1}\rangle$. Hence, $\kappa_{j_k,1} = \cdots = \kappa_{j_k,m_{j_k}} = w_0\varpi_k$.

Arrange the integers $j_1, \dots, j_{n-1}$ in an increasing sequence: $j_{l_1} < \cdots < j_{l_{n-1}}$.

We shall prove that if $k > j_l$, then $v_{km} \geq u_l$. Since $p_T(F_{\text{can}}, F_1, \dots, F_r) \neq 0$, we have $p_{\kappa_{km}}(F_k) \neq 0$, hence $p_{\kappa_{km}}$ does not vanish identically on the Richardson variety X_w^v, where $w = w_{\max}(i_1 \dots i_k)$ and $v = w_0(w_{\max}(i_{k+1} \dots i_r))^{-1}$. This means that $p_{\kappa_{km}}$ does not vanish identically on the positroid variety $\pi(X_w^v)$, where $\pi : F\ell(n) \to G_{i_k,n}$. By Lemma 3.3, there exists $u \in [v, w]$ such that $u\varpi_{i_k} = \kappa_{km}$. It follows that the maximal element x_l of $\mathcal{E}(w, \kappa_{km})$ is greater than u. But, since $k > j_l$, a reduced expression of w_0v^{-1} consists of letters taken from $i_{k+1} \dots i_r$, where no l appears. Thus, $w_0v^{-1}\varpi_l = \varpi_l$, so $v\varpi_l = w_0\varpi_l$, that is, $v \geq u_l$. Hence $x_1 \geq u \geq v \geq u_l$. We then conclude with Lemma 4.19.

Now, we consider subwords J_{km} with $k \leq j_{l_1}$. In this case, $k \leq j_t$ $(t \geq 1)$, so we have the inequalities $w\big(\mathbf{i}(J_{j_t,1} \cap [j_t])\big) \leq w_{\max}\big(\mathbf{i}(J_{km})\big)$, hence

$$w_{\max}\big(\mathbf{i}(J_{km})\big)\varpi_t \geq w\big(\mathbf{i}(J_{j_t,1} \cap [j_t])\big)\varpi_t = \kappa_{j_t,1} = w_0\varpi_t,$$

i.e. $w_{\max}\big(\mathbf{i}(J_{km})\big)\varpi_t = w_0\varpi_t$. So $w_{\max}\big(\mathbf{i}(J_{km})\big) = w_0$.

If $j_{l_t} < k \leq j_{l_{t+1}}$, then we have, in one hand, $w_{\max}\big(\mathbf{i}(J_{km})\big) \geq w\big(\mathbf{i}(J_{j_{l_p},1})\big)$ for every $p \geq t+1$, so $w_{\max}\big(\mathbf{i}(J_{km})\big)\varpi_{l_p} \geq \kappa_{l_p,1} = w_0\varpi_{l_p}$, hence $w_{\max}\big(\mathbf{i}(J_{km})\big)\varpi_{l_p} = w_0\varpi_{l_p}$. On the other hand, $w_{\max}\big(\mathbf{i}(J_{km})\big) \geq v_{km} \geq u_{l_q}$ for every $q \leq t$, hence $w_{\max}\big(\mathbf{i}(J_{km})\big)\varpi_{l_q} = w_0\varpi_{l_q}$. It follows that $w_{\max}\big(\mathbf{i}(J_{km})\big) = w_0$. □

Remark 4.22. The assumption $m_{j_k} > 0$ for every k is necessary: recall the tableau $T = 123{*}13{*}3{*}134{*}24{*}124$ of Example 4.14. It is standard of shape $(3213233213, 1111110000)$, and one may check that p_T does not vanish identically on $\Gamma_{\mathbf{i}}$. However, an optimal lifting of T is given by

$$\begin{aligned} J_{11} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{21} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{31} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{41} &= \{2,3,4,5,6,7,8,9,10\} \\ J_{51} &= \{\quad 3,4,5,6,7,8,9,10\} \\ J_{61} &= \{\quad 3, \qquad 6,7,8,9,10\} \end{aligned}$$

and we have $w_{\max}(\mathbf{i}(J_{61})) = s_1s_3s_2s_1s_3 = [4231] \neq w_0$, hence T is not w_0-standard.

5. Basis of $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$

Assume that $\mathbf{m}$ is regular. We shall prove that the w_0-standard monomials of shape $(\mathbf{i}, \mathbf{m})$ form a basis of the space of sections $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$. By Theorems 3.4 and 4.21, we just have to show that the restriction map

$$H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$$

is surjective. The idea is to find a sequence of varieties $\big(Y_{\mathbf{i}}^{u_t}\big)$, parametrized by $u_t \in S_n$, such that

- $Y_{\mathbf{i}}^{u_0} = Z_{\mathbf{i}}$ and $Y_{\mathbf{i}}^{u_N} = \Gamma_{\mathbf{i}}$,
- $Y_{\mathbf{i}}^{u_{t+1}}$ is a hypersurface in $Y_{\mathbf{i}}^{u_t}$,
- each restriction map $H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}})$ is surjective.

Example 5.1. Let $n = 4$ and $\mathbf{i} = 123212312$. Consider the following reduced expression

$$w_0 = s_1s_2s_1s_3s_2s_1 = s_{a_6}s_{a_5}\dots s_{a_1},$$

and set

$$\begin{cases} u_0 = e, \\ u_{t+1} = s_{a_{t+1}}u_t \quad \forall t \geq 0. \end{cases}$$

The sequence (u_t) is increasing, and $u_6 = w_0$. Thus, we obtain a sequence of opposite Schubert varieties

$$F\ell(n) = X^{u_0} \supset X^{u_1} \supset \dots \supset X^{u_6} = \{F_{\mathrm{op\,can}}\}.$$

Let $F = (F^1 \subset F^2 \subset F^3 \subset F^4 = k^4)$ be a flag.

- We have the equivalence
$$\begin{aligned} F \in X^{s_1} &\iff F^1 \in \langle e_2, e_3, e_4 \rangle \\ &\iff p_1(F) = 0. \end{aligned}$$
So X^{u_1} is defined inside X^{u_0} by the vanishing of $p_1 = p_{\kappa_0}$.
- Assume $F \in X^{u_1}$. Then
$$\begin{aligned} F \in X^{s_2 s_1} &\iff F^1 \in \langle e_3, e_4 \rangle \\ &\iff p_2(F) = 0, \text{ since we already know that } p_1(F) = 0. \end{aligned}$$
Hence X^{u_2} is defined inside X^{u_1} by the vanishing of $p_2 = p_{\kappa_1}$.
- Similarly, X^{u_3} is defined inside X^{u_2} by the vanishing of $p_3 = p_{\kappa_2}$.
- The opposite Schubert variety X^{u_4} is defined inside X^{u_3} by the vanishing of $p_{14} = p_{\kappa_3}$.
- X^{u_5} is defined inside X^{u_4} by the vanishing of $p_{24} = p_{\kappa_4}$.
- X^{u_6} is defined inside X^{u_5} by the vanishing of $p_{134} = p_{\kappa_5}$.

We then set $Y_{\mathbf{i}}^{u_t} = \mathrm{pr}_9^{-1}(X^{u_t})$. Thus, $Y_{\mathbf{i}}^{u_0} = Z_{\mathbf{i}}$, $Y_{\mathbf{i}}^{u_6} = \Gamma_{\mathbf{i}}$. Moreover, $Y_{\mathbf{i}}^{u_{t+1}}$ is defined inside $Y_{\mathbf{i}}^{u_t}$ by the vanishing of p_{κ_t}, where we view κ_t as a tableau of shape $(123212312, \mathbf{a}'_t)$, where
$$\mathbf{a}'_1 = \mathbf{a}'_2 = \mathbf{a}'_3 = 000000010,\ \mathbf{a}'_4 = \mathbf{a}'_5 = 000000001,\ \mathbf{a}'_6 = 000000100.$$

This example leads us to work with the following varieties. Consider the last projection $\mathrm{pr}_r : Z_{\mathbf{i}} \to F\ell(n)$. Fix $u \in S_n$ and a reduced decomposition
$$w_0 u = s_{k_l} s_{k_{l-1}} \dots s_{k_1}.$$
Consider the opposite Schubert variety $X^u \subset F\ell(n)$ and set
$$Y_{\mathbf{i}}^u = \mathrm{pr}_r^{-1}(X^u) \subset Z_{\mathbf{i}}.$$
In particular, $Y_{\mathbf{i}}^e = Z_{\mathbf{i}}$ and $Y_{\mathbf{i}}^{w_0} = \Gamma_{\mathbf{i}}$.

Proposition 5.2. *The variety $Y_{\mathbf{i}}^u$ is irreducible, and if $\mathbf{i}' = i_1 \dots i_r k_1 \dots k_l$, then the projection $F\ell(n)^{r+l} \to F\ell(n)^r$ onto the r first factors restricts to a morphism*
$$\varphi : \Gamma_{\mathbf{i}'} \to Y_{\mathbf{i}}^u$$
that is birational, hence surjective.

Proof. Recall that a flag F lies in X^u if and only if it can be connected to $F_{\mathrm{op\,can}}$ by a gallery of type $k_1 \dots k_l$. Hence $Y_{\mathbf{i}}^u$ consists of all galleries
$$F_{\mathrm{can}} \overset{i_1}{—} F_1 \overset{i_2}{—} \dots \overset{i_r}{—} F_r$$
that can be extended to a gallery of the form
$$F_{\mathrm{can}} \overset{i_1}{—} F_1 \overset{i_2}{—} \dots \overset{i_r}{—} F_r \overset{k_1}{—} \dots \overset{k_l}{—} F_{\mathrm{op\,can}}.$$
Thus, φ indeed takes values in $Y_{\mathbf{i}}^u$ and is surjective. The irreducibility of $Y_{\mathbf{i}}^u$ follows.

Moreover, in the diagram
$$\begin{array}{ccc} \Gamma_{\mathbf{i}'} \cong Z_{\mathbf{i}} \times_{F\ell(n)} Z_{k_l \dots k_1}(F_{\mathrm{op\,can}}) & \longrightarrow & Z_{k_l \dots k_1}(F_{\mathrm{op\,can}}) \\ \downarrow & & \downarrow \mathrm{pr}_l \\ Z_{\mathbf{i}} & \xrightarrow[\mathrm{pr}_r]{} & F\ell(n), \end{array}$$

pr_l is an isomorphism over C^u, and the morphism $\mathrm{id}\times(\mathrm{pr}_l^{-1}\circ\mathrm{pr}_r)$ from $\mathrm{pr}_r^{-1}(C^u)\subset Y_{\mathbf{i}}^u$ to $Z_{\mathbf{i}}\times Z_{k_l\dots k_1}(F_{\mathrm{op\,can}})$ is an inverse of φ over $\mathrm{pr}_r^{-1}(C^u)$, hence φ is birational. □

Corollary 5.3. *Take the notations of the previous proposition, and consider the jth projection $\mathrm{pr}_j : Z_{\mathbf{i}} \to F\ell(n)$. Then $\mathrm{pr}_j(Y_{\mathbf{i}}^u)$ is the Richardson variety X_y^x for $y = w_{max}(i_1 \dots i_j)$ and $x = w_0 w_{max}(i_{j+1}\dots i_r k_1 \dots k_l)^{-1}$.*

Proof. Note that $\mathrm{pr}_j(Y_{\mathbf{i}}^u) = \mathrm{pr}_j(\varphi(\Gamma_{\mathbf{i}'})) = \mathrm{pr}_j(\Gamma_{\mathbf{i}'})$ since $j\in[r]$. Proposition 1.6 leads to the result. □

Notations 5.4. As in Example 5.1, consider the reduced decomposition

$$w_0 = s_1(s_2s_1)\dots(s_{n-1}\dots s_1) = s_{a_N}\dots s_{a_1},$$

and set $u_t = s_{a_t}\dots s_1$, $u_0 = e$.

Consider the sequence of columns κ_t defined in the following way.

- The $n-1$ first columns are $\kappa_0 = 1,\ \kappa_1 = 2,\dots,\ \kappa_{n-2} = n-1$.
- The $n-2$ next columns are $1*n,\ 2*n,\dots,\ n-2*n$.
- The $n-3$ next ones are of size 3: $1*w_0\varpi_2,\ 2*w_0\varpi_2,\dots,\ n-3*w_0\varpi_2$.
- We proceed in the same way for the other columns until we get $\kappa_{N-1} = 1*w_0\varpi_{n-2}$.

We denote by b_t the size of κ_t, so that $\kappa_t = u_t\varpi_{b_t}$. We set $\kappa'_t = u_{t+1}\varpi_{b_t}$.

In the sequel, we say that a variety Y *is defined in X by the vanishing of f* if

$$y\in Y \iff y\in X \text{ and } f(y)=0.$$

This does not necessarily mean that f generates the ideal of Y in the coordinate ring of X.

Lemma 5.5. *For every $t\in[0,N-1]$, the opposite Schubert variety $X^{u_{t+1}}\subset F\ell(n)$ is defined inside X^{u_t} by the vanishing of p_{κ_t}.*

Proof. We begin by proving the following

Claim For every t, the opposite Schubert variety $X^{u_{t+1}\varpi_{b_t}}\subset G_{b_t,n}$ is defined inside $X^{u_t\varpi_{b_t}} = X^{\kappa_t}$ by the vanishing of p_{κ_t}.

Indeed, recall that a b_t-space V belongs to the opposite Schubert variety X^{κ_t} if and only if for every $\kappa\not\geq\kappa_t$, $p_\kappa(V)=0$, and similarly for $X^{\kappa'_t}$. Thus, we have to describe the set

$$E_t = \{\kappa\not\geq\kappa'_t \mid \kappa\geq\kappa_t\}.$$

We distinguish two cases.

- Case 1: $b_{t+1} = b_t$. Then $\kappa'_t = u_{t+1}\varpi_{b_t} = \kappa_{t+1}$. But κ_t is of the form $p*w_0\varpi_{b_t-1}$, and $\kappa_{t+1} = (p+1)*w_0\varpi_{b_t-1}$. So $\kappa_t<\kappa'_t$, hence $\kappa_t\in E_t$. Let $\kappa\in E_t$ with $\kappa\neq\kappa_t$. Then $\kappa>\kappa_t$, so $\kappa\geq\kappa_{t+1}$: a contradiction. Hence, the claim is proved in this case.
- Case 2: $b_{t+1} = b_t+1$. Then $\kappa'_t = u_{t+1}\varpi_{b_t} = w_0\varpi_{b_t} = (n-b_t+1)*w_0\varpi_{b_t-1}$, and $\kappa_t = (n-b_t)*w_0\varpi_{b_t-1}$. Again, $\kappa_t\in E_t$. If $\kappa\in E_t$ and $\kappa>\kappa_t$, then $\kappa = w_0\varpi_{b_t}$: a contradiction. This proves the claim.

Now, let q be the restriction to X^{u_t} of the canonical projection $F\ell(n)\to G_{b_t,n}$. We have to show that $X^{u_{t+1}} = q^{-1}(X^{\kappa'_t})$. since $u_t\varpi_{b_t}\neq\kappa'_t$, u_{t+1} is a minimal element of the poset $\{u\geq u_t,\ u\varpi_{b_t} = \kappa'_t\}$, so by Remark 4.10 $q^{-1}(X^{\kappa'_t}) = X^{u_{t+1}}$. □

Notation 5.6. For every $t \in [0, N-1]$, we set $l_t = j_{b_t}$, that is the largest integer j such that $i_j = b_t$.

Corollary 5.7. *With the notation of Lemma 5.5, the variety* $Y_{\mathbf{i}}^{u_{t+1}}$ *is defined inside* $Y_{\mathbf{i}}^{u_t}$ *by the vanishing of* p_T*, where* $T = \emptyset * \cdots * \kappa_t * \cdots * \emptyset$ *is a tableau of shape* $(\mathbf{i}, 0 \ldots 1 \ldots 0)$*, the 1 being at position* l_t*.*

Proof. Write $\kappa = \kappa_t$. Let γ be a gallery

$$F_{\mathrm{can}} \overset{i_1}{\text{---}} F_1 \overset{i_2}{\text{---}} \cdots \overset{b_t}{\text{---}} F_{k_t} \text{---} \cdots \text{---} F_r$$

in $Y_{\mathbf{i}}^{u_t}$. This gallery belongs to $Y_{\mathbf{i}}^{u_{t+1}}$ if and only if $F_r \in X^{u_{t+1}}$. Since we already know that $F_r \in X^{u_t}$, we have

$$\gamma \in Y_{\mathbf{i}}^{u_{t+1}} \iff p_\kappa(F_r) = 0 \iff p_\kappa(\pi_{b_t} F_r) = 0,$$

where the first equivalence follows from Lemma 5.5 and the second from the fact that κ is of size b_t. By definition of l_t, no adjacency after F_{j_t} is an b_t-adjacency, hence $\pi_{b_t} F_{j_t} = \pi_{b_t} F_{j_t+1} = \cdots = \pi_{b_t} F_r$, and therefore,

$$p_\kappa(F_r) = 0 \iff p_\kappa(F_{j_t}) = 0 \iff p_T(\gamma) = 0,$$

where $T = \emptyset * \cdots * \kappa * \cdots * \emptyset$ with κ in position l_t. □

Notations 5.8. We fix an $\mathbf{a} = a_1 \ldots a_r$ with $a_i > 0$ for every i. The associated line bundle $L_{\mathbf{i},\mathbf{a}}$ is very ample, so it induces an embedding of $Z_{\mathbf{i}}$ in some projective space $\mathbf{P}_{\mathbf{a}}$. We denote by R_t the homogeneous coordinate ring of $Y_{\mathbf{i}}^{u_t}$ viewed as a subvariety of $\mathbf{P}_{\mathbf{a}}$.

Remark 5.9. For the rest of this section, if a notion depends on an embedding, such as projective normality, or the homogeneous coordinate ring of a variety, it will be implicitly understood that we work with the line bundle $L_{\mathbf{i},\mathbf{a}}$.

The ring R_{t+1} is a quotient R_t/I_t, and we shall determine the ideal I_t. We begin by computing the equations of $Y_{\mathbf{i}}^{u_{t+1}}$ in an affine open set of $Y_{\mathbf{i}}^{u_t}$.

Definition 5.10. We shall define an affine open set Ω of $Z_{\mathbf{i}}$, isomorphic to the affine space k^r. This construction is taken from [15].

First, we define inductively a sequence of permutations (σ_j) with $\sigma_N = w_0$:

$$\begin{cases} \sigma_0 = e, \\ \sigma_{j+1} = \sigma_j * s_{i_{j+1}} \quad \forall j \geq 0. \end{cases}$$

Moreover, we set $v_{j+1} = \sigma_j^{-1} \sigma_{j+1} \in \{e, s_{i_{j+1}}\}$.

Next, consider the 1-parameter unipotent subgroup U_β associated to a root β, with its standard parametrization $\epsilon_\beta : k \to U_\beta$ (*i.e.* the matrix $\epsilon_\beta(x)$ has 1s on the diagonal, the entry corresponding to β equal to x, and 0s elsewhere). We also denote by $\alpha_1, \ldots, \alpha_{n-1}$ the simple roots and by P_j the minimal parabolic subgroup associated to α_j, *i.e.* the subgroup generated by B and by $U_{-\alpha_j}$.

We set $\beta_j = v_j(-\alpha_{i_j})$ and consider the morphism

$$\begin{array}{rcl} k^r & \to & P_{\mathbf{i}} = P_{i_1} \times \ldots P_{i_r} \\ (x_1, \ldots, x_r) & \mapsto & (A_1, \ldots, A_r) \end{array}$$

with $A_j = \epsilon_{\beta_j}(x_j) v_j$. Set $B_j = A_1 \ldots A_j$.

Finally, let

$$\begin{array}{rrcl}\varphi : & k^r & \to & Z_{\mathbf{i}} \\ & (x_1, \dots, x_r) & \mapsto & (\gamma_1, \dots, \gamma_r)\end{array}$$

for $\gamma_j = B_j F_{\mathrm{can}}$.

The image of φ is denoted by Ω: it is an open set in $Z_{\mathbf{i}}$, and $\varphi : k^r \to \Omega$ is an isomorphism.

Notation 5.11. Let $\kappa = k_1 \dots k_i$ and $\tau = t_1 \dots t_i$ be two columns of the same size. Given a matrix M, we denote by $M[\kappa, \tau]$ the determinant of the submatrix of M obtained by taking the rows $k_1, \dots, k_i$ and the columns $t_1, \dots, t_i$. Moreover, $M[\kappa, [i]]$ is simply denoted by $M[\kappa, i]$.

Example 5.12. We work on Example 5.1, where $\mathbf{i} = 123212312$, and we set $\mathbf{a} = 111111111$. The sequence (σ_j) is given by

$$\begin{array}{lll}\sigma_0 = [1234], & \sigma_1 = [2134], & \sigma_2 = [2314], \\ \sigma_3 = [2341], & \sigma_4 = [2431], & \sigma_5 = [4231], \\ \sigma_6 = [4321], & & \\ \sigma_7 = \sigma_8 = \sigma_9 = \sigma_6. & & \end{array}$$

and the sequence (v_j) is

$$\begin{array}{lll}v_1 = s_1, & v_2 = s_2, & v_3 = s_3, \\ v_4 = s_2, & v_5 = s_1, & v_6 = s_2, \\ v_7 = v_8 = v_9 = e. & & \end{array}$$

Let $T_0 = 2 * 23 * 234 * 24 * 4 * 34 * 234 * 4 * 34$. It can be shown that Ω is exactly the open set $\{\gamma \in Z_{\mathbf{i}} \mid p_{T_0}(\gamma) \neq 0\}$.

Now, direct computations show that the affine variety $Y_{\mathbf{i}}^{s_1} \cap \Omega \subset k^9$ is defined by the equation

$$Q(x_1, \dots, x_9) = x_8(x_1 x_6 + x_2) + x_1 x_5 + x_2 x_4 + x_3 = 0.$$

Since $Y_{\mathbf{i}}^{s_1} \cap \Omega$ is irreducible (as an open set of the irreducible $Y_{\mathbf{i}}^{s_1}$), this equation is also irreducible and generates the ideal of $Y_{\mathbf{i}}^{s_1} \cap \Omega$. Thus, if f is a linear combination of monomials p_T with T of shape $(\mathbf{i}, \mathbf{m})$ such that f vanishes identically on $Y_{\mathbf{i}}^{s_1}$, then $\frac{f}{T_0} \in k[x_1, \dots, x_9]$ vanishes on $Y_{\mathbf{i}}^{s_1} \cap \Omega$, hence

$$\frac{f}{T_0} \in Qk[x_1, \dots, x_9].$$

But we know that each coordinate x_j is a quotient f_j / T_0^k of degree 0 for an $f_j \in R_0 = k[Z_{\mathbf{i}}]$, and also that

$$x_8(x_1 x_6 + x_2) + x_1 x_5 + x_2 x_4 + x_3 = \frac{p_{T_1}}{p_{T_0}},$$

where $T_1 = 2 * 23 * 234 * 24 * 4 * 34 * 234 * 1 * 34$. It follows that f is a multiple of p_{T_1}, hence $f \in p_1 H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{a}'})$ where $\mathbf{a}' = 111111101$.

Hence, the ideal I_1 of $Y_{\mathbf{i}}^{u_1}$ in R_0 is $p_1 H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{a}'})$. We shall generalize this computation below.

Lemma 5.13. *For every j,*

$$U_{\beta_1} v_1 \dots U_{\beta_j} v_j = U_{\beta_1} U_{\sigma_1(\beta_2)} \dots U_{\sigma_{j-1}(\beta_j)} \sigma_j \subset B \sigma_j.$$

Proof. The equality follows from the formula

$$\sigma U_\beta = U_{\sigma(\beta)}\sigma, \quad \forall \sigma \in S_n.$$

For the inclusion, we proceed by induction over j. Since $\beta_1 = (i_1, i_1+1)$ and $v_1 = s_{i_1}$, $U_{\beta_1}v_1 \subset Bv_1 = B\sigma_1$.

Assume that the property holds for $j \geq 1$, that is $U_{\beta_1}v_1 \dots U_{\beta_j}v_j \subset B\sigma_j$.

If $\sigma_j s_{i_{j+1}} < \sigma_j$, then $\sigma_{j+1} = \sigma_j$, $v_{j+1} = e$, $\beta_{j+1} = (i_{j+1}+1, i_{j+1})$, and $\sigma_j(\beta_{j+1}) = (\sigma_j(i_{j+1}+1), \sigma_j(i_{j+1}))$.

$$\begin{aligned} \sigma_j s_{i_{j+1}} < \sigma_j &\iff \sigma_j(i_{j+1}) > \sigma_j(i_{j+1}+1) \\ &\iff U_{\sigma_j(\beta_{j+1})} \subset B. \end{aligned}$$

It follows that

$$\begin{aligned} U_{\beta_1}v_1 \dots U_{\beta_j}v_j U_{\beta_{j+1}}v_{j+1} &\subset B\sigma_j U_{\beta_{j+1}} \\ &\subset BU_{\sigma_j(\beta_{j+1})}\sigma_j \\ &\subset B\sigma_{j+1}. \end{aligned}$$

Similarly, if $\sigma_j s_{i_{j+1}} > \sigma_j$, then $\sigma_{j+1} = \sigma_j s_{i_{j+1}}$, $v_{j+1} = s_{i_j}$ and $\beta_{j+1} = \alpha_{i_{j+1}}$. Hence

$$\begin{aligned} U_{\beta_1}v_1 \dots U_{\beta_j}v_j U_{\beta_{j+1}}v_{j+1} &\subset B\sigma_j U_{\beta_{j+1}} s_{i_j} \\ &\subset BU_{\sigma_j\beta_{j+1}}\sigma_j s_{i_j} \\ &\subset B\sigma_{j+1}. \ \square \end{aligned}$$

Notation 5.14. Let κ be a column of size i. We set $O_\kappa = \{F \in F\ell(n) \mid p_\kappa(F) \neq 0\}$.

Proposition 5.15. *Let $w \in S_n$ and $\kappa = w\varpi_i \in I_{i,n}$. Then $X_w \cap O_\kappa = \coprod_{v \in \mathcal{E}(w,\kappa)} C_v$.*

Proof. Let q be the restriction to X_w of the canonical projection $F\ell(n) \to G_{i,n}$. Then we have $X_w \cap O_\kappa = q^{-1}(X_\kappa \cap q(O_\kappa))$. It is well known that $X_\kappa \cap q(O_\kappa) = C_\kappa$. But we have the equality

$$q^{-1}(C_\kappa) = \coprod_{v \in \mathcal{E}(w,\kappa)} C_v. \ \square$$

Proposition 5.16. *There exists a tableau T_0 of shape $(\mathbf{i}, \mathbf{a})$ such that*

$$\Omega = \{\gamma \in Z_{\mathbf{i}} \mid p_{T_0}(\gamma) \neq 0\}.$$

In particular, φ induces an isomorphism $\varphi^* : (R_0)_{(p_{T_0})} \to k[x_1, \dots, x_r]$, where $(R_0)_{(p_{T_0})}$ is the subring of elements of degree 0 in the localized ring $(R_0)_{p_{T_0}}$, i.e.

$$(R_0)_{(p_{T_0})} = \bigcup_{d \geq 0} \left\{ \frac{f}{p_{T_0}^d} \;\middle|\; f \in R_0 \textit{ is homogeneous of degree } d \right\}.$$

Proof. Let $T_0 = (\sigma_1\varpi_{i_1})^{*a_1} * (\sigma_2\varpi_{i_2})^{*a_2} * \dots * (\sigma_r\varpi_{i_r})^{*a_r}$. Then

$$\{\gamma \in Z_{\mathbf{i}} \mid p_{T_0}(\gamma) \neq 0\} = \prod_{j=1}^{r} \mathrm{pr}_j^{-1}\left(O_{\sigma_j\varpi_{i_j}}\right).$$

We know that

$$\mathrm{pr}_j(\Omega) = U_{\beta_1}v_1 \dots U_{\beta_j}v_j F_{\mathrm{can}}.$$

Thus, by Lemma 5.13, $\mathrm{pr}_j(\Omega) \subset B\sigma_j F_{\mathrm{can}} = C_{\sigma_j}$. But if $F \in C_{\sigma_j}$, then its i_j-th constituent F^{i_j} belongs to $C_{\sigma_j\varpi_{i_j}}$, so

$$p_{\sigma_j\varpi_{i_j}}(F) = p_{\sigma_j\varpi_{i_j}}(F^{i_j}) \neq 0.$$

This proves the inclusion

$$\Omega \subset \prod_{j=1}^{r} \mathrm{pr}_j^{-1}\left(O_{\sigma_j \varpi_{i_j}}\right).$$

For the opposite inclusion, we proceed by induction over r.

If $r = 1$, then $\Omega = \{F_{\mathrm{can}}\} \times U_{\beta_1} v_1 F_{\mathrm{can}}$, with $v_1 = s_{i_1}$ and $\beta_1 = \alpha_{i_1}$. Hence $\Omega = \{F_{\mathrm{can}}\} \times C_{s_{i_1}}$. By Lemma 5.15, $X_{s_{i_1}} \cap O_{s_{i_1}\varpi_{i_1}} = C_{s_{i_1}}$, so

$$\mathrm{pr}_1^{-1}(O_{\sigma_1\varpi_{i_1}}) = \{F_{\mathrm{can}} \overset{i_1}{\text{—}} F^1 \mid F^1 \in C_{s_{i_1}}\}.$$

Thus $\Omega = \mathrm{pr}_1^{-1}(O_{\sigma_1\varpi_{i_1}})$.

Let $r > 1$ and assume that the property holds for $r-1$. Let $\gamma \in \prod_{j=1}^{r} \mathrm{pr}_j^{-1}\left(O_{\sigma_j\varpi_{i_j}}\right)$. By induction, there exist $A_1, \ldots, A_{r-1}$ such that $A_j \in U_{\beta_j} v_j$ and $\gamma_j = A_1 \ldots A_j F_{\mathrm{can}}$ for $j \leq r-1$. Since $\gamma_{r-1} \overset{i_r}{\text{—}} \gamma_r$, there exists $p \in P_{i_r}$ such that $A_1 \ldots A_{r-1} p F_{\mathrm{can}} = \gamma_r$. Now, $P_{i_r} F_{\mathrm{can}}$ is a T-stable curve and we have

$$P_{i_r} F_{\mathrm{can}} = U_{-\alpha_{i_r}} F_{\mathrm{can}} \cup \{s_{i_r} F_{\mathrm{can}}\} = U_{\alpha_{i_r}} s_{i_r} F_{\mathrm{can}} \cup \{F_{\mathrm{can}}\}.$$

If $v_r = e$, then $\sigma_r = \sigma_{r-1}$ and $\beta_r = -\alpha_{i_r}$. By Lemma 5.13, $\gamma_r \in B\sigma_{r-1} p F_{\mathrm{can}}$. If $pF_{\mathrm{can}} = s_{i_r} F_{\mathrm{can}}$, then $\gamma_r \in C_{\sigma_r s_r}$. But by Lemma 5.15, γ_r belongs to a Schubert cell C_v with $v\varpi_{i_r} = \sigma_r \varpi_{i_r}$. Since $\sigma_r \varpi_{i_r} \neq \sigma_r s_r \varpi_{i_r}$, we have a contradiction. Hence $pF_{\mathrm{can}} \in U_{-\alpha_{i_r}} F_{\mathrm{can}}$. So we may choose p in U_{β_r}, which proves that $\gamma \in \Omega$.

If $v_r = s_{i_r}$, we prove similarly that $pF_{\mathrm{can}} \neq F_{\mathrm{can}}$, so we may choose p in $U_{\alpha_{i_r}} s_{i_r} = U_{\beta_r} v_r$, thus $\gamma \in \Omega$. □

Remark 5.17. Consider an arbitrary tableau T of shape $(\mathbf{i}, \mathbf{a})$. Then we may compute $\varphi^*\left(\frac{p_T}{p_{T_0}}\right)$ in the following way. Write

$$T = \kappa_{11} * \cdots * \kappa_{1a_1} * \cdots * \kappa_{r1} * \cdots * \kappa_{ra_r},$$

then

$$\varphi^*\left(\frac{p_T}{p_{T_0}}\right)(x_1, \ldots, x_r) = B_1[\kappa_{11}, i_1] \ldots B_1[\kappa_{1a_1}, i_1] \ldots B_r[\kappa_{r1}, i_r] \ldots B_r[\kappa_{ra_r}, i_r].$$

Proposition 5.18. *Denote by $Q_{ij} \in k[x_1, \ldots, x_r]$ the entries of B_r:*

$$B_r = \begin{pmatrix} Q_{1,1} & Q_{2,1} & \cdots & \cdots & Q_{n-1,1} & 1 \\ \vdots & \vdots & & & 1 & 0 \\ \vdots & \vdots & & \iddots & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots & \vdots \\ Q_{n-1,1} & 1 & \iddots & & \vdots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

The polynomials Q_{ij} all have distinct linear parts.

Proof. It suffices to prove that $B_r w_0 \in B$, but this follows from Lemma 5.13:

$$B_r \in U_{\beta_1} v_1 \ldots U_{\beta_r} v_r F_{\mathrm{can}} \subset B\sigma_r = Bw_0.$$

We may obtain the linear part of the Q_{ij} by derivating B_r. From the expression $B_r = \epsilon_{\beta_1}(x_1)v_1 \dots \epsilon_{\beta_r}(x_r)v_r$, we see that

$$\frac{\partial B_r}{\partial x_j}(0) = E_{\sigma_{j-1}(\beta_j)} w_0.$$

This already proves that the linear parts of the Q_{ij} are distinct. We shall now prove that every elementary matrix E_{kl} occurs as a derivative of B_r, that is, each pair $(i, i+1)$ equals some $\sigma_{j-1}(\beta_j)$, or equivalently, $U_{\beta_1} U_{\sigma_1\beta_2} \dots U_{\sigma_{r-1}\beta_r} = U$ (where U is the unipotent part of B). Since $\mathrm{pr}_r(\Omega) = C_{w_0}$, we have

$$U_{\beta_1} U_{\sigma_1(\beta_2)} \dots U_{\sigma_{r-1}(\beta_r)} F_{\mathrm{can}} = U_{\beta_1} v_1 \dots U_{\beta_r} v_r F_{\mathrm{can}} = U w_0 F_{\mathrm{can}}.$$

Since the stabilizer of $w_0 F_{\mathrm{can}}$ in U is trivial, we conclude that $U_{\beta_1} U_{\sigma_1\beta_2} \dots U_{\sigma_{r-1}\beta_r} = U$. □

Proposition 5.19. *The ideal of $Y_{\mathbf{i}}^{u_{t+1}} \cap \Omega$ in the coordinate ring of $Y_{\mathbf{i}}^{u_t} \cap \Omega$ is generated by Q_{κ_{t1}, b_t}, where κ_{t1} is the first entry of the column κ_t. Moreover,*

$$Q_{\kappa_{t1}, b_t} = \varphi^* \left(\frac{p_{T_t}}{p_{T_0}} \right)$$

where T_t is the tableau obtained from T_0 by replacing its last column of size b_t by κ_t.

Moreover, the varieties $Y_{\mathbf{i}}^{u_t} \cap \Omega$ are isomorphic to affine spaces.

Proof. We alredady know that $Y_{\mathbf{i}}^{u_{t+1}}$ is defined inside $Y_{\mathbf{i}}^{u_t}$ by the vanishing of p_{κ_t}: given a gallery $\gamma = (F_{\mathrm{can}} \overset{i_1}{\text{—}} F_1 \overset{i_2}{\text{—}} \dots \overset{i_r}{\text{—}} F_r)$ in $Y_{\mathbf{i}}^{u_t}$, we know by Corollary 5.7 that $\gamma \in Y_{\mathbf{i}}^{u_{t+1}}$ if and only if $p_{\kappa_t}(F_{l_t}) = 0$.

In Ω, this corresponds to the vanishing of $B_{l_t}[\sigma_{l_t} \varpi_{b_t}, b_t]$. Now, as in the proof of Lemma 5.13,

$$B_r = B_{l_t} b v_{l_{t+1}} \dots v_r$$

for some $b \in U$. The jth column of $B_{l_t} b$ is then a linear combination of the columns $1, \dots, j$ of B_{l_t}. So

$$(B_{l_t} b)[\sigma_{l_t} \varpi_{b_t}, b_t] = B_{l_t}[\sigma_{l_t} \varpi_{b_t}, b_t].$$

Moreover, by definition of l_t, the permutation $v_{l_{t+1}} \dots v_r$ stabilizes the fundamental weight column ϖ_{b_t}, so $B_{l_t} b$ and B_r have the same first b_t columns up to a permutation, hence

$$B_{l_t}[\sigma_{l_t} \varpi_{b_t}, b_t] = \pm B_r[\sigma_{l_t} \varpi_{b_t}, b_t].$$

A straightforward computation shows that this determinant is $\pm Q_{\kappa_{t1}, b_t}$.

To prove that $\varphi^* \left(\frac{p_{T_t}}{p_{T_0}} \right) = \pm Q_{\kappa_{t1}, b_t}$, note that

$$\varphi^* \left(\frac{p_{T_t}}{p_{T_0}} \right) = B_1[\sigma_1 \varpi_{i_1}, i_1] \dots B_{j_t}[\sigma_{j_t} \varpi_{b_t}, b_t] \dots B_r[\sigma_r \varpi_{i_r}, i_r].$$

Now, by Lemma 5.13, $B_j = b_j \sigma_j$ for some $b_j \in B$. So, for $j \neq l_t$,

$$B_j[\sigma_j \varpi_{i_j}, i_j] = \pm b_j[\sigma_j \varpi_{i_j}, \sigma_j \varpi_{i_j}] = \pm 1.$$

Hence $Y_{\mathbf{i}}^{u_{t+1}} \cap \Omega$ is defined by the equation $Q_{\kappa_{t1}, b_t} = 0$. But this polynomial is of the form $x_{p_t} - Q'$ for some variable $x_{p_t} \in \{x_1, \dots, x_r\}$, so we may substitute x_{p_t} by Q' in the coordinate ring of $Y_{\mathbf{i}}^{u_t} \cap \Omega$ to obtain the coordinate ring of $Y_{\mathbf{i}}^{u_{t+1}} \cap \Omega$. Thus, by induction over t, we see that the coordinate ring of $Y_{\mathbf{i}}^{u_{t+1}} \cap \Omega$ is isomorphic to $k[x_i \mid i \neq p_0, \dots, p_t]$. In particular, this ring is a Unique Factorization Domain.

Therefore, the irreducible polynomial Q_{κ_{t1},b_t} generates the ideal of $Y_{\mathbf{i}}^{u_{t+1}}$ in the coordinate ring of $Y_{\mathbf{i}}^{u_t}$. □

Notations 5.20. We set $\mathbf{a}'_t = 0 \dots -1 \dots 0$, the -1 again being at position l_t. Let S_t be the R_t-graded module associated to the coherent sheaf $L_{\mathbf{i},\mathbf{a}'_t}$, that is,

$$S_t = \bigoplus_{d=0}^{+\infty} H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},d\mathbf{a}+\mathbf{a}'_t}).$$

Assumption 5.21. We assume that $\mathbf{a} + \sum_{t=0}^{N-1} \mathbf{a}'_t$ is regular.

Corollary 5.22. *Denote by $\mathcal{O}_{Y_{\mathbf{i}}^{u_t}}$ the structural sheaf of $Y_{\mathbf{i}}^{u_t}$ and assume that $Y_{\mathbf{i}}^{u_t}$ is projectively normal. Then the sequence of R_t-modules*

$$(*) \qquad 0 \to S_t \to R_t \to R_{t+1} \to 0$$

is exact, where the first map is the multiplication by p_{κ_t} and the second is the natural projection.

The exact sequence $(*)$ induces an exact sequence of sheaves of $\mathcal{O}_{Y_{\mathbf{i}}^{u_t}}$-modules

$$0 \to L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t} \to L_{\mathbf{i},\mathbf{m}} \to (L_{\mathbf{i},\mathbf{m}})_{|Y_{\mathbf{i}}^{u_{t+1}}} \to 0$$

and a long exact sequence in cohomology

$$(**) \qquad 0 \longrightarrow H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}) \longrightarrow H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) \longrightarrow H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \longrightarrow$$

$$\longrightarrow H^1(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{i-1}(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \longrightarrow$$

$$\longrightarrow H^i(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}) \longrightarrow H^i(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) \longrightarrow H^i(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \longrightarrow$$

$$\longrightarrow H^{i+1}(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}) \longrightarrow \cdots$$

Proof. Since $Y_{\mathbf{i}}^{u_t}$ is projectively normal, we know that

$$R_t = \bigoplus_{d=0}^{+\infty} H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},d\mathbf{a}}),$$

hence the sequence

$$0 \longrightarrow S_t \xrightarrow{\mu} R_t \xrightarrow{q} R_{t+1} \longrightarrow 0$$

is well defined. Moreover, we already know by Corollary 5.7 that $q \circ \mu = 0$.

Let f be an homogeneous element of degree d in R_t, and suppose that $q(f) = 0$, that is, f vanishes identically on $Y_{\mathbf{i}}^{u_{t+1}}$. Then $\frac{f}{p_{T_0}^d}$ vanishes identically on $Y_{\mathbf{i}}^{u_{t+1}} \cap \Omega$, hence $\varphi^*\left(\frac{f}{p_{T_0}^d}\right) \in k[x_1, \ldots, x_r]$ is a multiple of $Q_{\kappa_{t1}, b_t} = \varphi^*\left(\frac{p_{T_t}}{p_{T_0}}\right)$. It follows that f is a multiple p_{T_t}, hence $f \in p_{\kappa_t} S_t = \mu(S_t)$.

If we consider the coherent sheaves associated to these R_t-modules and tensor them by $L_{\mathbf{i},\mathbf{m}}$, then we get the exact sequence of sheaves of $\mathcal{O}_{Y_{\mathbf{i}}^{u_t}}$-modules

$$0 \to L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t} \to L_{\mathbf{i},\mathbf{m}} \to (L_{\mathbf{i},\mathbf{m}})_{|Y_{\mathbf{i}}^{u_{t+1}}} \to 0,$$

which gives the long exact sequence $(**)$. □

Theorem 5.23.

$(1)_t$ *For every t, the variety $Y_{\mathbf{i}}^{u_t}$ is projectively normal.*

$(2)_t$ *For every $i > 0$, and every $\mathbf{m}$ such that $\mathbf{m}_t = \mathbf{m} + \mathbf{a}'_0 + \cdots + \mathbf{a}'_{t-1}$ is regular, $H^i(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) = 0$. In particular, $H^i(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},d\mathbf{a}}) = 0$ for every $d \in \mathbf{Z}_{\geq 0}$.*

$(3)_t$ *If $t \geq 0$ and $\mathbf{m}_{t+1}$ is regular, then the restriction map $H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}})$ is surjective.*

Proof. We proceed by induction over t.

For $t = 0$, $Y_{\mathbf{i}}^{u_0} = Z_{\mathbf{i}}$. By Theorem 2.6, $Z_{\mathbf{i}}$ is projectively normal, and $H^i(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) = 0$ for $i > 0$. Assume now that $\mathbf{m}_1 = \mathbf{m} + \mathbf{a}'_0$ is regular. Since $Z_{\mathbf{i}}$ is projectively normal, by Corollary 5.22, we have the exact sequence

$$0 \to H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}_1}) \to H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_1}, L_{\mathbf{i},\mathbf{m}}) \to H^1(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}_1}).$$

Since $H^1(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}_1}) = 0$, the restriction map $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_1}, L_{\mathbf{i},\mathbf{m}})$ is surjective.

Assume that the theorem is true for a $t \geq 0$.

We shall prove that $(1)_{t+1}$ is true. By induction, $Y_{\mathbf{i}}^{u_t}$ is projectively normal, so the sequence

$$0 \to H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}) \to H^0(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \to H^1(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t})$$

is exact. Moreover, for $d \in \mathbf{Z}_{\geq 0}$, by $(2)_t$ we have $H^1(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},d\mathbf{a}}) = 0$, hence an exact sequence

$$0 \to S_t \to R_t \to \bigoplus_{d \geq 0} H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},d\mathbf{a}}) \to 0.$$

Since the sequence

$$0 \to S_t \to R_t \to R_{t+1} \to 0$$

is also exact, we have $R_{t+1} = \bigoplus_{d \geq 0} H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},d\mathbf{a}})$, that is, $Y_{\mathbf{i}}^{u_{t+1}}$ is projectively normal.

We now prove that $(2)_{t+1}$ is true. Let $\mathbf{m}$ be such that $\mathbf{m}_{t+1}$ is regular. Note that $\mathbf{m}_t$ is also regular. Since $Y_{\mathbf{i}}^{u_t}$ is projectively normal, we have the exact sequence

$$H^i(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) \to H^i(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \to H^{i+1}(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}).$$

Now, by $(2)_t$,

$$\begin{array}{lll} \mathbf{m}_t \text{ regular} & \Longrightarrow & H^i(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}}) = 0, \\ \mathbf{m}_{t+1} \text{ regular} & \Longrightarrow & H^{i+1}(Y_{\mathbf{i}}^{u_t}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_t}) = 0. \end{array}$$

Thus, $H^i(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) = 0$.

Finally, we prove $(3)_{t+1}$. Since $Y_{\mathbf{i}}^{u_{t+1}}$ is projectively normal, we have the exact sequence

$$0 \to H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_{t+1}}) \to H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_{t+2}}, L_{\mathbf{i},\mathbf{m}}) \to H^1(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_{t+1}}).$$

Since $\mathbf{m}_{t+2}$ is regular, we have by $(2)_{t+1}$ that $H^1(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}+\mathbf{a}'_{t+1}}) = 0$, hence the restriction map $H^0(Y_{\mathbf{i}}^{u_{t+1}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(Y_{\mathbf{i}}^{u_{t+2}}, L_{\mathbf{i},\mathbf{m}})$ is surjective. □

Corollary 5.24. *If* $\mathbf{m}_N = \mathbf{m} + \sum_{t=0}^{N-1} \mathbf{a}'_t$ *is regular, then a basis of* $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ *is given by the* w_0*-standard monomials of shape* $(\mathbf{i}, \mathbf{m})$.

Proof. Since the restriction $H^0(Z_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}}) \to H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ is surjective, the standard monomials p_T that do not vanish identically on $\Gamma_{\mathbf{i}}$ form a generating set. By Theorem 4.21, these monomials are exactly the w_0-standard monomials. By Theorem 3.4, these monomials are linearly independent. □

Proposition 5.25. *Let* p_T *be a standard monomial of shape* $(\mathbf{i}, \mathbf{m})$, *with* $\mathbf{m}$ *arbitrary. Then* p_T *decomposes as a linear combination of* w_0*-standard monomials on* $\Gamma_{\mathbf{i}}$.

Proof. With the notation of Theorem 4.21, the result is true if $\mathbf{m}_N$ is regular. If this is not the case, then we set $\mathbf{b} = b_1 \dots b_r$ with

$$b_j = \begin{cases} N & \text{if } j = l_t \text{ for some } t, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{m}' = \mathbf{m} + \mathbf{b}$. Now, $\mathbf{m}'$ satisfies the assumption of Theorem 4.21. We multiply p_T by $p_{\kappa_1}^N \dots p_{\kappa_{n-1}}^N$ to obtain a new monomial p'_T, where $\kappa_j = w_0\varpi_j$. So p'_T is of shape $(\mathbf{i}, \mathbf{m}')$ and does not vanish identically on $\Gamma_{\mathbf{i}}$. Now, p'_T may be non standard, so we decompose it as a linear combination of w_0-standard monomials of shape $(\mathbf{i}, \mathbf{m}')$ on $\Gamma_{\mathbf{i}}$, thanks to Corollary 5.24:

$$p_T(p_{\kappa_1} \dots p_{\kappa_{n-1}})^N = p_{T'} = \sum_{T''} a_{T''}\, p_{T''}.$$

Since a w_0-standard monomial does not vanish identically on $\Gamma_{\mathbf{i}}$, the columns κ that are in position j_k in a tableau T'' are maximal, *i.e.* equal to $w_0\varpi_k$. Hence we may factor this linear combination by $(p_{\kappa_1} \dots p_{\kappa_{n-1}})^N$, so that p_T is a linear combination of w_0-standard monomials. □

Corollary 5.26. *A basis of* $H^0(\Gamma_{\mathbf{i}}, L_{\mathbf{i},\mathbf{m}})$ *is given by the* w_0*-standard monomials of shape* $(\mathbf{i}, \mathbf{m})$. □

Remark 5.27. In the regular case ($m_i > 0$ for every i), the basis given by standard monomials is compatible with $\Gamma_{\mathbf{i}}$: this is no longer the case if $\mathbf{m}$ is not regular, see Remark 4.22.

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