GENERAL THEORY OF A POSTERIORI ERROR ESTIMATION FOR CONVEX VARIATIONAL PROBLEMS

General variational problem

Now, we shortly discuss the general theory of a posteriori error control for convex variational problems. In the framework of this theory we are able to derive computable upper bounds for the errors for problems of the type

$$\inf_{\mathbf{v}\in\mathbf{V}} \mathbf{J}(\mathbf{v},\mathbf{\Lambda v}), \quad \mathbf{J}(\mathbf{v},\mathbf{\Lambda v}) := \mathbf{G}(\mathbf{\Lambda v}) + \mathbf{F}(\mathbf{v}),$$

where $\mathbf{\Lambda} : \mathbf{V} \longrightarrow \mathbf{Y}$ is a linear continuous operator from a Banach space $\mathbf{V}$ to another Banach space $\mathbf{Y}$ and $\mathbf{J} : \mathbf{Y} \longrightarrow \mathbb{R}$ and $\mathbf{F} : \mathbf{V} \longrightarrow \mathbb{R}$ are convex l.s.c. functionals.

In particular, if

$$\mathbf{\Lambda v} = \nabla \mathbf{v}, \quad \mathbf{G}(\mathbf{y}) = (\mathbf{A y}, \mathbf{y}), \quad \mathbf{F}(\mathbf{v}) = (\mathbf{f}, \mathbf{v}),$$

then we arrive to the variational formulation of the problem

$$\mathbf{div}\, \mathbf{A} \nabla \mathbf{u} + \mathbf{f} = \mathbf{0}$$

with certain boundary conditions.
Many other problems have the above form, were

G is the energy functional whose form is dictated by the dissipative properties of a media.
F is the functional associated with external forces.

Such problems as **diffusion**
linear elasticity,
biharmonic problems,
Kirhghoff and Mindlin plates,
deformation theory of elastoplasticity,
p-Laplace equation,
Stokes problem,
nonlinear models in the theory of viscous fluids and many other problems in mechanics and physics can be presented in such a general form.

In such models the structure of the "energy functional" G plays crucial role in all the parts of the mathematical analysis: existence and differentiability properties of minimizers and estimates of deviations from the minimizers.

Dual (polar) functionals

Hereafter $\mathbf{V}^*$ contains all linear continuous functionals defined on $\mathbf{V}$. The elements of $\mathbf{V}^*$ are marked by stars,
$\langle \mathbf{v}^*, \mathbf{v} \rangle$ is called the **duality pairing** of the spaces $\mathbf{V}$ and $\mathbf{V}^*$.
Let $\mathbf{J} : \mathbf{V} \longrightarrow \mathbb{R}$, then $\mathbf{J}^*$ defined by the relation

$$\mathbf{J}^*(\mathbf{v}^*) = \sup_{\mathbf{v} \in \mathbf{V}} \{ \langle \mathbf{v}^*, \mathbf{v} \rangle - \mathbf{J}(\mathbf{v}) \}$$

is called **dual** to $\mathbf{J}$.
If $\mathbf{J}$ is a smooth function that increases at infinity faster than any linear function, then $\mathbf{J}^*$ is the Legendre transform of $\mathbf{J}$. The above general definition comes from Young and Fenchel. The functional $\mathbf{J}^*$ is also called **polar** to $\mathbf{J}$.

Uniformly convex functionals

Let a proper l.s.c. functional $\boldsymbol{\Upsilon} : \mathbf{Y} \to \overline{\mathbb{R}}$ be subject to the conditions

$$\boldsymbol{\Upsilon}(\mathbf{y}) \geq \mathbf{0},\ \forall \mathbf{y} \in \mathbf{Y}, \qquad \boldsymbol{\Upsilon}(\mathbf{y}) = \mathbf{0} \iff \mathbf{y} = \mathbf{0}_{\mathbf{Y}}.$$

Definition

A convex functional $\mathbf{J} : \mathbf{Y} \to \overline{\mathbb{R}}$ is called uniformly convex in $\mathcal{B}(0_{\mathbf{Y}}, \delta)$ if there exists a functional $\boldsymbol{\Upsilon}_\delta$ such that $\boldsymbol{\Upsilon}_\delta \not\equiv \mathbf{0}$ and for all $\mathbf{y_1}, \mathbf{y_2} \in \mathcal{B}(\mathbf{0_Y}, \delta)$ the following inequality holds:

$$\mathbf{J}\left(\frac{\mathbf{y_1} + \mathbf{y_2}}{\mathbf{2}}\right) + \boldsymbol{\Upsilon}_\delta(\mathbf{y_1} - \mathbf{y_2}) \leq \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{J}(\mathbf{y_1}) + \mathbf{J}(\mathbf{y_2})\right). \tag{1}$$

The functional $\boldsymbol{\Upsilon}_\delta$ enforces standard convexity inequality. For this reason, it is called a **forcing** functional.

It is clear that any uniformly convex functional is convex in $\mathcal{B}(0_{\mathbf{Y}}, \delta)$. Now we establish two important inequalities that hold for uniformly convex functionals.

Theorem

If $\mathbf{J} : \mathbf{Y} \to \overline{\mathbb{R}}$ *is uniformly convex in* $\mathcal{B}(0_{\mathbf{Y}}, \delta)$ *and Gâteaux differentiable in* $\mathcal{B}(0_{\mathbf{Y}}, \delta)$*, then for any* $\mathbf{y}, \mathbf{z} \in \mathcal{B}(\mathbf{0}_{\mathbf{Y}}, \delta)$ *the following relations hold:*

$$\mathbf{J}(\mathbf{z}) \geq \mathbf{J}(\mathbf{y}) + \langle \mathbf{J}'(\mathbf{y}), \mathbf{z} - \mathbf{y} \rangle + \mathbf{2}\boldsymbol{\Upsilon}_{\delta}(\mathbf{z} - \mathbf{y})$$

and

$$\langle \mathbf{J}'(\mathbf{z}) - \mathbf{J}'(\mathbf{y}), \mathbf{z} - \mathbf{y} \rangle \geq \mathbf{2}\boldsymbol{\Upsilon}_{\delta}(\mathbf{z} - \mathbf{y}) + \mathbf{2}\boldsymbol{\Upsilon}_{\delta}(\mathbf{y} - \mathbf{z}).$$

Deviations from the minimizer

Theorem

Let a functional $\mathbf{J}$ *be uniformly convex in* $\mathcal{B}(0_{\mathbf{Y}},\delta)$ *and* $\mathbf{y_m} \in \mathcal{B}(\mathbf{0_Y},\delta)$ *be the minimizer of* $\mathbf{J}$.

$$\Upsilon_\delta(\mathbf{z}-\mathbf{y_m}) \leq \frac{1}{2}\left(\mathbf{J}(\mathbf{z})-\mathbf{J}(\mathbf{y_m})\right), \quad \forall \mathbf{z} \in \mathcal{B}(\mathbf{0_Y},\delta). \tag{2}$$

Estimate (2) is the first step in deriving a posteriori error estimates of the functional type by means of the variational techniques. It shows that deviations from the minimizer (measured in terms of the functional $\boldsymbol{\Upsilon}_{\delta}$) are controlled by the difference of the functionals.

Example. Power growth functionals

Let

$$\mathbf{G}(\mathbf{y}) = \frac{1}{\alpha}\int_{\Omega} |\mathbf{y}|^{\alpha}\,\mathbf{dx} \qquad \mathbf{F}(\mathbf{v}) = \int_{\Omega} \mathbf{fv}\mathbf{dx},$$

where $\alpha > 1$. Then Problem $\mathcal{P}$ is to minimize the functional

$$\mathbf{J}_{\alpha}(\mathbf{v}) := \int_{\Omega} \left(\frac{1}{\alpha}|\nabla \mathbf{v}|^{\alpha} + \mathbf{fv}\right)\mathbf{dx}$$

over the space $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^{\alpha}(\Omega) \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$.

Problem $\mathcal{P}^*$ is to maximize the functional

$$\mathbf{I}^*_{\boldsymbol{\alpha}_*}(\mathbf{y}^*) \;=\; -\frac{\mathbf{1}}{\boldsymbol{\alpha}_*}\int_{\boldsymbol{\Omega}} \;|\mathbf{y}^*|^{\boldsymbol{\alpha}^*}\;\mathbf{dx}$$

over the set

$$\mathbf{Q}^*_{\mathbf{f}} = \left\{ \mathbf{y}^* \in \mathbf{Y}^* := \mathbf{L}^{\boldsymbol{\alpha}_*}(\boldsymbol{\Omega}, \mathbb{R}^{\mathbf{n}}) | \int_{\boldsymbol{\Omega}} \mathbf{y}^* \cdot \nabla \mathbf{w} \mathbf{dx} = \int_{\boldsymbol{\Omega}} \mathbf{fwdx} \;\forall \mathbf{w} \in \mathbf{V} \right\}.$$

For $\alpha \geq 2$ uniform convexity of $\mathbf{G}(\mathbf{y})$ follows from the first Clarkson's inequality

$$\int_\Omega \left|\frac{\mathbf{y_1}+\mathbf{y_2}}{2}\right|^\alpha \mathbf{dx} + \int_\Omega \left|\frac{\mathbf{y_1}-\mathbf{y_2}}{2}\right|^\alpha \mathbf{dx} \leq \frac{1}{2}\int_\Omega \left(|\mathbf{y_1}|^\alpha + |\mathbf{y_2}|^\alpha\right)\mathbf{dx},$$

which is valid for all $\mathbf{y_1},\ \mathbf{y_2} \in \mathbf{Y}$.

See **S. L. Sobolev.** *Some Applications of Functional Analysis in Mathematical Physics*. Hence, we observe that in this case

$$\Upsilon_{\ominus}(\mathbf{z}) = \frac{1}{\alpha}\,\|\mathbf{z}\|^\alpha_{\alpha,\Omega}\,.$$

and

$$\frac{1}{\alpha 2^\alpha}\int_\Omega |\nabla(\mathbf{v}-\mathbf{u})|^\alpha \mathbf{dx} \leq \frac{1}{2}\left(\mathbf{J}_\alpha(\mathbf{v}) - \mathbf{I}^*_\alpha(\mathbf{q}^*)\right),\ \forall q^* \in \mathbf{Q}^*_\mathbf{f},$$

General form of the functional a posteriori estimate

Theorem (2)

Assume that the above conditions on $\mathbf{F}$ *and* $\mathbf{G}$ *are satisfied and*
(i) $\mathbf{G}$ *is uniformly convex on a ball* $B(0,\delta)$*,*
(ii) the solution $\mathbf{u}$ *of Problem* $\mathcal{P}$ *and an element* $\mathbf{v}\in\mathbf{V}$ *are such, that* $\boldsymbol{\Lambda}\mathbf{u}$, $\boldsymbol{\Lambda}\mathbf{v}\in\mathbf{B}(\mathbf{0},\delta)$*. Then, for any* $\mathbf{y}^*\in\mathbf{Y}^*$

$$\boldsymbol{\Phi}_\delta\left(\boldsymbol{\Lambda}(\mathbf{v}-\mathbf{u})\right)\leq\mathbf{M}_\oplus(\mathbf{v},\mathbf{y}^*):=\mathbf{D}_\mathbf{F}(\boldsymbol{\Lambda}^*\mathbf{y}^*,\mathbf{v})+\mathbf{D}_\mathbf{G}(\mathbf{y}^*,\boldsymbol{\Lambda}\mathbf{v}) \qquad (3)$$

where $\mathbf{D}_\mathbf{F}(\boldsymbol{\Lambda}^*\mathbf{y}^*,\mathbf{v}):=\frac{1}{2}\left(\mathbf{F}(\mathbf{v})+\mathbf{F}^*(\boldsymbol{\Lambda}^*\mathbf{y}^*)-\langle\boldsymbol{\Lambda}^*\mathbf{y}^*,\mathbf{v}\rangle\right)$ *and* $\mathbf{D}_\mathbf{G}(\mathbf{y}^*,\boldsymbol{\Lambda}\mathbf{v}):=\frac{1}{2}\left(\mathbf{G}(\boldsymbol{\Lambda}\mathbf{v})+\mathbf{G}^*(-\mathbf{y}^*)+\langle\langle\mathbf{y}^*,\boldsymbol{\Lambda}\mathbf{v}\rangle\rangle\right)$*.*

Proof is presented in S. R. *Math. Comput.*, 2000.

Example. Diffusion problem with Robin conditions

Consider the variational problems for the functional

$$\mathbf{J}(\mathbf{v}, \nabla \mathbf{v}) = \int_{\boldsymbol{\Omega}} \left(\frac{\mathbf{1}}{\mathbf{2}}|\nabla \mathbf{v}|^{\mathbf{2}} + \frac{\delta}{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}}\right)\mathbf{dx} + \int\limits_{\partial_{\mathbf{2}}\boldsymbol{\Omega}} \left(\frac{\alpha}{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}} - \mathbf{g}\mathbf{v}\right)\mathbf{ds}\,.$$

Our problem is to minimize **J** on the set of functions vanishing at $\partial_1\Omega$.

Minimizer **u** of this variational problem is related to the system

$$\begin{aligned} -\boldsymbol{\Delta}\mathbf{u} + \delta\mathbf{u} = \mathbf{0}, &\qquad \text{in } \boldsymbol{\Omega}, \\ \frac{\partial \mathbf{u}}{\partial \nu} + \alpha\mathbf{u} - \mathbf{g} = \mathbf{0}, &\qquad \text{on } \partial_{\mathbf{2}}\boldsymbol{\Omega}. \end{aligned}$$

On $\partial_2\Omega$ the solution satisfies the so–called Robin boundary condition. Let us show that the respective functional a posteriori estimate for the problem with Robin type boundary conditions easily follows from the above general estimate.

We set

$$\mathbf{\Lambda v} := \nabla \mathbf{v},$$

$$\mathbf{G}(\mathbf{\Lambda w}) = \int_{\mathbf{\Omega}} \frac{\mathbf{1}}{\mathbf{2}} |\nabla \mathbf{v}|^{\mathbf{2}} \mathbf{dx}$$

and

$$\mathbf{F}(\mathbf{v}) = \int_{\mathbf{\Omega}} \frac{\delta}{\mathbf{2}} |\mathbf{v}|^{\mathbf{2}} \mathbf{dx} + \int\limits_{\partial_{\mathbf{2}}\mathbf{\Omega}} (\frac{\alpha}{\mathbf{2}} |\mathbf{v}|^{\mathbf{2}} - \mathbf{gv})\, \mathbf{ds}\,.$$

Since

$$\int_{\boldsymbol{\Omega}} \mathbf{y}^* \cdot \nabla \mathbf{v} \mathbf{dx} = \int_{\boldsymbol{\Omega}} -\mathrm{div}\mathbf{y}^* \mathbf{v} \mathbf{dx} + \int\limits_{\partial_2 \boldsymbol{\Omega}} (\mathbf{y}^* \nu) \mathbf{v} \mathbf{ds},$$

we observe that $\boldsymbol{\Lambda}^* \mathbf{y}^* = \{-\mathrm{div}\mathbf{y}^* \mid_{\boldsymbol{\Omega}}, \mathbf{y}^* \cdot \nu \mid_{\partial_2 \boldsymbol{\Omega}}\}$.
In the considered case,

$$\langle \mathbf{y}^*, \mathbf{y} \rangle := \int_{\boldsymbol{\Omega}} \mathbf{y}^* \cdot \mathbf{y} \, \mathbf{dx};$$

$$\mathbf{G}^*(-\mathbf{y}^*) = \sup_{\mathbf{y}} \int_{\boldsymbol{\Omega}} (-\mathbf{y}^* \cdot \mathbf{y} - \frac{\mathbf{1}}{\mathbf{2}} |\mathbf{y}|^2) \mathbf{dx} = \int_{\boldsymbol{\Omega}} \frac{\mathbf{1}}{\mathbf{2}} |\mathbf{y}^*|^2 \mathbf{dx}.$$

Therefore,

$$\mathbf{G}(\boldsymbol{\Lambda}\mathbf{v}) + \mathbf{G}^*(-\mathbf{y}^*) + \langle \mathbf{y}^*, \boldsymbol{\Lambda}\mathbf{v} \rangle = \int_{\boldsymbol{\Omega}} (\frac{\mathbf{1}}{\mathbf{2}}|\nabla \mathbf{v}|^{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}}|\mathbf{y}^*|^{\mathbf{2}} + \nabla \mathbf{v} \cdot \mathbf{y}^*)\mathbf{dx}.$$

Next, in general,

$$\langle\langle \boldsymbol{\Lambda}^* \mathbf{y}^*, \mathbf{v} \rangle\rangle =< -\mathrm{div}\mathbf{y}^*, \mathbf{v} >_{\mathbf{H}^{-\mathbf{1}}(\boldsymbol{\Omega})} + < \mathbf{y}^* \cdot \nu, \mathbf{v} >_{\mathbf{H}^{-\mathbf{1}/\mathbf{2}}(\partial_{\mathbf{2}}\boldsymbol{\Omega})} \cdot$$

However, if we assume that $\mathbf{y}^*$ is sufficiently regular, then

$$\langle\langle \boldsymbol{\Lambda}^* \mathbf{y}^*, \mathbf{v} \rangle\rangle = \int_{\boldsymbol{\Omega}} -\mathrm{div}\mathbf{y}^* \mathbf{v} \mathbf{dx} + \int_{\partial_{\mathbf{2}}\boldsymbol{\Omega}} \mathbf{y}^* \cdot \mathbf{n} \mathbf{v} \mathbf{ds}.$$

Now,

$$\mathbf{F}^*(\mathbf{\Lambda}^*\mathbf{y}^*) = \sup_{\mathbf{v}}\{\int_{\mathbf{\Omega}} -\mathrm{div}\mathbf{y}^*\mathbf{v}\mathbf{dx} + \int\limits_{\partial_2\mathbf{\Omega}} \mathbf{y}^* \cdot \mathbf{nvds} - \mathbf{F}(\mathbf{v})\} =$$

$$\sup_{\mathbf{v}}\{\int_{\Omega} -\mathrm{div}\mathbf{y}^*\mathbf{v}\mathbf{dx} + \int\limits_{\partial_2\mathbf{\Omega}} \mathbf{y}^* \cdot \mathbf{nvds} - \int_{\mathbf{\Omega}} \frac{\delta}{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}}\mathbf{dx} - \int\limits_{\partial_2\mathbf{\Omega}} (\frac{\alpha}{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}} - \mathbf{gv})\,\mathbf{ds} \leq$$

$$\sup_{\mathbf{v}\in\mathbf{L}_2(\mathbf{\Omega})} \int_{\mathbf{\Omega}} (-\mathrm{div}\mathbf{y}^*\mathbf{v} - \frac{\delta}{\mathbf{2}}|\mathbf{v}|^{\mathbf{2}})\mathbf{dx} + \sup_{\varrho\in\mathbf{L}_2(\partial_2\mathbf{\Omega})} \int\limits_{\partial_2\mathbf{\Omega}} ((\mathbf{y}^* \cdot \nu)\varrho - \frac{\alpha}{\mathbf{2}}|\varrho|^{\mathbf{2}} + \mathbf{g}\varrho)\mathbf{ds}$$

$$\sup_{\mathbf{v}\in\mathbf{L}_2(\Omega)}\int_{\Omega}(-\mathrm{div}\mathbf{y}^*\mathbf{v}-\frac{\delta}{2}|\mathbf{v}|^2)\mathbf{dx}=\int_{\Omega}\frac{1}{2\delta}|\mathrm{div}\mathbf{y}^*|^2\mathbf{dx},$$

$$\sup_{\varrho\in\mathbf{L}_2(\partial_2\Omega)}\int\limits_{\partial_2\Omega}((\mathbf{y}^*\cdot\nu)\varrho-\frac{\alpha}{2}|\varrho|^2+\mathbf{g}\varrho)\mathbf{ds}=\int\limits_{\partial_2\Omega}\frac{1}{2\alpha}|\mathbf{y}^*\cdot\nu+\mathbf{g}|^2\mathbf{ds}.$$

Hence,

$$\mathbf{F}^*(\mathbf{\Lambda}^*\mathbf{y}^*)\leq\int_{\Omega}\frac{1}{2\delta}|\mathrm{div}\mathbf{y}^*|^2\mathbf{dx}+\int\limits_{\partial_2\Omega}\frac{1}{2\alpha}|\mathbf{y}^*\cdot\nu+\mathbf{g}|^2\mathbf{ds}.$$

Now,

$$\mathbf{F}(\mathbf{v}) = \int_{\Omega} \frac{\delta}{2}|\mathbf{v}|^2\mathbf{dx} + \int\limits_{\partial_2\Omega} (\frac{\alpha}{2}|\mathbf{v}|^2 - \mathbf{gv})\,\mathbf{ds}\,,$$

$$\langle\langle \mathbf{\Lambda}^*\mathbf{y}^*, \mathbf{v}\rangle\rangle = \int_{\Omega} -\mathrm{div}\mathbf{y}^*\mathbf{vdx} + \int\limits_{\partial_2\Omega} (\mathbf{y}^* \cdot \nu)\mathbf{vds},$$

$$\mathbf{F}^*(\mathbf{\Lambda}^*\mathbf{y}^*) \leq \int_{\Omega} \frac{1}{2\delta}|\mathrm{div}\mathbf{y}^*|^2\mathbf{dx} + \int\limits_{\partial_2\Omega} \frac{1}{2\alpha}|\mathbf{y}^* \cdot \nu + \mathbf{g}|^2\mathbf{ds}.$$

Therefore,

$$\mathbf{F}(\mathbf{v}) + \mathbf{F}^*(\mathbf{\Lambda}^*\mathbf{y}^*) - \langle\langle \mathbf{\Lambda}^*\mathbf{y}^*, \mathbf{v}\rangle\rangle \leq \int_{\Omega} \frac{1}{2\delta}(\mathrm{div}\mathbf{y}^* + \delta\mathbf{v})^2\mathbf{dx} +$$

$$\int\limits_{\partial_2\Omega} \Big(\frac{\alpha}{2}|\mathbf{v}|^2 + \frac{1}{2\alpha}|\mathbf{y}^* \cdot \nu + \mathbf{g}|^2 - (\mathbf{y}^* \cdot \nu + \mathbf{g})\mathbf{v}\Big)\mathbf{ds}.$$

We obtain

$$\mathbf{F}(\mathbf{v}) + \mathbf{F}^*(\mathbf{\Lambda}^*\mathbf{y}^*) - \langle\langle \mathbf{\Lambda}^*\mathbf{y}^*, \mathbf{v}\rangle\rangle \le \int_{\mathbf{\Omega}} \frac{\mathbf{1}}{\mathbf{2}\delta}(\mathrm{div}\mathbf{y}^* + \delta\mathbf{v})^{\mathbf{2}}\mathbf{dx} + \int\limits_{\partial_{\mathbf{2}}\mathbf{\Omega}} \frac{\mathbf{1}}{\mathbf{2}\alpha}|\mathbf{y}^* \cdot \nu + \mathbf{g} - \alpha\mathbf{v}|^{\mathbf{2}}\mathbf{ds}.$$

$$\mathbf{G}(\mathbf{\Lambda v}) + \mathbf{G}^*(-\mathbf{y}^*) + \langle \mathbf{y}^*, \mathbf{\Lambda v}\rangle = \int_{\mathbf{\Omega}} \frac{\mathbf{1}}{\mathbf{2}}|\nabla\mathbf{v} + \mathbf{y}^*|^{\mathbf{2}}\mathbf{dx}.$$

Hence, the error is majorated by

$$\int_{\mathbf{\Omega}}\left(\frac{1}{2\delta}(\mathrm{div}\mathbf{y}^*+\delta\mathbf{v})^2+\frac{1}{2}|\nabla\mathbf{v}+\mathbf{y}^*|^2\right)\mathbf{dx}+\int\limits_{\partial_2\mathbf{\Omega}}\frac{1}{2\alpha}|\mathbf{y}^*\cdot\nu+\mathbf{g}-\alpha\mathbf{v}|^2\mathbf{ds}$$

We observe that the Majorant vanishes if and only if

$$\begin{aligned}&\mathrm{div}\mathbf{y}^*+\delta\mathbf{v}=\mathbf{0}\qquad \text{in } \mathbf{\Omega},\\&\mathbf{y}^*\cdot\nu+\mathbf{g}-\alpha\mathbf{v}=\mathbf{0}\qquad \text{on } \partial_2\mathbf{\Omega},\\&\mathbf{y}^*=-\nabla\mathbf{v}\qquad \text{in } \mathbf{\Omega}.\end{aligned}$$

These relations mean that

$$\begin{aligned}&-\mathbf{\Delta}\mathbf{v}+\delta\mathbf{v}=\mathbf{0}\qquad \text{in } \mathbf{\Omega},\\&\frac{\partial\mathbf{v}}{\partial\nu}+\alpha\mathbf{v}=\mathbf{g}\qquad \text{on } \partial_2\mathbf{\Omega},\end{aligned}$$

i.e., since **v** vanishes at $\partial_1\Omega$ it is but the exact solution.

Maxwell's equation.

Note that the majorant for the Maxwell's problem can be also derived from the general a posteriori estimate obtained for convex variational problems related to the functional

$$\mathbf{G}(\mathbf{\Lambda v}) + \mathbf{F}(\mathbf{v}).$$

In the case considered,

$$\mathbf{\Lambda} = \mathbf{curl},$$

$$\mathbf{G}(\mathbf{y}) = \int_{\Omega} \frac{1}{2}\mu^{-1}|\mathbf{y}|^2\,\mathbf{dx}, \quad \text{and} \quad \mathbf{F}(\mathbf{v}) = \int_{\Omega} (\frac{1}{2}\mathbf{k}^2|\mathbf{v}|^2 - \mathbf{j}\cdot\mathbf{v})\,\mathbf{dx}.$$

FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR PROBLEMS IN THE THEORY OF VISCOUS INCOMPRESSIBLE FLUIDS

”Fluid dynamicists were divided into hydraulic engineers who observe what cannot be explained and mathematicians who explain things that cannot be observed.”

Sir Cyril Hinshelwood

Mathematical models in the theory of viscous fluids

Certainly the most known model is the Navier–Stokes problem: find $\mathbf{u}(\mathbf{x},\mathbf{t}) \in \overset{\circ}{\mathbf{J}}{}_{\mathbf{2}}^{\mathbf{1}}(\mathbf{\Omega})$ and $\mathfrak{p}(x,t) \in \overset{\circ}{\mathbf{L}}_{\mathbf{2}}(\mathbf{\Omega})$ such that

$$\begin{aligned}
&\mathbf{u_t} - \nu \mathbf{\Delta u} + (\mathbf{u} \cdot \nabla \mathbf{u}) = \mathbf{f} - \nabla \mathfrak{p} \qquad \text{in } \mathbf{\Omega},\\
&\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{U}(\mathbf{x}),\\
&\mathbf{u} = \mathbf{u_0} \quad \text{on } \mathbf{\Gamma_D}\\
&\nabla \mathbf{u} \cdot \boldsymbol{\nu} + \mathfrak{p}\boldsymbol{\nu} = \mathbf{g_N} \quad \text{on} \mathbf{\Gamma_N}.
\end{aligned}$$

For the first equation it is also used the form

$$\mathbf{u_t} - \nu \mathbf{\Delta u} + \mathrm{Div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{f} - \nabla \mathfrak{p} \qquad \text{in } \mathbf{\Omega},$$

From the mathematical point of view NS is still a mystery. Existence of a unique solution in 3D is not yet proved even for $(0, T] \times \mathbb{R}^n$.

It is known that for sufficiently regular solenoidal $\mathbf{U}(\mathbf{x})$ there exists a weak Leray-Hopf solution, i.e., a function

$$\mathbf{u} \in \mathbf{L}^{\infty}(\mathbf{0}, \mathbf{T}; \mathbf{L}^{\mathbf{2}}(\mathbb{R}^{\mathbf{n}})) \cap \mathbf{L}^{\mathbf{2}}(\mathbf{0}, \mathbf{T}; \mathbf{H}^{\mathbf{1}}(\mathbb{R}^{\mathbf{n}}))$$

Proving (or presenting a contr–example) of that NS equation possesses in $(0, T] \times \mathbb{R}^n$ a smooth solution provided that initial data are sufficiently regular forms one of the

Millennium Prize Problems

stated by the Clay Mathematical Institute.

From the introduction to the Third Millennium Prize Problem:

"...Although these (NS) equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations."

At present all the results of such a type are conditional:
e.g., it is possible to prove that (H. Beirao da Veiga,G. Galdi)

$$\mathbf{u} \in \mathbf{C}^{\infty}(\mathbf{0}, \mathbf{T}; \mathbf{L}^{\mathbf{s}}(\mathbb{R}^{\mathbf{n}})$$

provided that

$$\nabla \mathbf{u} \in \mathbf{L}^{\mathbf{r}}(\mathbf{0}, \mathbf{T}; \mathbf{L}^{\mathbf{s}}(\mathbb{R}^{\mathbf{n}})), \qquad \frac{\mathbf{2}}{\mathbf{r}} + \frac{\mathbf{n}}{\mathbf{s}} = \mathbf{2}, \quad \mathbf{s} > \frac{\mathbf{n}}{\mathbf{2}}.$$

In mathematical modeling *linearizations* of NS equations are often used.

Stokes Problem

$$\begin{aligned}
&\mathbf{u_t} - \ \nu\mathbf{\Delta u}+ = \mathbf{f} - \nabla\mathfrak{p} \qquad \text{in } \mathbf{\Omega},\\
&\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{U}(\mathbf{x}),\\
&\mathbf{u} = \mathbf{u_0} \quad \text{on } \mathbf{\Gamma_D}\\
&\nabla\mathbf{u}\cdot\boldsymbol{\nu} + \mathfrak{p}\boldsymbol{\nu} = \mathbf{g_N} \quad \text{on}\mathbf{\Gamma_N}
\end{aligned}$$

and

Oseen Problem

$$\begin{aligned}
&\mathbf{u_t} - \ \nu\mathbf{\Delta u} + \text{div}(\mathbf{a}\otimes\mathbf{u}) = \mathbf{f} - \nabla\mathfrak{p} \qquad \text{in } \mathbf{\Omega},\\
&\mathbf{u}(\mathbf{x},\mathbf{0}) = \mathbf{U}(\mathbf{x}), \qquad \text{div}\mathbf{a} = \mathbf{0},\\
&\mathbf{u} = \mathbf{u_0} \quad \text{on } \mathbf{\Gamma_D}\\
&\nabla\mathbf{u}\cdot\boldsymbol{\nu} + \mathfrak{p}\boldsymbol{\nu} = \mathbf{g_N} \quad \text{on}\mathbf{\Gamma_N}
\end{aligned}$$

problems.

Semidiscrete approximations of NS equation:

For example:

$$\frac{\mathbf{u^k} - \mathbf{u^{k-1}}}{\mathbf{\Delta t}} - \ \nu \mathbf{\Delta u^k} + \mathrm{div}(\mathbf{u^{k-1}} \otimes \mathbf{u^{k-1}}) = \mathbf{f} - \nabla \mathfrak{p}^{\mathbf{k}} \qquad \text{in } \mathbf{\Omega},$$

$$\mathrm{div}\mathbf{u^k} = \mathbf{0}$$

and

$$\frac{\mathbf{u^k} - \mathbf{u^{k-1}}}{\mathbf{\Delta t}} - \ \nu \mathbf{\Delta u^k} + \mathrm{div}(\mathbf{u^{k-1}} \otimes \mathbf{u^{k}}) = \mathbf{f} - \nabla \mathfrak{p}^{\mathbf{k}} \qquad \text{in } \mathbf{\Omega},$$

$$\mathrm{div}\mathbf{u^k} = \mathbf{0}.$$

Generalized Stokes Problem

$$\alpha \mathbf{u} - \ \nu \boldsymbol{\Delta} \mathbf{u} + = \mathbf{f} - \nabla \mathfrak{p} \qquad \text{in } \boldsymbol{\Omega},$$

$$\mathbf{u} = \mathbf{u_0} \quad \text{on } \boldsymbol{\Gamma_D}$$

$$\nabla \mathbf{u} \cdot \boldsymbol{\nu} + \mathfrak{p} \boldsymbol{\nu} = \mathbf{g_N} \quad \text{on} \boldsymbol{\Gamma_N}$$

Generalized Oseen Problem

$$\begin{aligned}
&\alpha\mathbf{u} - \ \nu\boldsymbol{\Delta}\mathbf{u} + \operatorname{div}(\mathbf{a}\otimes\mathbf{u}) = \mathbf{f} - \nabla\mathfrak{p} \qquad \text{in } \boldsymbol{\Omega},\\
&\operatorname{div}\mathbf{a} = \mathbf{0},\\
&\mathbf{u} = \mathbf{u_0} \quad \text{on } \boldsymbol{\Gamma_D}\\
&\nabla\mathbf{u}\cdot\boldsymbol{\nu} + \mathfrak{p}\boldsymbol{\nu} = \mathbf{g_N} \quad \text{on}\boldsymbol{\Gamma_N}
\end{aligned}$$

Another version of the generalized Stokes problem is related to models of fluids with polymerization. One of the most simple models of such a type can be found in e.g., J. Bonvin, M. Picasso, and R. Stenberg, *Comput. Methods Appl. Mech. Engrg.*(2001). It can be presented in the form

$$\begin{aligned} \mathrm{Div}\boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0}, && \text{in } \Omega, && (4)\\ \mathrm{div}\mathbf{u} &= \mathbf{0} && \text{in } \Omega, && (5)\\ \boldsymbol{\sigma} &= -\mathfrak{p}\mathbb{I} + æ + \nu\nabla\mathbf{u}, && \text{in } \Omega, && (6)\\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, && (7) \end{aligned}$$

where æ is a given tensor–valued function such that $\mathrm{tr}æ = 0$.

Note that $\boldsymbol{\sigma}$ is decomposed into the spherical and deviatoric parts, respectively.

Generalized solution $\mathbf{u}$ of the system (4)–(7) is a function in $\overset{\circ}{\mathbf{J}}{}^{\mathbf{1}}_{\mathbf{2}}(\boldsymbol{\Omega})$ satisfying that the integral identity

$$\int_{\boldsymbol{\Omega}} \nu\nabla\mathbf{u} : \nabla\mathbf{w} + \text{æ} : \nabla\mathbf{w}\mathbf{dx} = \int_{\boldsymbol{\Omega}} \mathbf{f}\cdot\mathbf{w}\,\mathbf{dx}, \quad \mathbf{w} \in \overset{\circ}{\mathbf{J}}{}^{\mathbf{1}}_{\mathbf{2}}(\boldsymbol{\Omega}). \; (8)$$

This function minimizes the functional

$$\mathbf{I}(\mathbf{w}) := \int_{\Omega} \left(\frac{\nu}{\mathbf{2}}|\nabla \mathbf{w}|^{\mathbf{2}} + æ : \nabla \mathbf{w}\right) \mathbf{dx} - \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, \mathbf{dx} \tag{9}$$

over the space $\overset{\circ}{\mathbf{J}}{}^{\mathbf{1}}_{\mathbf{2}}(\mathbf{\Omega})$.

Generalized Newtonian Fluids

In these models the basic equation is as follows:

$$\begin{aligned}&\mathbf{u_t} - \operatorname{div}\boldsymbol{\sigma} + (\mathbf{u}\cdot\nabla\mathbf{u}) = \mathbf{f} - \nabla\mathfrak{p} \qquad \text{in } \boldsymbol{\Omega},\\ &\boldsymbol{\sigma} \in \partial\pi(\nabla\mathbf{u}),\end{aligned}$$

where π is the so–called *dissipative potential*.
Newtonian models refer to

$$\pi = \frac{1}{2}\nu|\nabla\mathbf{u}|^2.$$

One of the most known examples of such a fluid is the Bingham fluid, where

$$\pi = \frac{1}{2}\nu|\nabla\mathbf{u}|^2 + \mathbf{k}_*|\nabla\mathbf{u}|.$$

Numerical methods in CFD

V. Girault and P. A. Raviart. Springer, 1986
M. Gunzburger. Academic press. 1989.
R. Glowinski. North–Holland, 2003.
R. Rannacher. Birkhauser, 2000.

A posteriori error estimates and mesh adaptive FEM

A posteriori methods for FEM in CFD can be found e.g. in publications (see the list of references at the end) of:
M. Ainsworth, C. Bernardi, R. E. Bank, M. Boman,
L. Demkowicz, P. Devloo, C. Johnson, J. T. Oden,
R. Rannacher, T. Strouboulis, R. Verfürth, B. D. Welfert, W. Wu...

Typical indicators used in mesh refinement

Usually automatic mesh adaptation is guided by "cell indicators" computed with help of approximate solution $(\mathbf{u_h}, \mathfrak{p}_\mathbf{h})$.

Vorticity indicator: $\|\nabla \times \mathbf{u_h}\|_\mathbf{T}.$

Pressure–gradient indicator $\|\nabla \mathfrak{p}_\mathbf{h}\|_\mathbf{T}.$

Energy–norm error indicator

$$\|\mathrm{div}\mathbf{u_h}\|_{\mathbf{T}} + \\ + \| -\nu\Delta\mathbf{u_h} + \mathbf{u_h}\cdot\nabla\mathbf{u_y} + \nabla\mathfrak{p}_{\mathbf{h}} + \mathbf{f}\|_{\mathbf{T}} + \\ + h^{1/2}\|j(\mathbf{u_h},\mathfrak{p}_{\mathbf{h}})\|_{\partial\mathbf{T}},$$

where $\mathbf{j}(\mathbf{u_h},\mathfrak{p}_{\mathbf{h}})$ are certain jumps on the edges.

WE WILL DISCUSS A DIFFERENT APPROACH TO THE DERIVATION OF A POSTERIORI ESTIMATES BASED ON THE ANALYSIS OF THE RESPECTIVE DIFFERENTIAL FORMULATION

STOKES PROBLEM

$$
\begin{aligned}
-\nu\boldsymbol{\Delta}\mathbf{u} &= \mathbf{f} - \nabla\mathfrak{p} && \text{in } \boldsymbol{\Omega}, && (10)\\
\operatorname{div}\mathbf{u} &= \mathbf{0} && \text{in } \boldsymbol{\Omega}, && (11)\\
\mathbf{u} &= \mathbf{u_0} && \text{on } \partial\boldsymbol{\Omega}, && (12)
\end{aligned}
$$

$\mathbf{u_0}$ is a given function such that $\operatorname{div}\mathbf{u_0} = \mathbf{0}$.

Generalized solution

Find $\mathbf{u} \in \overset{\circ}{\mathbf{J}}{}^1_2(\boldsymbol{\Omega}) + \mathbf{u_0}$ such that

$$\nu \int_{\boldsymbol{\Omega}} \nabla \mathbf{u} : \nabla \mathbf{w}\, \mathbf{dx} = \int_{\boldsymbol{\Omega}} \mathbf{f} \cdot \mathbf{w}\, \mathbf{dx} \quad \forall \mathbf{w} \in \overset{\circ}{\mathbf{J}}{}^1_2(\boldsymbol{\Omega}). \tag{13}$$

$$\nu \int_{\boldsymbol{\Omega}} \nabla \mathbf{u} : \nabla \mathbf{w}\, \mathbf{dx} = \int_{\boldsymbol{\Omega}} (\mathbf{f} - \nabla \mathfrak{p}) \cdot \mathbf{w}\, \mathbf{dx} \quad \forall \mathbf{w} \in \mathbf{V_0}. \tag{14}$$

Here $\mathfrak{p} \in \overset{\circ}{L}_2(\Omega)$ is the unknown pressure.

Variational formulation

$$\inf_{\mathbf{w}\in \overset{\circ}{\mathbf{J}}{}^{1}_{2}(\boldsymbol{\Omega})+\mathbf{u_0}} \mathbf{I}(\mathbf{w}), \quad \mathbf{I}(\mathbf{w}) = \int\limits_{\boldsymbol{\Omega}} \left(\frac{\nu}{\mathbf{2}}|\nabla \mathbf{w}|^{\mathbf{2}} - \mathbf{f}\cdot\mathbf{w}\right) \mathbf{dx} \tag{15}$$

Let $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^1_2(\mathbf{\Omega}) + \mathbf{u_0}$ be an approximate solution.

Our aim is to derive a computable functional $\mathbf{M}_\oplus(\mathbf{v})$ such that

$$\|\mathbf{u} - \mathbf{v}\| \leq \mathbf{M}_\oplus \mathbf{v}$$

$$\mathbf{M}_\oplus(\mathbf{v_k}) \to \mathbf{0} \quad \text{as } \mathbf{v_k} \longrightarrow \mathbf{u} \text{ in } \overset{\circ}{\mathbf{J}}{}^1_2(\mathbf{\Omega}).$$

Such functional can be derived both from the **integral identity** and from the **variational formulation**.

Error majorant $\mathbf{M}_\oplus$ for div-free approx.

For $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^{\mathbf{1}}_{2}(\boldsymbol{\Omega})$ we have (see S. R. (2002) [29])

$$\nu\|\nabla(\mathbf{u}-\mathbf{v})\| \leq \|\nu\nabla\mathbf{v}-\boldsymbol{\tau}\| + \mathbf{C}_{\boldsymbol{\Omega}}\|\mathrm{div}\boldsymbol{\tau}+\mathbf{f}-\nabla\mathbf{q}\|$$

where $\mathbf{C}_{\boldsymbol{\Omega}}$ is a constant in the Friederichs inequality,

$$\|\mathbf{w}\| \leq \mathbf{C}_{\boldsymbol{\Omega}}\|\nabla\mathbf{w}\| \qquad \forall\mathbf{w} \in \overset{\circ}{\mathbf{H}}{}^{\mathbf{1}}(\boldsymbol{\Omega},\mathbb{R}^{\mathbf{n}}),$$

$\boldsymbol{\tau} \in \boldsymbol{\Sigma}_{\mathrm{div}} := \{\boldsymbol{\tau} \in \mathbf{L}^{\mathbf{2}},\ \mathrm{div}\boldsymbol{\tau} \in \mathbf{L}^{\mathbf{2}}\}$, $\mathfrak{q} \in H^1 \cap \overset{\circ}{L}_2(\Omega)$.

Quadratic form of the Majorant

$$\nu^2\|\nabla(\mathbf{u}-\mathbf{v})\|^2 \leq (1+\beta)\|\nu\nabla\mathbf{v}-\boldsymbol{\tau}\|^2+\frac{1+\beta}{\beta}\mathbf{C}_{\boldsymbol{\Omega}}^2\|\mathrm{div}\boldsymbol{\tau}+\mathbf{f}-\nabla\mathbf{q}\|^2$$

Here β is any positive real number.
Denote the right–hand side by

$$\mathbf{M}_{\oplus}(\mathbf{v},\boldsymbol{\tau},\mathbf{q},\beta,\mathbf{C}_{\boldsymbol{\Omega}},\mathbf{f})$$

New formulation of the Stokes problem

For any $\beta > 0$ Stokes problem can be considered as a solution of the NEW variational problem for the quadratic functional $\mathbf{M}_{\oplus}$:

$$\inf_{\substack{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_{\mathrm{div}}(\boldsymbol{\Omega}),\\ \mathsf{q}\in\overset{\circ}{L}_2(\Omega)\cap H^1(\Omega),\\ \mathbf{v}\in\overset{\circ}{\mathbf{J}}{}_{\mathbf{2}}^{\mathbf{1}}(\boldsymbol{\Omega})+\mathbf{u_0}}} \mathbf{M}_{\oplus}(\mathbf{v},\boldsymbol{\tau},\mathbf{q},\beta,\mathbf{C}_{\boldsymbol{\Omega}},\mathbf{f})$$

New variational problem has important properties:

- **Infimum of new functional is known, it is zero!;**
- **Infimum is attained if and only if $\mathbf{v} = \mathbf{u}$, $\boldsymbol{\tau} = \nu\nabla\mathbf{u}$, and $\mathfrak{q} = \mathfrak{p}$;**
- **For any v, $\boldsymbol{\tau}$, and $\mathfrak{q}$ the functional $\mathbf{M}_{\oplus}(\mathbf{v}, \boldsymbol{\tau}, \mathbf{q}, \beta, \mathbf{C}_{\boldsymbol{\Omega}}, \mathbf{f})$ gives the**
  radius of the ball (in energy space) centered at v that contains true velocity field u !

We may also rewrite this estimate in a different form, which is valid for a wider set of pressure functions $\mathfrak{q}$. For this purpose, we set $\boldsymbol{\tau} = \boldsymbol{\eta} + \mathfrak{q}\mathbb{I}$, where $\mathbb{I}$ is the unit tensor, $\mathfrak{q} \in \overset{\circ}{L}_2(\Omega)$, and $\boldsymbol{\eta} \in \Sigma_{\rm div}(\boldsymbol{\Omega})(\boldsymbol{\Omega})$. Then, the estimate has a somewhat different form

$$\nu \,\|\nabla(\mathbf{u}-\mathbf{v})\| \leq \|\nu\nabla\mathbf{v}-\boldsymbol{\eta}-\mathfrak{q}\mathbb{I}\| + \mathbf{C}_{\boldsymbol{\Omega}}\|{\rm div}\boldsymbol{\eta}+\mathbf{f}\|. \tag{16}$$

Above estimates are valid **only for solenoidal approximations**.

HOW TO DERIVE A POSTERIORI ESTIMATES FOR NON–SOLENOIDAL APPROXIMATIONS?

Lemma. For any $\mathbf{g} \in \overset{\circ}{\mathbf{L}}_\mathbf{2}(\mathbf{\Omega})$ there exists a function $\bar{\mathbf{u}} \in V_0$ satisfying the relation $\mathrm{div}\bar{\mathbf{u}} = g$ and the condition

$$\|\nabla\bar{\mathbf{u}}\| \leq \kappa_{\mathbf{\Omega}}\|\mathbf{g}\|, \tag{17}$$

where κ_Ω is a positive constant that depends on Ω.

This Lemma (see the proof in the paper by O. Ladyzhenskaya and V. Solonnikov [25] 1976) implies several important corollaries.

Ladyzhenskaya–Babuvska–Brezzi (LBB)–condition

There exists a positive constant **C** such that

$$\inf_{\mathbf{g}\in\overset{\circ}{\mathbf{L}}_2(\mathbf{\Omega})\ \mathbf{g}\neq\mathbf{0}}\quad\sup_{\mathbf{w}\in\mathbf{V_0}\ \mathbf{w}\neq\mathbf{0}}\ \frac{\displaystyle\int_{\mathbf{\Omega}}\mathbf{g}\,\mathrm{div}\mathbf{w}\,\mathbf{dx}}{\|\phi\|\ \|\nabla\mathbf{w}\|}\geq\mathbf{C}\,. \qquad (18)$$

It is easy to show that Lemma implies (18).

For $\mathbf{g} \in \overset{\circ}{\mathbf{L}}_2(\boldsymbol{\Omega})$ we have $\mathbf{v_g} \in \mathbf{V_0}$ that meets the conditions

$$\mathrm{div} v_g = g, \quad \|\nabla v_\phi\| \leq \kappa_\Omega \|g\|.$$

Therefore,

$$\sup_{\mathbf{v} \in \mathbf{V_0}, \mathbf{w} \neq \mathbf{0}} \frac{\int_\Omega \mathbf{g} \mathrm{div} \mathbf{v}\, \mathbf{dx}}{\|\nabla \mathbf{v}\| \, \|\mathbf{g}\|} \geq \frac{\int_\Omega \mathbf{g} \mathrm{div} \mathbf{v_g}\, \mathbf{dx}}{\|\nabla \mathbf{v_g}\| \, \|\mathbf{g}\|} = \frac{\|\mathbf{g}\|}{\|\nabla \mathbf{v_g}\|} \geq \frac{\mathbf{1}}{\kappa_{\boldsymbol{\Omega}}}$$

and, consequently, (18) holds with $\mathbf{C} = \kappa_\Omega^{-1}$.

Note that (18) can be presented in the form

$$\sup_{\mathbf{w}\in\mathbf{V_0},\ \mathbf{w}\neq\mathbf{0}} \frac{\int\limits_{\Omega} \mathbf{g}\,\mathrm{div}\mathbf{w}\,\mathbf{dx}}{\|\nabla\mathbf{w}\|} \geq \mathbf{C}\,\|\,\mathbf{g}\| \qquad \text{for all } \mathbf{g}\in\overset{\circ}{\mathbf{L}}_{\mathbf{2}}(\mathbf{\Omega}).$$

The expression in the left–hand side of the above inequality is the norm of $\nabla\mathbf{g}$ in the space topologically dual to $\mathbf{V_0}$, namely

$$[\![\,\nabla\mathbf{g}\,]\!] := \sup_{\mathbf{w}\in\mathbf{V_0}} \frac{<\nabla\mathbf{g},\mathbf{w}>}{\|\nabla\mathbf{w}\|}.$$

Then, we arrive to the Nečas inequality

$$\|\mathbf{p}\| \leq \kappa_{\boldsymbol{\Omega}} [\![\nabla \mathfrak{p}]\!] \qquad \forall \; \mathbf{p} \in \overset{\circ}{\mathbf{L}}_{\mathbf{2}}(\boldsymbol{\Omega}), \tag{19}$$

A simple proof for Lipschitz domains is given in the paper by J. Bramble, 2002 [7] , where it is also shown that the well–known Korn's inequality follows from the inf-sup condition.

Constants C and κ_Ω play an important role in the numerical analysis of the Stokes problem as well as in the theoretical one. They affect the stability of mixed–type formulations and the efficiency of iteration methods.

Estimates of the distance to the set of solenoidal functions

Another corollary of the above Lemma is the projection type inequality

$$\inf_{\mathbf{v}\in \overset{\circ}{\mathbf{J}}{}^1_2(\boldsymbol{\Omega})} \|\nabla(\widehat{\mathbf{v}}-\mathbf{v})\| \leq \kappa_{\boldsymbol{\Omega}}\|\mathrm{div}\widehat{\mathbf{v}}\|, \tag{20}$$

which shows that the distance to $\overset{\circ}{\mathbf{J}}{}^1_2(\boldsymbol{\Omega})$ is measured by the value of $\|\mathrm{div}\widehat{\mathbf{v}}\|$.
Proof is presented in
S. R. (2002) [29, 32] where also the following a posteriori estimate was derived

$$\nu\|\nabla(\mathbf{u}-\widehat{\mathbf{v}})\| \leq \|\nu\nabla\widehat{\mathbf{v}}-\boldsymbol{\tau}\| + \mathbf{C}_{\boldsymbol{\Omega}}\|\mathrm{div}\boldsymbol{\tau}+\mathbf{f}-\nabla\mathbf{q}\| + \mathbf{2}\nu\kappa_{\boldsymbol{\Omega}}\|\mathrm{div}\widehat{\mathbf{v}}\|,$$

Three terms in the right–hand side of the estimate present three natural parts of the error, namely
errors in the constitutive (Newton) law
error in the differential equation
error arose due to non-solenoidality of approximations.

Another formulation of the Stokes problem

$$\inf_{\substack{\boldsymbol{\tau}\in\Sigma_{\rm div}(\Omega),\\ \mathsf{q}\in \overset{\circ}{L}_2(\Omega)(\Omega)\cap H^1(\Omega),\ \mathbf{v}\in\mathbf{V_0}+\mathbf{u_0}}} \mathcal{M}_{\oplus}(\mathbf{v},\boldsymbol{\tau},\mathbf{q},\beta,\mathbf{C_\Omega},\mathbf{f})$$

where

$$\mathcal{M}_{\oplus}(\mathbf{v},\boldsymbol{\tau},\mathbf{q},\beta,\mathbf{C_\Omega},\mathbf{f}) := \\ \|\nu\nabla(\widehat{\mathbf{v}})-\boldsymbol{\tau}\| + \mathbf{C_\Omega}\|{\rm div}\boldsymbol{\tau}+\mathbf{f}-\nabla\mathbf{q}\|+\mathbf{2}\kappa_{\mathbf{\Omega}}\nu\|{\rm div}\widehat{\mathbf{v}}\|$$

Estimates for pressure

Estimates of $\|\mathfrak{p}-\mathfrak{q}\|$ can be also derived with the help of Lemma. Since $(\mathfrak{p}-\mathfrak{q}) \in \overset{\circ}{L}_2(\Omega)$, we know that

$$\mathrm{div}\widetilde{\mathbf{w}} = \mathfrak{p}-\mathfrak{q}, \quad \text{and} \quad \|\nabla(\widetilde{\mathbf{w}})\| \leq \kappa_\Omega \|\mathfrak{p}-\mathfrak{q}\|,$$

for some $\widetilde{\mathbf{w}} \in V_0$. Hence,

$$\|\mathfrak{p}-\mathfrak{q}\|^2 \;\; = \int_\Omega \mathrm{div}\widetilde{\mathbf{w}}(\mathfrak{p}-\mathfrak{q})\,dx$$

Recall that u satisfies the equation and, therefore,

$$\int_\Omega (\mathfrak{p}-\mathfrak{q})\mathrm{div}\widetilde{\mathbf{w}}\,dx = \int\limits_\Omega (\nu\nabla\mathbf{u} : \nabla\widetilde{\mathbf{w}} - \mathbf{f}\cdot\widetilde{\mathbf{w}} - \mathfrak{q}\mathrm{div}\widetilde{\mathbf{w}})\,\mathbf{dx} =$$

$$= \int_\Omega \nu\nabla(\mathbf{u}-\widehat{\mathbf{v}}) : \nabla\widetilde{\mathbf{w}}\mathbf{dx} + \int_{\boldsymbol{\Omega}} (\nu\nabla\widehat{\mathbf{v}} : \nabla\widetilde{\mathbf{w}} + \mathbf{f}\cdot\widetilde{\mathbf{w}} - \mathfrak{q}\mathrm{div}\widetilde{\mathbf{w}})\,\mathbf{dx}.$$

We have

$$\int_\Omega \nu\nabla(\mathbf{u}-\widehat{\mathbf{v}}):\nabla\widetilde{\mathbf{w}}\,\mathbf{dx} \leq \kappa_{\boldsymbol{\Omega}}\nu\|\nabla(\mathbf{u}-\widehat{\mathbf{v}})\|\,\|\mathfrak{p}-\mathfrak{q}\|$$

and

$$\begin{aligned}
&\int_\Omega \left(\nu\nabla\widehat{\mathbf{v}}:\nabla\widetilde{\mathbf{w}}+f\cdot\widetilde{\mathbf{w}}-\mathfrak{q}\mathrm{div}\widetilde{\mathbf{w}}\right)dx = \\
&\qquad = \int_\Omega \left(\nu\nabla\widehat{\mathbf{v}}-\boldsymbol{\tau}-\mathfrak{q}\mathbb{I}\right):\nabla\widetilde{\mathbf{w}}dx - \int_\Omega \left(\mathrm{Div}\boldsymbol{\tau}+f\right)\cdot\widetilde{\mathbf{w}}dx \leq \\
&\qquad \leq \Big(\|\nu\nabla\widehat{\mathbf{v}}-\boldsymbol{\tau}-\mathfrak{q}\mathbb{I}\|+\nu C_\Omega\|\mathrm{Div}\boldsymbol{\tau}+f\|\Big)\kappa_\Omega\|\mathfrak{p}-\mathfrak{q}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathfrak{p}-\mathfrak{q}\| \leq\ & \kappa_\Omega\Big(\nu\|\nabla(\mathbf{u}-\widehat{\mathbf{v}})\|+\|\nu\nabla\widehat{\mathbf{v}}-\boldsymbol{\tau}-\mathfrak{q}\mathbb{I}\|+\mathbf{C}_{\boldsymbol{\Omega}}\|\mathrm{Div}\boldsymbol{\tau}+\mathbf{f}\|) \leq \\
& \leq 2\kappa_\Omega\Big(\|\nu\nabla(\widehat{\mathbf{v}})-\boldsymbol{\tau}-\mathfrak{q}\mathbb{I}\|+C_\Omega\|\mathrm{Div}\boldsymbol{\tau}+f\|+\nu\kappa_\Omega\|\mathrm{div}\widehat{\mathbf{v}}\|)
\end{aligned}$$

and we arrive at the estimate

$$\frac{1}{2\kappa_\Omega}\|\mathfrak{p}-\mathfrak{q}\| \leq \|\nu\nabla(\widehat{\mathbf{v}}) - \boldsymbol{\tau} - \mathfrak{q}\mathbb{I}\| + C_\Omega\|\mathrm{Div}\boldsymbol{\tau} + f\| + \nu\kappa_\Omega\|\mathrm{div}\widehat{\mathbf{v}}\|.$$

It is easy to observe that the right–hand side consists of the same terms as in the estimate for velocity and vanishes if and only if,

$$\widehat{\mathbf{v}} = \mathbf{u}, \quad \boldsymbol{\tau} = \boldsymbol{\sigma}, \quad \text{and} \quad \mathfrak{p} = \mathfrak{q}.$$

However, in this case, the dependence of the penalty multipliers on the constant κ_Ω is stronger.

Estimates for stresses

Let $\boldsymbol{\tau} \in \Sigma$ be an approximation of $\boldsymbol{\sigma}$. We have

$$\|\boldsymbol{\tau} - \boldsymbol{\sigma}\| = \|\boldsymbol{\tau} + \mathfrak{p}\mathbb{I} - \nu\nabla\mathbf{u}\| \leq$$
$$\leq \|\boldsymbol{\tau} + \mathfrak{q}\mathbb{I} - \nu\nabla\widehat{\mathbf{v}}\| + \nu\|\nabla(\widehat{\mathbf{v}} - \mathbf{u})\| + \sqrt{\mathbf{d}}\|\mathfrak{p} - \mathfrak{q}\|. \quad (21)$$

We conclude that

$$\|\boldsymbol{\tau} - \boldsymbol{\sigma}\| \leq 2(1 + \sqrt{d}\kappa_\Omega)\|\boldsymbol{\tau} + \mathfrak{q}\mathbb{I} - \nu\nabla(\widehat{\mathbf{v}})\| +$$
$$+ C_\Omega(1 + 2\sqrt{d}\kappa_\Omega)\|\mathrm{Div}\boldsymbol{\tau} + f\| + 2\nu\kappa_\Omega(1 + \sqrt{d}\kappa_\Omega)\|\mathrm{div}\widehat{\mathbf{v}}\|.$$

Now it is not difficult to estimate the deviation $\boldsymbol{\tau} - \boldsymbol{\sigma}$ in the norm of $H(\Omega, \mathrm{Div})$. However, the estimate has a more symmetric form if the deviation is expressed in terms of the norm

$$|[\,\eta\,|]_{\mathrm{Div}} := \|\eta\| + C_\Omega \|\mathrm{Div}\eta\|.$$

In this case,

$$\bar{c}\;|[\boldsymbol{\tau} - \boldsymbol{\sigma}|]_{\mathrm{Div}} \leq \|\boldsymbol{\tau} + \mathfrak{q}\mathbb{I} - \nu\nabla\widehat{\mathbf{v}}\| + C_\Omega \|\mathrm{Div}\boldsymbol{\tau} + f\| + \nu\|\mathrm{div}\widehat{\mathbf{v}}\|, \quad (22)$$

where $\bar{c} = \frac{1}{2(1+\sqrt{d}\kappa_\Omega)}$.

Problems for almost incompressible fluids

Models of almost incompressible fluids are often used for constructing sequences of functions converging to a solution of the Stokes problem. In this case, the incompressibility condition is replaced by the term that contains the divergence with a large multiplier. Namely, we must find $\mathbf{u}_\delta \in \mathbf{V}$ satisfying the integral identity

$$\int_{\mathbf{\Omega}} \big(\nu\nabla(\mathbf{u}_\delta) : \nabla\mathbf{w} + \frac{\mathbf{1}}{\delta}\mathrm{div}\mathbf{u}_\delta \, \mathrm{div}\mathbf{w}\big)\mathbf{dx} = \int_{\mathbf{\Omega}} \mathbf{f} \cdot \mathbf{w}\,\mathbf{dx}, \quad \mathbf{w} \in \mathbf{V_0},$$

and the boundary condition

$$\mathbf{u}_\delta = \mathbf{u_0} \ \partial\mathbf{\Omega}.$$

If $\delta \to 0$, then $\mathbf{u}_\delta \longrightarrow \mathbf{u}$ in the H^1
$\mathfrak{p}_\delta = -\frac{1}{\delta}\mathrm{div}\mathbf{u}_\delta \in \overset{\circ}{\mathbf{L}}_\mathbf{2}(\boldsymbol{\Omega})$ converges to $\mathfrak{p}$ in $\mathbf{L}_\mathbf{2}$.
Now $\mathbf{M}_\oplus$ immediately gives the estimate of the difference between $\mathbf{u}$ and $\mathbf{u}_\delta$. Let us set in

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\delta := \nu\nabla(\mathbf{u}_\delta)$$

and $q = p_\delta$. In this case,

$$\|\nu\nabla(\mathbf{u}_\delta) - \boldsymbol{\tau}_\delta\| = \mathbf{0}$$

and

$$[\![\, \mathrm{div}\boldsymbol{\tau}_\delta + f - \nabla\mathfrak{p}_\delta \,]\!] = \sup_{w \in V_0} \frac{\int_\Omega \left(-\nu\nabla(\mathbf{u}_\delta) : \nabla(\mathbf{w}) + \mathbf{f}\cdot\mathbf{w} + \mathfrak{p}_\delta\mathrm{div}\mathbf{w}\right) dx}{\|\nabla w\|}$$

$$= 0.$$

Modeling error

Thus, we conclude that

$$\frac{\mathbf{1}}{\mathbf{2}}\|\nabla(\mathbf{u}-\mathbf{u}_\delta)\| \leq \kappa_{\boldsymbol{\Omega}}\|\mathrm{div}\mathbf{u}_\delta\|, \tag{23}$$

Oseen equation

$$\begin{aligned} -\nu\boldsymbol{\Delta}\mathbf{u} + \mathrm{div}(\mathbf{a}\otimes\mathbf{u}) &= \mathbf{f} - \nabla\mathfrak{p} && \mathrm{in}\Omega, \\ \mathrm{div}\mathbf{u} &= \mathbf{0} && \mathrm{in}\Omega, \\ \mathbf{u} &= \mathbf{0} && \mathrm{on}\partial\Omega. \end{aligned}$$

Let $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^{\mathbf{1}}_{\mathbf{2}}(\boldsymbol{\Omega})$ and $\boldsymbol{\tau} \in \mathbf{H}(\boldsymbol{\Omega}, \mathrm{div})$ and $\mathfrak{q} \in \overset{\circ}{L}_2(\Omega) \cap H^1$.
Then (see S. R. (2002) [32]),

$$\nu\|\nabla(\mathbf{u}-\mathbf{v})\| \leq \|\boldsymbol{\tau} - \nu\nabla\mathbf{v}\| + \mathbf{C}_{\boldsymbol{\Omega}}\|\mathbf{f} - \nabla\mathbf{q} - \mathrm{div}(\mathbf{a}\otimes\mathbf{v}) + \mathrm{div}\boldsymbol{\tau}\|,$$

GENERALIZATIONS

$$\begin{tikzcd}
\mathbf{H} & \mathbf{V_0} \arrow[l, "\mathbf{B}"'] \arrow[r, "\mathbf{\Lambda}"] \arrow[d, Leftrightarrow] & \mathbf{U} \quad (\mathbf{Y}, \mathbf{Y}^*) \\
\mathbf{H} \arrow[r, "\mathbf{B}^*"] & \mathbf{V}_\mathbf{0}^* & \mathbf{U} \arrow[l, "\mathbf{\Lambda}^*"']
\end{tikzcd}$$

Find $\mathfrak{p} \in H$ and $\mathbf{u} \in \mathcal{V}_\mathbf{0}$ that satisfy the relation

$$(\mathcal{A}\mathbf{\Lambda}\mathbf{u}, \mathbf{\Lambda}\mathbf{w}) + \langle \mathbf{f} - \mathbf{B}^*\mathbf{p}, \mathbf{w} \rangle = \mathbf{0} \quad \forall \mathbf{w} \in \mathbf{V_0},$$

where

$$\mathcal{V}_\mathbf{0} = \mathbf{KerB} := \left\{ \mathbf{v} \in \mathbf{V_0} \ \middle| \ \mathbf{Bv} = \mathbf{0} \right\}.$$

Let the operator $\mathbf{B}$ possesses the following property: there exists a constant κ such that for any
$\mathbf{g} \in \mathbf{Im}\,\mathbf{B} := \{\mathbf{z} \in \mathbf{H} \mid \exists \mathbf{v} \in \mathbf{V_0} : \mathbf{Bv} = \mathbf{z}\}$
one can find $\mathbf{u_g} \in \mathbf{V_0}$ such that

$$\mathbf{Bu_g} = \mathbf{g} \quad \text{and} \quad \|\mathbf{u_g}\|_{\mathbf{V}} \leq \kappa \|\mathbf{g}\|.$$

Estimate of the deviation from u

$$\| \Lambda(\mathbf{u}-\widehat{\mathbf{v}}) \| \leq$$

$$\leq 2\sqrt{\nu_2}\kappa\|\mathbf{B}\widehat{\mathbf{v}}\| + \| \mathcal{A}\Lambda\widehat{\mathbf{v}} - \mathbf{y} \|_* + \frac{1}{\sqrt{\nu_1}} [\![\mathbf{f} + \Lambda^*\mathbf{y} - \mathbf{B}^*\mathbf{q}]\!] .$$

$\| \mathbf{y} \| := (\mathcal{A}\mathbf{y},\mathbf{y})^{1/2}$, $\| \mathbf{y} \|_* := (\mathcal{A}^{-1}\mathbf{y},\mathbf{y})^{1/2}$

$$\begin{cases} \langle \Lambda^*\boldsymbol{\sigma} + \mathbf{f} - \mathbf{B}^*\mathbf{p}, \mathbf{w}\rangle = \mathbf{0} \qquad \forall \mathbf{w} \in \mathbf{V_0}, \\ \boldsymbol{\sigma} = \mathcal{A}\Lambda\mathbf{u}, \\ \mathbf{B}\mathbf{v} = \mathbf{0}. \end{cases}$$

FINAL REMARKS

1. Any problem that has a functional error majorant has a **new variational formulation** convenient for the error control.

2. It looks most probable that all such majorants consist of terms that can be thought of **as penalties for the unconformity** in each of the basic relations with multipliers defined by **constants in the embedding inequalities** for spaces arising in the mathematical formulation of the problem.

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