

**A POSTERIORI ERROR CONTROL for
PDE's.
PART 2**

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Lecture goal

In the lecture, we present **Functional A Posteriori Estimate** for the several elliptic problem starting with

$$\Delta u + f = 0, \quad \Omega \quad u = 0 \quad \partial\Omega.$$

We discuss their meaning, principal features and practical implementation.

FUNCTIONAL A POSTERIORI ERROR ESTIMATES. FIRST GLANCE

Functional A Posteriori Estimates

Functional A Posteriori Estimate is a computable majorant of the difference between exact solution \mathbf{u} and any conforming approximation \mathbf{v} having the general form:

$$\Phi(\mathbf{u} - \mathbf{v}) \leq \mathbf{M}(\mathcal{D}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}! \quad (1)$$

\mathcal{D} is the data set (coefficients, domain, parameters, etc.),
 $\Phi : \mathbf{V} \rightarrow \mathbb{R}_+$ is a given functional.

\mathbf{M} must be computable and continuous in the sense that

$$\mathbf{M}(\mathcal{D}, \mathbf{v}) \rightarrow \mathbf{0}, \quad \text{if } \mathbf{v} \rightarrow \mathbf{u}$$

Types of Φ

■ Energy norm

$$\Phi(\mathbf{u} - \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_{\Omega}$$

■ Local norm

$$\Phi(\mathbf{u} - \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_{\omega}$$

■ Goal-oriented quantity

$$\Phi(\mathbf{u} - \mathbf{v}) = (\ell, \mathbf{u} - \mathbf{v})$$

Functional a posteriori estimate gives complete solution of the error control problem from the viewpoint of the MATHEMATICAL THEORY of PDE's

METHODS OF THE DERIVATION.

**These estimates are derived by purely functional methods:
using the the duality theory in the calculus of variations
or
analysis of integral identities.**

Variational method 96'-97':

Exploits variational structure of the original First publications:
[8, 9, 10, 11, 12, 13, 19] .

A systematic exposition of the variational approach to deriving
Functional a Posteriori Estimates can be found in

P. Neittaanmaki and S. Repin. Elsevier, NY, 2004

Nonvariational method 2000':

Derives a posteriori estimates by certain transformations of integral
identities. Basic idea of the method is presented in S. R. (2001)
[16]

Let us consider both methods in application to our basic problem

Variational Method

Let \mathbf{u} be a (generalized) solution of the problem

$$\Delta \mathbf{u} + \mathbf{f} = \mathbf{0}, \quad \Omega \quad \mathbf{u} = \mathbf{0} \quad \partial\Omega.$$

As we have seen in Lecture 1, this problem is equivalent to the following variational problem:

Problem \mathcal{P} . Find $\mathbf{u} \in \mathbf{V}_0 := \overset{\circ}{\mathbf{H}}^1(\Omega)$ such that

$$\mathbf{J}(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{V}_0} \mathbf{J}(\mathbf{v}),$$

where

$$\mathbf{J}(\mathbf{v}) = \frac{1}{2} \|\nabla \mathbf{v}\|^2 - (\mathbf{f}, \mathbf{v}).$$

By the reasons that we discussed earlier this problem has a unique solution.

Lagrangian

Note that

$$\mathbf{J}(\mathbf{v}) = \sup_{\mathbf{y} \in \mathbf{Y}} \mathbf{L}(\nabla \mathbf{v}, \mathbf{y}), \quad \mathbf{L}(\nabla \mathbf{v}, \mathbf{y}) = \int_{\Omega} \left(\nabla \mathbf{v} \cdot \mathbf{y} - \frac{1}{2} |\mathbf{y}|^2 - \mathbf{f} \mathbf{v} \right) \mathbf{d}\mathbf{x}$$

where $\mathbf{Y} = \mathbf{L}^2(\Omega, \mathbb{R}^n)$. Indeed, the value of the above supremum cannot exceed the one we obtain if for almost all $\mathbf{x} \in \Omega$ solve the pointwise problems

$$\sup_{\mathbf{y}(\mathbf{x})} (\nabla \mathbf{v})(\mathbf{x}) \cdot \mathbf{y}(\mathbf{x}) - \frac{1}{2} |\mathbf{y}(\mathbf{x})|^2 \quad \mathbf{x} \in \Omega$$

whose upper bound is attained if set $\mathbf{y}(\mathbf{x}) = (\nabla \mathbf{v})(\mathbf{x})$. Since $\nabla \mathbf{v} \in \mathbf{Y}$, we observe that the respective maximizer belongs to \mathbf{Y} and, therefore

$$\sup_{\mathbf{y} \in \mathbf{Y}} \mathbf{L}(\nabla \mathbf{v}, \mathbf{y}) = \mathbf{L}(\nabla \mathbf{v}, \nabla \mathbf{v}) = \mathbf{J}(\mathbf{v}).$$

Minimax Formulations

Then, the original problem comes in the **minimax** form:

$$(\mathcal{P}) \quad \inf_{\mathbf{v} \in \mathbf{V}_0} \sup_{\mathbf{y} \in \mathbf{Y}} \mathbf{L}(\nabla \mathbf{v}, \mathbf{y})$$

If the order of inf and sup is changed, then we arrive at the so-called **dual problem**

$$(\mathcal{P}^*) \quad \sup_{\mathbf{y} \in \mathbf{Y}} \inf_{\mathbf{v} \in \mathbf{V}_0} \mathbf{L}(\nabla \mathbf{v}, \mathbf{y})$$

Note that

$$\begin{aligned} \inf_{\mathbf{v} \in \mathbf{V}_0} \int_{\Omega} \left(\nabla \mathbf{v} \cdot \mathbf{y} - \frac{1}{2} |\mathbf{y}|^2 - \mathbf{f} \mathbf{v} \right) dx &= -\frac{1}{2} \|\mathbf{y}\|^2 + \inf_{\mathbf{v} \in \mathbf{V}_0} \int_{\Omega} (\nabla \mathbf{v} \cdot \mathbf{y} - \mathbf{f} \mathbf{v}) dx = \\ &= \begin{cases} -\frac{1}{2} \|\mathbf{y}\|^2 & \text{if } \mathbf{y} \in \mathbf{Q}_f := \{\mathbf{y} \in \mathbf{Y} \mid \mathbf{div} \mathbf{y} + \mathbf{f} = \mathbf{0}\} \\ -\infty & \text{if } \mathbf{y} \notin \mathbf{Q}_f \end{cases} \end{aligned}$$

Dual Problem

Thus, we observe that the dual problem has the form: find $\mathbf{p} \in \mathbf{Q}_f$ such that

$$-I^*(\mathbf{p}) = \sup_{\mathbf{y} \in \mathbf{Q}_f} -I^*(\mathbf{y})$$

where

$$I^*(\mathbf{q}) = \frac{1}{2} \|\mathbf{q}\|^2$$

How are these two problems related?

First, we establish one relation that holds regardless of the structure of the Lagrangian.

Sup Inf and Inf Sup

Lemma

Let $\mathbf{L}(\mathbf{x}, \mathbf{y})$ be a functional defined on the elements of two nonempty sets \mathbf{X} and \mathbf{Y} . Then

$$\sup_{\mathbf{y} \in \mathbf{Y}} \inf_{\mathbf{x} \in \mathbf{X}} \mathbf{L}(\mathbf{x}, \mathbf{y}) \leq \inf_{\mathbf{x} \in \mathbf{X}} \sup_{\mathbf{y} \in \mathbf{Y}} \mathbf{L}(\mathbf{x}, \mathbf{y}). \quad (2)$$

Proof

It is easy to see that

$$\mathbf{L}(\mathbf{x}, \mathbf{y}) \geq \inf_{\xi \in \mathbf{X}} \mathbf{L}(\xi, \mathbf{y}), \quad \forall \mathbf{x} \in \mathbf{X}, \mathbf{y} \in \mathbf{Y}.$$

Taking the supremum over $\mathbf{y} \in \mathbf{Y}$, we obtain

proof

$$\sup_{y \in Y} L(x, y) \geq \sup_{y \in Y} \inf_{\xi \in X} L(\xi, y), \quad \forall x \in X.$$

The left-hand side depends on x , while the right-hand side is a number. Thus, we may take infimum over $x \in X$ and obtain the inequality

$$\inf_{x \in X} \sup_{y \in Y} L(x, y) \geq \sup_{y \in Y} \inf_{\xi \in X} L(\xi, y).$$

Therefore, we always have

$$\sup \mathcal{P}^* \leq \inf \mathcal{P}$$

Duality relations

However, in our case we have a stronger relation, namely

$$\sup \mathcal{P}^* = \inf \mathcal{P}$$

To prove this fact, we note that

$$\int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{V}_0.$$

Therefore $\mathbf{p} = \nabla \mathbf{u} \in \mathbf{Q}_f$ and

$$-\mathbf{l}^*(\mathbf{p}) = -\frac{1}{2} \|\nabla \mathbf{u}\|^2 = \int_{\Omega} \left(\frac{1}{2} |\nabla \mathbf{u}|^2 - |\nabla \mathbf{u}|^2 \right) d\mathbf{x} = \int_{\Omega} \left(\frac{1}{2} |\nabla \mathbf{u}|^2 - \mathbf{f} \mathbf{u} \right) d\mathbf{x} = \mathbf{J}(\mathbf{u}).$$

Let us recall the estimate established in Part 1:

$$\frac{1}{2} \|\nabla(\mathbf{u} - \mathbf{v})\|^2 \leq \mathbf{J}(\mathbf{v}) - \mathbf{J}(\mathbf{u}).$$

Since $\mathbf{J}(\mathbf{u}) = -\mathbf{I}^*(\mathbf{p})$, we have

$$\frac{1}{2} \|\nabla(\mathbf{u} - \mathbf{v})\|^2 \leq \mathbf{J}(\mathbf{v}) + \mathbf{I}^*(\mathbf{p}) \leq \mathbf{J}(\mathbf{v}) + \mathbf{I}^*(\mathbf{q}) \quad \forall \mathbf{q} \in \mathbf{Q}_f.$$

Reform this estimate by using the fact that $\mathbf{q} \in \mathbf{Q}_f$.

$$\begin{aligned} \mathbf{J}(\mathbf{v}) + \mathbf{I}^*(\mathbf{q}) &= \frac{1}{2} \|\nabla \mathbf{v}\|^2 - (\mathbf{f}, \mathbf{v}) + \frac{1}{2} \|\mathbf{q}\|^2 \\ &= \frac{1}{2} \|\nabla \mathbf{v}\|^2 + \frac{1}{2} \|\mathbf{q}\|^2 - (\nabla \mathbf{v}, \mathbf{q}) = \\ &= \frac{1}{2} \|\nabla \mathbf{v} - \mathbf{q}\|^2. \end{aligned}$$

Now, we have

$$\|\nabla(\mathbf{v} - \mathbf{u})\| \leq \|\nabla\mathbf{v} - \mathbf{q}\| \quad \forall \mathbf{q} \in \mathbf{Q}_f.$$

Take arbitrary $\mathbf{y} \in \mathbf{L}^2(\Omega)$. Then,

$$\|\nabla(\mathbf{v} - \mathbf{u})\| \leq \|\nabla\mathbf{v} - \mathbf{y}\| + \inf_{\mathbf{q} \in \mathbf{Q}_f} \|\mathbf{y} - \mathbf{q}\|.$$

How to estimate the above infimum?

Various methods give one and the same answer:

$$\inf_{\mathbf{q} \in \mathbf{Q}_f} \|\mathbf{y} - \mathbf{q}\| \leq \mathbf{I} \mathbf{div} \mathbf{y} + \mathbf{f} \quad \mathbf{y} \in \mathbf{L}^2(\Omega), \quad (3)$$

$$\inf_{\mathbf{q} \in \mathbf{Q}_f} \|\mathbf{y} - \mathbf{q}\| \leq \mathbf{C}_\Omega \|\mathbf{div} \mathbf{y} + \mathbf{f}\| \quad \mathbf{y} \in \mathbf{H}(\Omega, \mathbf{div}), \quad (4)$$

Proof

To prove these estimates we consider an auxiliary problem

$$\Delta \eta + \mathbf{f} + \mathbf{div} \mathbf{y} = 0 \quad \Omega \quad \eta = 0 \quad \partial\Omega.$$

$$\int_{\Omega} \nabla \eta \cdot \nabla \mathbf{w} \, dx = \int_{\Omega} (\mathbf{f} \mathbf{w} - \mathbf{y} \cdot \nabla \mathbf{w}) \, dx$$

$$\int_{\Omega} \overbrace{(\nabla \eta + \mathbf{y})}^{\bar{\mathbf{q}}} \cdot \nabla \mathbf{w} \, dx = \int_{\Omega} \mathbf{f} \mathbf{w} \, dx \quad \forall \mathbf{w} \in \mathbf{V}_0$$

Thus, $\bar{\mathbf{q}} \in Q_f$!!!

Since η is a solution of the boundary-value problem with right-hand side $\operatorname{div} \mathbf{y} + \mathbf{f} \in \mathbf{H}^{-1}$, we have

$$\|\nabla \eta\| \leq \mathbf{I} \operatorname{div} \mathbf{y} + \mathbf{f} \mathbf{I},$$

Then

$$\inf_{\mathbf{q} \in \mathbf{Q}_f} \|\mathbf{y} - \mathbf{q}\| \leq \|\mathbf{y} - \bar{\mathbf{q}}\| = \|\nabla \eta\| \leq \mathbf{I} \operatorname{div} \mathbf{y} + \mathbf{f} \mathbf{I}.$$

Here

$$\mathbf{I} \operatorname{div} \mathbf{y} + \mathbf{f} \mathbf{I} = \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{\int_{\Omega} (\mathbf{y} \cdot \nabla \mathbf{w} - \mathbf{f} \mathbf{w}) \, dx}{\|\nabla \mathbf{w}\|}$$

$\mathbf{y} \in \mathbf{H}(\Omega, \text{div})$

If \mathbf{y} has a square summable divergence, then we have

$$\|\mathbf{div} \mathbf{y} + \mathbf{f}\| = \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{\int_{\Omega} (\mathbf{div} \mathbf{y} + \mathbf{f}) \mathbf{w} \, dx}{\|\nabla \mathbf{w}\|} \leq \mathbf{C}_{\Omega} \|\mathbf{div} \mathbf{y} + \mathbf{f}\|,$$

where \mathbf{C}_{Ω} is the constant in the Friedrichs–Steklov inequality for the domain Ω . We observe that

a "noncomputable" negative norm has been estimated by a "computable" one without an attraction of Galerkin orthogonality and local (mesh-dependent) constants.

Thus, for any $\mathbf{y} \in \mathbf{H}(\Omega, \mathbf{div})$ we obtain

$$\begin{aligned} \|\nabla(\mathbf{v} - \mathbf{u})\| &\leq \|\nabla\mathbf{v} - \mathbf{y}\| + \inf_{\mathbf{q} \in \mathbf{Q}_f} \|\mathbf{y} - \mathbf{q}\| \leq \\ &\|\nabla\mathbf{v} - \mathbf{y}\| + \mathbf{C}_\Omega \|\mathbf{div}\mathbf{y} + \mathbf{f}\|. \end{aligned}$$

Above presented *modus operandi* can be viewed as a simplest version of the variational approach to the derivation of Functional Error Majorants.

Nonvariational method in the simplest case

Let us expose its simplest version adapted to our model problem.
We have

$$\int_{\Omega} \nabla(\mathbf{u} - \mathbf{v}) \nabla \mathbf{w} \, dx = \int_{\Omega} (\mathbf{f} \mathbf{w} - \nabla \mathbf{v} \cdot \nabla \mathbf{w}) \, dx$$

In order to get an upper bound of $\|\nabla(\mathbf{u} - \mathbf{v})\|$ we use the relation

$$\int_{\Omega} (\operatorname{div} \mathbf{y} \mathbf{w} + \nabla \mathbf{w} \cdot \mathbf{y}) \, dx = 0 \quad \forall \mathbf{w} \in \mathbf{V}_0$$

valid for any $\mathbf{y} \in \mathbf{H}(\Omega, \operatorname{div})$.

We have

$$\begin{aligned} & \int_{\Omega} (\nabla \mathbf{v} \cdot \nabla \mathbf{w} - \mathbf{f} \mathbf{w}) \, \mathbf{d}\mathbf{x} = \\ & \int_{\Omega} (\nabla \mathbf{v} \cdot \nabla \mathbf{w} - \mathbf{f} \mathbf{w} - (\mathbf{div} \mathbf{w} + \nabla \mathbf{w} \cdot \mathbf{y})) \, \mathbf{d}\mathbf{x} = \\ & \int_{\Omega} ((\nabla \mathbf{v} - \mathbf{y}) \cdot \nabla \mathbf{w} - (\mathbf{f} + \mathbf{div}) \mathbf{w}) \, \mathbf{d}\mathbf{x} \leq \\ & \|\nabla \mathbf{v} - \mathbf{y}\| \|\nabla \mathbf{w}\| + \|\mathbf{f} + \mathbf{div}\| \|\mathbf{w}\| \leq \\ & \leq (\|\nabla \mathbf{v} - \mathbf{y}\| + \mathbf{C}_{\Omega} \|\mathbf{f} + \mathbf{div}\|) \|\nabla \mathbf{w}\|. \end{aligned}$$

Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

$$\int_{\Omega} |\nabla(\mathbf{u} - \mathbf{v})|^2 \mathbf{d}\mathbf{x} \leq (\|\nabla\mathbf{v} - \mathbf{y}\| + \mathbf{C}_{\Omega}\|\mathbf{f} + \mathbf{div}\mathbf{y}\|)\|\nabla(\mathbf{u} - \mathbf{v})\|.$$

Thus, we find that

$$\|\nabla(\mathbf{u} - \mathbf{v})\| \leq \|\nabla\mathbf{v} - \mathbf{y}\| + \mathbf{C}_{\Omega}\|\mathbf{f} + \mathbf{div}\mathbf{y}\|.$$

Functional error estimate. Meaning and properties

For the problem

$$\Delta \mathbf{u} + \mathbf{f} = \mathbf{0}, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega$$

we have obtained the estimate

$$\|\nabla(\mathbf{u} - \mathbf{v})\| \leq \|\nabla \mathbf{v} - \mathbf{y}\| + \mathbf{C}_\Omega \|\operatorname{div} \mathbf{v} + \mathbf{f}\| \quad (5)$$

The estimate is valid for any $\mathbf{v} \in \mathbf{V}_0$ and $\mathbf{y} \in \mathbf{H}(\Omega, \operatorname{div})$

Two terms in the right-hand side have a clear sense: they **present measures of the errors in two basic relations**

$$\mathbf{p} = \nabla \mathbf{u}, \quad \operatorname{div} \mathbf{p} + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega$$

that jointly form the equation.

The estimate is sharp

If set $\mathbf{v} = \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$, we obtain the energy estimate for the generalized solution

$$\|\nabla \mathbf{u}\| \leq \mathbf{C}_\Omega \|\mathbf{f}\|.$$

Therefore, no constant less than \mathbf{C}_Ω can be stated in the second term.

If set $\mathbf{y} = \nabla \mathbf{u}$, then the inequality holds as the equality.

Thus, we see that the estimate (5) is **sharp** in the sense that the multipliers of both terms **cannot be taken smaller** and in the set of admissible \mathbf{y} there **exists a function that makes the inequality hold as equality**.

The estimate as a quadratic functional

By means of the algebraic Young's inequality

$$2\mathbf{ab} \leq \beta\mathbf{a}^2 + \frac{1}{\beta}\mathbf{b}^2, \quad \beta > 0$$

we rewrite this estimate in the form

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{v})\|^2 &\leq \\ &\leq (1 + \beta)\|\nabla\mathbf{v} - \mathbf{y}\|^2 + \frac{1 + \beta}{\beta}\mathbf{C}_\Omega^2\|\mathbf{div}\mathbf{y} + \mathbf{f}\|^2 \end{aligned} \tag{6}$$

For any β the right-hand side is a quadratic functional. This property makes it possible to apply well known methods for the minimization with respect to \mathbf{y} .

Deviation Majorant

Denote the right-hand side of (6) by \mathcal{M}_{\oplus} , i.e.,

$$\mathcal{M}_{\oplus}(\mathbf{v}, \mathbf{y}, \beta, \mathbf{C}_{\Omega}, \mathbf{f}) := (\mathbf{1} + \beta) \|\nabla \mathbf{v} - \mathbf{y}\|^2 + \frac{\mathbf{1} + \beta}{\beta} \mathbf{C}_{\Omega}^2 \|\mathbf{div} \mathbf{v} + \mathbf{f}\|^2.$$

This functional provides an upper bound for the norm of the deviation of \mathbf{v} from \mathbf{u} . Therefore, it is natural to call it the

Deviation Majorant.

BVP $\Delta u + f = 0$ has another variational formulation

$$\begin{array}{l} \inf \\ \mathbf{v} \in \mathbf{V}_0, \\ \beta > 0, \\ \mathbf{y} \in \mathbf{H}(\Omega, \text{div}), \end{array} \quad \mathcal{M}_{\oplus}(\mathbf{v}, \mathbf{y}, \beta, \mathbf{C}_{\Omega}, \mathbf{f})$$

- Minimum of this functional is **zero**;
- it is attained if and only if $\mathbf{v} = \mathbf{u}$ and $\mathbf{y} = \mathbf{A}\nabla\mathbf{u}$!;
- \mathcal{M}_{\oplus} contains only one global constant \mathbf{C}_{Ω} , which is problem independent;

HOW TO USE FUNCTIONAL A POSTERIORI ERROR ESTIMATES?

In principle, one can select certain sequences of subspaces $\{\mathbf{V}_{hk}\} \in \mathbf{V}_0$ and $\{\mathbf{Y}_{hk}\} \in \mathbf{H}(\Omega, \text{div})$ and minimize the Error Majorant with respect to these subspaces

$$\begin{aligned} & \inf_{\mathbf{v} \in \mathbf{V}_{hk},} \mathcal{M}_{\oplus}(\mathbf{v}, \mathbf{y}, \beta, \mathbf{C}_{\Omega}, \mathbf{f}) \\ & \beta > 0, \\ & \mathbf{y} \in \mathbf{Y}_{hk}, \end{aligned}$$

If the subspaces are limit dense, then we would obtain a sequence of approximate solutions $(\mathbf{v}_k, \mathbf{y}_k)$ and the sequence of numbers

$$\gamma_k := \inf_{\beta > 0} \mathcal{M}_{\oplus}(\mathbf{v}_k, \mathbf{y}_k, \beta, \mathbf{C}_{\Omega}, \mathbf{f}) \rightarrow 0$$

Consider **CONFORMING FEM APPROXIMATIONS**.

We have 3 basic ways to use the deviation estimate:

- (a) **Direct** (via flux averaging on the mesh \mathcal{T}_h);
- (b) **One step delay** (via flux averaging on the mesh \mathbf{h}_{ref});
- (c) **Minimization** (minimization via \mathbf{y}).

(a) Use recovered gradients

Let $\mathbf{u}_h \in \mathbf{V}_h$, then

$$\mathbf{p}_h := \nabla \mathbf{u}_h \in \mathbf{L}_2(\Omega, \mathbb{R}^d), \quad \mathbf{p}_h \notin \mathbf{H}(\Omega, \text{div}).$$

Use an averaging operator $\mathbf{G}_h : \mathbf{L}_2(\Omega, \mathbb{R}^d) \rightarrow \mathbf{H}(\Omega, \text{div})$ and have a **directly computable estimate**

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\| \leq \|\nabla \mathbf{u}_h - \mathbf{G}_h \mathbf{p}_h\| + \mathbf{C}_\Omega \|\text{div} \mathbf{G}_h \mathbf{p}_h + \mathbf{f}\|$$

(b) Use recovered gradients from $\mathcal{T}_{h_{ref}}$

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \dots$ be a sequence of approximations on meshes \mathcal{T}_{h_k} . Compute $\mathbf{p}_k := \nabla \mathbf{u}_k$, average it by \mathbf{G}_k and for \mathbf{u}_{k-1} use the estimate

$$\|\mathbf{u} - \mathbf{u}_{k-1}\| \leq \|\nabla \mathbf{u}_{k-1} - \mathbf{G}_k \mathbf{p}_k\| + \mathbf{C}_\Omega \|\operatorname{div} \mathbf{G}_k \mathbf{p}_k + \mathbf{f}\|$$

This estimate gives

a quantitative form of the Runge's rule.

(c) Minimize \mathcal{M}_{\oplus} with respect to y .

Select a certain subspace \mathbf{Y}_{τ} in $\mathbf{H}(\Omega, \mathbf{div})$. **Generally, \mathbf{Y}_{τ} may be constructed on another mesh \mathcal{T}_{τ} and with help of different trial functions.** Then

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\| \leq \inf_{\mathbf{y}_h \in \mathbf{Y}_h} \{ \|\nabla \mathbf{u}_h - \mathbf{y}_h\| + \mathbf{C}_{\Omega} \|\mathbf{div} \mathbf{y}_h + \mathbf{f}\| \}$$

The wider $\mathbf{Y}_h \subset \mathbf{H}(\Omega, \mathbf{div})$ the sharper is the upper bound.

Quadratic type functional

From the technical point of view it is better to square both parts of the estimate and apply minimization to a quadratic functional, namely

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}_h)\|^2 \leq & \inf_{\mathbf{y}_h \in \mathbf{Y}_h} \left\{ (\mathbf{1} + \beta) \|\nabla \mathbf{u}_h - \mathbf{y}_h\| + \right. \\ & \left. + \mathbf{C}_\Omega \left(\mathbf{1} + \frac{\mathbf{1}}{\beta} \right) \|\operatorname{div} \mathbf{y}_h + \mathbf{f}\|^2 \right\} \end{aligned}$$

Here, the positive parameter β can be also used to minimize the right-hand side.

Simple 1-D problem

$$\begin{aligned}(\alpha(x) u')' &= f(x), \\ u(a) &= 0, \quad u(b) = u_b.\end{aligned}$$

It is equivalent to the variational problem

$$J(\mathbf{v}) = \int_a^b \left(\frac{1}{2} \alpha(x) |\mathbf{v}'|^2 + f(x)\mathbf{v} \right) dx.$$

Assume that the coefficient α belongs to $\in \mathbf{L}^\infty$ and bounded from below by a positive constant. Now

$$\mathbf{V}_0 + \mathbf{u}_0 = \{ \mathbf{v} \in \mathbf{H}^1(a, b) \mid \mathbf{v}(a) = 0, \mathbf{v}(b) = \mathbf{u}_b \}.$$

Deviation Majorant

$$\mathcal{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}) = (1+\beta) \left(\int_a^b |\alpha \mathbf{v}' - \mathbf{y}|^2 \, d\mathbf{x} + \frac{\mathbf{C}_{(a,b)}^2}{\beta} \int_a^b |\mathbf{y}' - \mathbf{f}|^2 \, d\mathbf{x} \right) \quad (7)$$

In this simple model, \mathbf{u} can be presented in the form

$$\mathbf{u}(\mathbf{x}) = \int_a^{\mathbf{x}} \frac{\mathbf{1}}{\alpha(\mathbf{t})} \int_a^{\mathbf{t}} \mathbf{f}(\mathbf{z}) \, d\mathbf{z} \, d\mathbf{t} + \frac{\mathbf{x}}{\mathbf{b}} \left(\mathbf{u}_b - \int_a^b \frac{\mathbf{1}}{\alpha(\mathbf{t})} \int_a^{\mathbf{t}} \mathbf{f}(\mathbf{z}) \, d\mathbf{z} \, d\mathbf{t} \right).$$

what gives an opportunity to verify how error estimation methods work.

Approximations

Let \mathbf{V}_h be made of piecewise- \mathbf{P}^1 continuous functions on uniform splittings of the interval and consider approximations of the following types:

- Galerkin approximations;
- Approximations very close to Galerkin (sharp);
- Approximations which are "good" but not Galerkin;
- Coarse (rough) approximations.

Our aim is to show that the Deviation Majorant can be effectively used as an error estimation instrument in all the above cases.

Computation of the Majorant

To find a sharp upper bound, we minimize \mathcal{M}_{\oplus} with respect to \mathbf{y} and β starting from the function $\mathbf{y}_0 = \mathbf{G}(\mathbf{v}')$, where \mathbf{G} is a simple averaging operator, e.g, defined by the relations

$$\mathbf{G}(\mathbf{v}')(x_i) = \frac{1}{2}(\mathbf{v}'(x_i - \mathbf{0}) + \mathbf{v}'(x_i + \mathbf{0})),$$

By the quantity

$$\inf_{\beta > 0} \mathcal{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}_0),$$

we obtain a coarse upper bound of the error. It is further improved by minimizing \mathcal{M}_{\oplus} with respect to \mathbf{y} .

Example

Let $\alpha(x) = 1$, $f(x) = c$, $a = 0$, $b = 1$, $u_b = 1$, e.g., we consider the problem

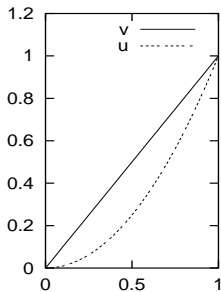
$$u'' = 2, \quad u(0) = 0, \quad u(1) = 1.$$

In this case, $C_{(a,b)} = 1/\pi$ and

$$u = \frac{c}{2}x^2 + \left(1 - \frac{c}{2}\right)x, \quad u' = cx + 1 - \frac{c}{2}.$$

Take a **rough** approximation $v = x$. Then

$$\|(u - v)'\|^2 = \int_0^1 c^2(x - 0.5)^2 dx = c^2/12 \approx 0.083c^2.$$



Exact solution and an approximation.

Various y give different upper bounds

(a) Take $\mathbf{y} = \mathbf{v}' = \mathbf{1}$, then the first term in

$$\mathcal{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}) = (\mathbf{1} + \beta) \left(\int_0^1 |\mathbf{v}' - \mathbf{y}|^2 \, dx + \frac{1}{\pi^2 \beta} \int_0^1 |\mathbf{y}' - \mathbf{c}|^2 \right) dx.$$

vanishes and we have $\mathcal{M}_{\oplus} \rightarrow \mathbf{c}^2/\pi^2 \approx \mathbf{0.101c}^2$; as $\beta \rightarrow +\infty$.

We see that this upper bound overestimates true error. Note that in this case, all sensible averagings of $\mathbf{v}' = \mathbf{1}$ give exactly the same function: $\mathbf{G}(\mathbf{1}) = \mathbf{1}$! Therefore,

$$\mathbf{G}(\mathbf{v}') - \mathbf{v}' \equiv \mathbf{0}$$

and formally averaging based indicator "**does not see the error**".

For the choice $\mathbf{y} = \mathbf{v}'$ the Majorant give a certain upper bound of the error (which is not so bad), but the integrand cannot indicate the distribution of local errors. Indeed, we have

$$\mathcal{M}_{\oplus} = \frac{1}{\pi^2} \int_0^1 \mathbf{c}^2 \mathbf{d}\mathbf{x}.$$

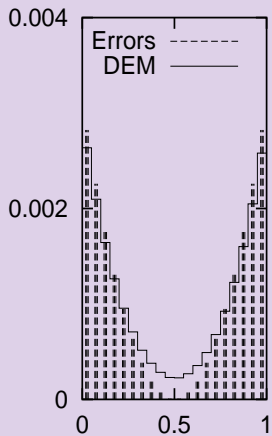
Hence, the integrand of the Majorant is a constant function, but the error is distributed in accordance with a parabolic law:

$$(\mathbf{u} - \mathbf{v})' = \mathbf{c}(\mathbf{x} - 0.5)^2.$$

(b). Take $\mathbf{y} = \mathbf{c}x + \mathbf{1} - \mathbf{c}/2$. Then, $\mathbf{y}' = \mathbf{c}$ and the second term of the majorant vanishes. We have (for $\beta = 0$)

$$\mathcal{M}_{\oplus} = \int_0^1 \mathbf{c}^2 (\mathbf{x} - 1/2)^2 d\mathbf{x} = \mathbf{c}^2/12.$$

We observe that both the global error and the error distribution are exactly reproduced. In real life computations such an "ideal" function \mathbf{y} may be unattainable. However, using direct minimization procedures we make the Majorant close to the exact error. In this elementary example, we have minimized the Majorant on a uniform mesh with 20 intervals using piecewise affine approximations of \mathbf{y} . The elementwise error distribution obtained as the result of this procedure is exposed on the next picture.



True errors and those computed by the Majorant.

To give further illustrations, we consider the functions

$$\mathbf{u}_\delta = \mathbf{u} + \delta\varphi,$$

where δ is a number and φ is a certain function (perturbation).

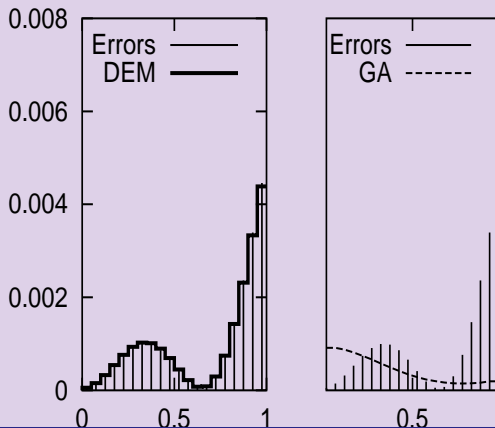
Approximate solutions (whose errors are measured) are piecewise affine continuous interpolants of \mathbf{u}_δ defined on a uniform mesh with 20 subintervals. We take $\varphi = \mathbf{x} \sin(\pi\mathbf{x})$ and $\delta = 0.1, 0.01, 0.001$, and 0.

Table:

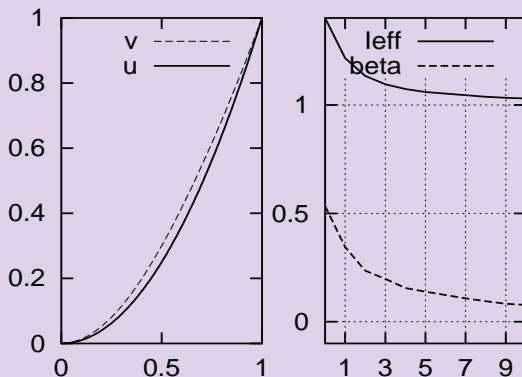
δ	e	$2\mathcal{M}_\oplus$	$2\mathcal{M}_\ominus$	i_{eff}	i_{esh}
0.1	0.019692	0.019743	0.019683	1.003	1.018
0.01	0.001022	0.001025	0.001013	1.003	1.011
0.001	0.000835	0.000839	0.000827	1.005	1.002
0	0.000833	0.000836	0.000825	1.004	1.002

In this experiment the Majorant was computed for $\frac{1}{2}\|\mathbf{e}\|^2$.

Error estimation for $\delta = 0.1$

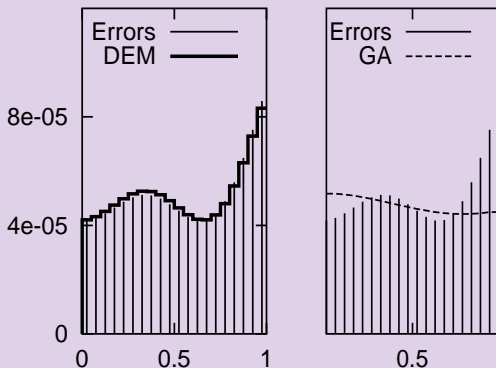


Functions \mathbf{u} , \mathbf{v} and \mathbf{i}_{eff} for $\delta = 0.1$

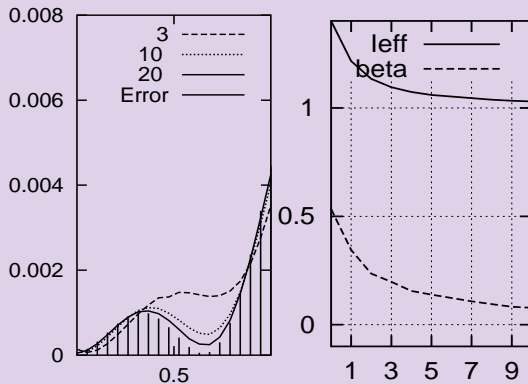


Error estimation for $\delta = 0.01$

A more precise approximation.

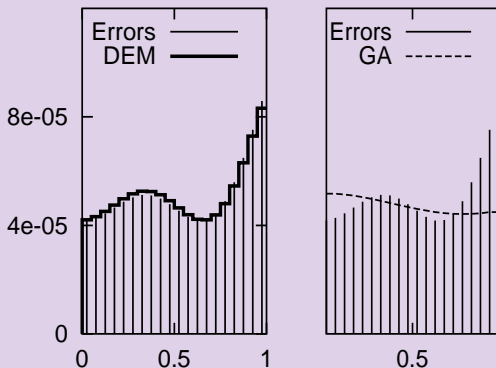


Functions $e(\mathbf{y})$, β and i_{eff} for $\delta = 0.1$

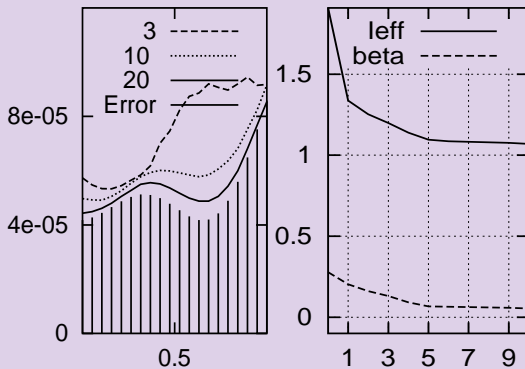


Error estimation for $\delta = 0.01$

A more precise approximation.

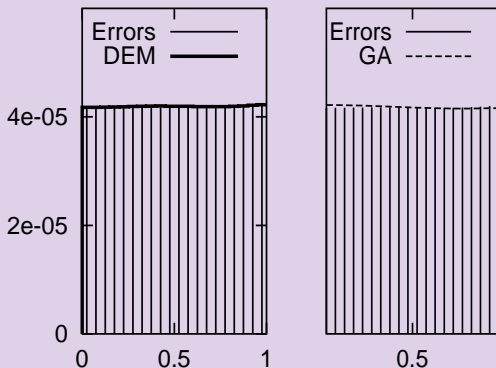


Functions $e(\mathbf{y})$, β and i_{eff} for $\delta = 0.01$

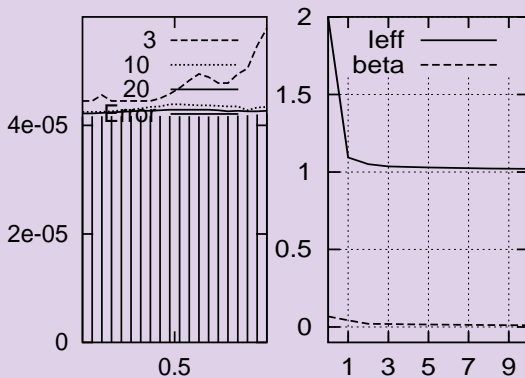


Error estimation for $\delta = 0.001$

Sharp approximation.

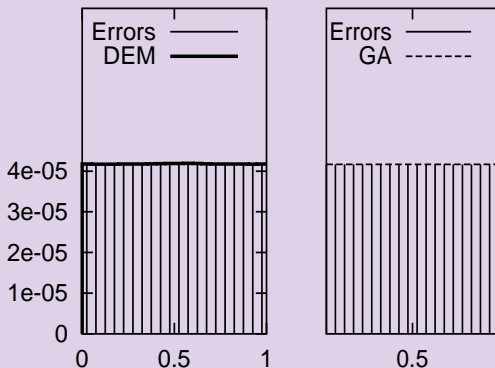


Functions $e(\mathbf{y})$, β and i_{eff} for $\delta = 0.001$

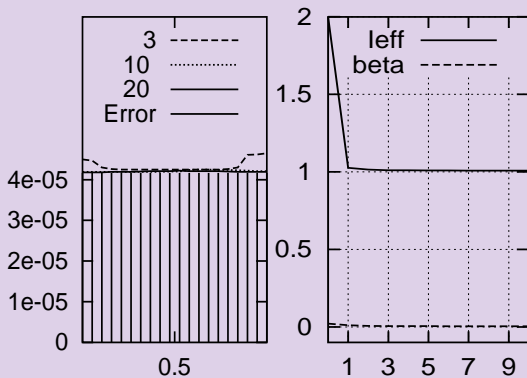


Error estimation for $\delta = 0$

Interpolant of the exact solution.



Functions $e(\mathbf{y})$, β and i_{eff} for $\delta = 0$



A POSTERIORI ERROR ESTIMATES FOR A DIFFUSION TYPE PROBLEM.

Elliptic problem with a lower term

For elliptic problems with lower terms it is easy to obtain estimates without C_Ω .

$$\begin{aligned}\Delta \mathbf{u} - \varrho \mathbf{u} + \mathbf{f} &= \mathbf{0}, & \varrho > 0, \\ \mathbf{u} &= \mathbf{u}_0 \quad \text{on } \partial\Omega.\end{aligned}$$

Such estimates can be derived by both *variational* and *non-variational* method. Let $\mathbf{w} \in \mathbf{V}_0 := \mathring{\mathbf{H}}^1(\Omega)$. We have

$$\begin{aligned}& \int_{\Omega} \nabla(\mathbf{u} - \mathbf{v}) \cdot \nabla \mathbf{w} \, dx + \varrho \int_{\Omega} (\mathbf{u} - \mathbf{v}) \mathbf{w} \, dx = \\ &= \int_{\Omega} (\mathbf{f} \mathbf{w} - \nabla \mathbf{v} \cdot \nabla \mathbf{w}) \, dx - \varrho \int_{\Omega} \mathbf{v} \mathbf{w} \, dx.\end{aligned}$$

Use the integral identity for $\mathbf{y} \in \mathbf{H}(\Omega, \mathbf{div})$:

$$\int_{\Omega} (\nabla \mathbf{w} \cdot \mathbf{y} + \mathbf{w} \mathbf{div} \mathbf{y}) \, d\mathbf{x} = 0 \quad \forall \mathbf{w} \in \mathbf{V}_0.$$

$$\begin{aligned} \int_{\Omega} \nabla(\mathbf{u} - \mathbf{v}) \cdot \nabla \mathbf{w} \, d\mathbf{x} + \varrho \int_{\Omega} (\mathbf{u} - \mathbf{v}) \mathbf{w} \, d\mathbf{x} &= \\ \int_{\Omega} (\mathbf{f} + \mathbf{div} \mathbf{y} - \varrho \mathbf{v}) \mathbf{w} \, d\mathbf{x} + \int_{\Omega} (\mathbf{y} - \nabla \mathbf{v}) \cdot \nabla \mathbf{w} \, d\mathbf{x} &\leq \\ \leq \|\mathbf{f} + \mathbf{div} \mathbf{y} - \varrho \mathbf{v}\| \|\mathbf{w}\| + \|\nabla \mathbf{v} - \mathbf{y}\| \|\nabla \mathbf{w}\|. \end{aligned}$$

Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and note that

$$\begin{aligned} & \| \mathbf{f} + \mathbf{div} \boldsymbol{\rho} \mathbf{v} \| \| \mathbf{u} - \mathbf{v} \| + \| \nabla \mathbf{v} - \mathbf{y} \| \| \nabla (\mathbf{u} - \mathbf{v}) \| = \\ & = \frac{1}{\varrho} \| \mathbf{f} + \mathbf{div} \boldsymbol{\rho} \mathbf{v} \| \varrho \| \mathbf{u} - \mathbf{v} \| + \| \nabla \mathbf{v} - \mathbf{y} \| \| \nabla (\mathbf{u} - \mathbf{v}) \| \leq \\ & \leq \left(\frac{1}{\varrho^2} \| \mathbf{f} + \mathbf{div} \boldsymbol{\rho} \mathbf{v} \|^2 + \| \nabla \mathbf{v} - \mathbf{y} \|^2 \right)^{1/2} \| \mathbf{u} - \mathbf{v} \| \end{aligned}$$

where

$$\| \mathbf{u} - \mathbf{v} \|^2 = \int_{\Omega} (|\nabla(\mathbf{u} - \mathbf{v})|^2 + \varrho |\mathbf{u} - \mathbf{v}|^2) \mathbf{d}x.$$

Then, we obtain the estimate

$$\| \mathbf{u} - \mathbf{v} \|^2 \leq \frac{1}{\varrho^2} \| \mathbf{f} + \mathbf{div} \mathbf{v} - \varrho \mathbf{v} \|^2 + \| \nabla \mathbf{v} - \mathbf{y} \|^2$$

By the variational method this estimate was derived in [8]. Also, it readily follows from the general variational method (see [13]) of a posteriori error estimation that we discuss later. This estimate has no \mathbf{C}_Ω . However, in practice, it may give *big overestimation* if ϱ is small due to large penalty at the first term.

**A POSTERIORI ERROR ESTIMATES THE A MAXWELL'S
PROBLEM.**

A simple version of the Maxwell's system

Quite analogously functional a posteriori estimate can be derived for the elliptic problem

$$\begin{aligned}\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{u} + \mathbf{k}^2 \mathbf{u} &= \mathbf{f} \\ \mathbf{u} \times \boldsymbol{\nu} &= \mathbf{0} \quad \text{on } \Gamma\end{aligned}$$

This equation with $\mathbf{k} > \mathbf{0}$ arises if the evolutionary Maxwell problem is solved by a semidiscrete scheme.

If $\mathbf{k} = \mathbf{0}$ than we have the simplest *magnetostatic* problem.

Here ν denotes the unit outward normal to $\partial\Omega$. By $V(\Omega)$ we denote the Hilbert space $\mathbf{H}(\Omega, \mathbf{curl})$ defined as the closure of smooth functions with respect to the norm

$$\|\mathbf{w}\|_{\mathbf{curl}} := \left(\|\mathbf{w}\|_{\Omega}^2 + \|\mathbf{curl} \mathbf{w}\|_{\Omega}^2 \right)^{1/2}.$$

Let

$$\mathbf{V}_0(\Omega) := \{ \mathbf{w} \in \mathbf{H}(\Omega, \mathbf{curl}) \mid \mathbf{w} \times \nu = \mathbf{0} \text{ on } \partial\Omega \}$$

and

$$\ell(\mathbf{w}) := \int_{\Omega} \mathbf{j} \cdot \mathbf{w} \, dx.$$

Approximation methods for the Maxwell's equation were investigated by many authors

see, e.g., [G. Haase, M. Kuhn, U. Langer](#) [5],

[R. Hiptmair](#) [6],

[P. Monk](#) [7].

A posteriori estimates (**residual approach**) can be found in

[R. Beck, R. Hiptmaier, R. Hoppe, and B. Wohlmuth](#) [1]

[S. Cochez–Dhondt and S. Nicaise](#) [3]

and **equilibrated residual approach** in [D. Braess and J. Schoberl](#) [2].

By the **functional approach** a posteriori estimates were derived in [S. R.](#) [17]. Below the arguments of the paper are shortly reproduced.

Case I.

Assume that $\mathbf{k} > 0$. Integral identity is as follows:

$$\int_{\Omega} \left(\mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{w} + \mathbf{k}^2 \mathbf{u} \cdot \mathbf{w} \right) \mathbf{d}\mathbf{x} = \ell(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_0. \quad (8)$$

Let $\mathbf{v} \in \mathbf{V}_0(\Omega)$. Then from (8) it follows that

$$\begin{aligned} & \int_{\Omega} \left(\mu^{-1} \mathbf{curl} (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{curl} \mathbf{w}) + \mathbf{k}^2 (\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} \right) \mathbf{d}\mathbf{x} = \\ & = \int_{\Omega} \left(\mathbf{j} \cdot \mathbf{w} - (\mathbf{curl} \mathbf{v}) \cdot (\mathbf{curl} \mathbf{w}) - \mathbf{k}^2 \mathbf{v} \cdot \mathbf{w} \right) \mathbf{d}\mathbf{x} \quad \forall \mathbf{w} \in \mathbf{V}_0(\Omega), \end{aligned} \quad (9)$$

Let $\mathbf{y} \in \mathbf{V}(\Omega)$. By (9) we obtain

$$\begin{aligned} & \int_{\Omega} \left(\mu^{-1} \mathbf{curl}(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{curl} \mathbf{w}) + \mathbf{k}^2 (\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} \right) \mathbf{d}\mathbf{x} = \\ & = \int_{\Omega} \left((\mathbf{j} - \mathbf{k}^2 \mathbf{v} - \mathbf{curl} \mathbf{y}) \cdot \mathbf{w} + (\mathbf{y} - \mu^{-1} \mathbf{curl} \mathbf{v}) \cdot \mathbf{curl} \mathbf{w} \right) \mathbf{d}\mathbf{x} \quad (10) \end{aligned}$$

Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$. We have

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 & := \int_{\Omega} \left(\mu^{-1} |\mathbf{curl}(\mathbf{u} - \mathbf{v})|^2 + \mathbf{k}^2 |\mathbf{u} - \mathbf{v}|^2 \right) \mathbf{d}\mathbf{x} = \\ & = \int_{\Omega} \left((\mathbf{j} - \mathbf{k}^2 \mathbf{v} - \mathbf{curl} \mathbf{y}) \cdot (\mathbf{u} - \mathbf{v}) + \right. \\ & \quad \left. + (\mathbf{y} - \mu^{-1} \mathbf{curl} \mathbf{v}) \cdot \mathbf{curl}(\mathbf{u} - \mathbf{v}) \right) \mathbf{d}\mathbf{x}. \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &\leq \left\| \frac{1}{\mathbf{k}} (\mathbf{j} - \mathbf{k}^2 \mathbf{v} - \mathbf{curl} \mathbf{y}) \right\| \|\mathbf{k}(\mathbf{u} - \mathbf{v})\| + \\ &\quad + \|\mu^{1/2}(\mathbf{y} - \mu^{-1} \mathbf{curl} \mathbf{v})\| \|\mu^{-1/2} \mathbf{curl}(\mathbf{u} - \mathbf{v})\|. \end{aligned}$$

From the above relation, it follows that

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &\leq \\ &\leq \mathbf{M}_{\oplus}^2(\mathbf{v}, \mathbf{y}) := \left\| \frac{1}{\mathbf{k}} (\mathbf{j} - \mathbf{k}^2 \mathbf{v} - \mathbf{curl} \mathbf{y}) \right\|^2 + \|\mu^{1/2}(\mathbf{y} - \mu^{-1} \mathbf{curl} \mathbf{v})\|^2. \end{aligned} \tag{11}$$

It is easy to see that

$$\inf_{\substack{\mathbf{v} \in \mathbf{V}_0, \\ \mathbf{y} \in \mathbf{H}(\Omega, \mathbf{curl})}} \mathbf{M}_{\oplus}(\mathbf{v}, \mathbf{y}) = \mathbf{0}$$

and the exact lower bound is attained if and only if

$$\mathbf{curl} \mathbf{y} + \mathbf{k}^2 \mathbf{v} = \mathbf{j} \quad \text{a.e. in } \Omega \quad (12)$$

$$\mathbf{y} = \mu^{-1} \mathbf{curl} \mathbf{v} \quad \text{a.e. in } \Omega. \quad (13)$$

By assumption, $\mathbf{v} \times \nu = \mathbf{0}$ on $\partial\Omega$. Therefore, (12) and (13) mean that \mathbf{v} coincides with the exact solution \mathbf{u} and \mathbf{y} with $\mu^{-1}\mathbf{curl}\mathbf{u}$. For any $\mathbf{y} \in \mathbf{V}_0$ the quantity $\mathbf{M}_{\oplus}^2(\mathbf{v}, \mathbf{y})$ gives an upper bound of the error. It is easy to observe that

$$\begin{aligned} \inf_{\mathbf{y} \in \mathbf{V}_0} \mathbf{M}_{\oplus}^2(\mathbf{v}, \mathbf{y}) &\leq \mathbf{M}_{\oplus}^2(\mathbf{v}, \mu^{-1}\mathbf{curl}\mathbf{u}) = \\ &= \left\| \frac{1}{\mathbf{k}} (\mathbf{j} - \mathbf{k}^2\mathbf{v} - \mathbf{curl}\mu^{-1}\mathbf{curl}\mathbf{u}) \right\|^2 + \left\| \mu^{-1/2}\mathbf{curl}(\mathbf{u} - \mathbf{v}) \right\|^2 = \\ &= \left\| \mathbf{k}(\mathbf{u} - \mathbf{v}) \right\|^2 + \left\| \mu^{-1/2}\mathbf{curl}(\mathbf{u} - \mathbf{v}) \right\|^2 = \left\| \mathbf{u} - \mathbf{v} \right\|^2. \end{aligned}$$

Therefore, the upper bound is **sharp**.

Lower bound

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &\geq \mathbf{M}_{\ominus}^2(\mathbf{v}, \mathbf{w}) := \\ &= \int_{\Omega} \left(2\mathbf{j} \cdot \mathbf{w} - \mu^{-1} |\operatorname{curl} \mathbf{w}|^2 - \mathbf{k}^2 |\mathbf{w}|^2 - 2\mu^{-1} \operatorname{curl} \mathbf{v} \cdot \operatorname{curl} \mathbf{w} - 2\mathbf{k}^2 \mathbf{v} \cdot \mathbf{w} \right) dx. \end{aligned} \tag{14}$$

For any $\mathbf{w} \in \mathbf{V}_0$ the quantity $\mathbf{M}_{\ominus}^2(\mathbf{v}, \mathbf{w})$ gives a lower bound of the error. Certainly, the sharpest bound is given by

$$\sup_{\mathbf{w} \in \mathbf{V}_0} \mathbf{M}_{\ominus}^2(\mathbf{v}, \mathbf{w}).$$

It is not difficult to prove that this quantity coincides with the squared error (to prove that it suffices to set $\mathbf{w} = \mathbf{u} - \mathbf{v}$).

Practically computable **two-sided bounds** of the error can be determined if $\mathbf{M}_{\oplus}^2(\mathbf{v}, \mathbf{y})$ is minimized over a finite dimensional subspace $\mathbf{V}_m \subset \mathbf{V}(\Omega)$ and $\mathbf{M}_{\ominus}^2(\mathbf{v}, \mathbf{w})$ is maximized over a finite dimensional subspace $\mathbf{V}_{0m} \subset \mathbf{V}_0$, $\dim \mathbf{V}_{0m} = m$. Then, finding the quantities

$$\sup_{\mathbf{w} \in \mathbf{V}_{0m}} \mathbf{M}_{\ominus}^2(\mathbf{v}, \mathbf{w}) \quad \text{and} \quad \inf_{\mathbf{y} \in \mathbf{V}_m} \mathbf{M}_{\oplus}^2(\mathbf{v}, \mathbf{y})$$

requires solving quadratic type problems.

Case II. If $\mathbf{k} = 0$ then an additional compatibility condition

$$\operatorname{div} \mathbf{j} = 0 \quad \text{in } \Omega, \quad \mathbf{j} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega \quad (15)$$

is necessary to have a well-posed problem. In this case, the respective generalized solution is defined by the integral identity

$$\int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{w} \, dx = \ell(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_0. \quad (16)$$

Since $\operatorname{curl} \nabla \psi = 0$, the solution should be understood as an element of the factor space where the functions are equivalent if their difference is a gradient field.

To ensure the uniqueness of a solution the Coulomb gauge condition

$$\mathbf{u} \in \mathbf{V}_{00} := \left\{ \mathbf{v} \in \mathbf{V}_0 \mid \int_{\Omega} \mathbf{v} \cdot \nabla \phi \, d\mathbf{x} = 0 \quad \forall \phi \in \overset{\circ}{\mathbf{W}}_2^1(\Omega) \right\} \quad (17)$$

is usually attracted.

First, we recall one result in the theory of functions in $H(\Omega, \mathbf{curl})$ (see, e.g., V. Girault and M. Raviart [4], P. Monk [7]).

Lemma

Let Ω be a Lipschitz simply connected bounded domain and $\mathbf{div} \mathbf{w} = \mathbf{0}$ in Ω . There exist a constant \mathbf{C}_{Ω} such that

$$\|\mathbf{w}\|_{\Omega} \leq \mathbf{C}_{\Omega} \|\mathbf{curl} \mathbf{w}\|_{\Omega} \quad \forall \mathbf{w} \in \mathbf{V}_0. \quad (18)$$

Lemma shows that the seminorm $\|[\]\| := \|\mathbf{curl}\|$ is a norm on the subspace of solenoidal functions in \mathbf{V}_0 .

Functional a posteriori estimate. Case II.

We have the following upper bound of the error:

$$\|[\mathbf{u} - \mathbf{v}]\| \leq \mathbf{C}_\Omega \|\mathbf{j} - \mathbf{curl} \mathbf{y}\|_\Omega + \|\mathbf{curl} \mathbf{v} - \mathbf{y}\|_\Omega. \quad (19)$$

Note that in (19) \mathbf{v} appears only as the argument of the operator **curl**. Therefore, it should be understood as an estimate of the factor–norm for the functions in the factor space equivalent up to a gradient field.

FUNCTIONAL A POSTERIORI ESTIMATES FOR LINEAR ELLIPTIC PROBLEMS IN GENERAL FORM.

”Nonvariational” method in the general form

Below we discuss the general approach to deriving **two-sided functional estimates of the deviations** from exact solutions of linear elliptic type problems having the operator form

$$\Lambda^* \mathcal{A} \Lambda u + \ell = 0$$

where Λ and \mathcal{A} are linear bounded operators and \mathcal{A} is symmetric and positive definite.

Abstract elliptic problem

Many problems can be presented in the following form: **find** $\mathbf{u} \in \mathbf{V}_0 + \mathbf{u}_0$ **such that**

$$(\mathcal{A}\Lambda\mathbf{u}, \Lambda\mathbf{w}) + \langle \ell, \mathbf{w} \rangle = 0 \quad \forall \mathbf{w} \in \mathbf{V}_0. \quad (20)$$

Here \mathbf{V}_0 is a subspace of a reflexive Banach space \mathbf{V} ,

e.g., $\mathbf{V} = \mathbf{H}^1$, $\mathbf{V}_0 = \overset{\circ}{\mathbf{H}}^1$.

$\Lambda : \mathbf{V} \rightarrow \mathbf{U}$ is a bounded linear operator, e.g. $\Lambda = \nabla$.

\mathbf{U} is a Hilbert space with scalar product (\cdot, \cdot) and norm $\|\cdot\|$,

e.g., $\mathbf{U} = \mathbf{L}^2$.

$\ell \in \mathbf{V}_0^*$, is a linear functional in the dual space, e.g., in \mathbf{H}^{-1} . In general, we may assume that

$$\langle \ell, \mathbf{w} \rangle = (\mathbf{f}, \mathbf{w}) + (\mathbf{g}, \Lambda\mathbf{w}).$$

$\mathcal{A} \in \mathcal{L}(\mathbf{U}, \mathbf{U})$ is a self-adjoint operator.

Assumptions

We assume that

$$\mathbf{V} \text{ is compactly embedded in } \mathbf{U} \quad (21)$$

and the operators \mathbf{A} and \mathcal{A} satisfy the relations

$$\mathbf{c}_1 \|\mathbf{y}\|^2 \leq (\mathcal{A}\mathbf{y}, \mathbf{y}) \leq \mathbf{c}_2 \|\mathbf{y}\|^2, \quad \forall \mathbf{y} \in \mathbf{U}, \quad (22)$$

$$\|\mathbf{A}\mathbf{w}\| \geq \mathbf{c}_3 \|\mathbf{w}\|_{\mathbf{V}}, \quad \forall \mathbf{w} \in \mathbf{V}_0, \quad (23)$$

For our analysis, it is convenient to introduce two more norms:

$$\| \mathbf{y} \| := (\mathcal{A}\mathbf{y}, \mathbf{y})^{1/2}, \quad \| \mathbf{y} \|_* := (\mathcal{A}^{-1}\mathbf{y}, \mathbf{y})^{1/2},$$

where $\mathcal{A}^{-1} : \mathbf{U} \rightarrow \mathbf{U}$

is the operator inverse to \mathcal{A} . The respective spaces \mathbf{Y} and \mathbf{Y}^* contain elements of \mathbf{U} equipped with the norms $\| \cdot \|$ and $\| \cdot \|_*$, respectively.

Let $\mathbf{\Lambda}^*$ be the operator conjugate to $\mathbf{\Lambda}$, i.e.,

$$(\mathbf{y}, \mathbf{\Lambda}\mathbf{w}) = \langle \mathbf{\Lambda}^*\mathbf{y}, \mathbf{w} \rangle, \quad \forall \mathbf{w} \in \mathbf{V}_0. \quad (24)$$

A general method of the derivation of a posteriori estimates by transformation of integral identities was suggested in [S. R. \(2001\)](#) [16]. We follow the lines of this paper.

Let $\mathbf{v} \in \mathbf{V}_0 + \mathbf{u}_0$ be an approximation.

$$(\mathcal{A}\Lambda(\mathbf{u} - \mathbf{v}), \Lambda\mathbf{w}) + \langle \ell, \mathbf{w} \rangle + (\mathcal{A}\Lambda\mathbf{v}, \Lambda\mathbf{w}) = 0 \quad \forall \mathbf{w} \in \mathbf{V}_0,$$

Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

$$\| \Lambda(\mathbf{u} - \mathbf{v}) \|^2 = | \langle \ell, \mathbf{u} - \mathbf{v} \rangle + (\mathcal{A}\Lambda\mathbf{v}, \Lambda(\mathbf{u} - \mathbf{v})) | .$$

By $(\mathbf{y}, \Lambda\mathbf{w}) = \langle \Lambda^*\mathbf{y}, \mathbf{w} \rangle$ we obtain

$$\| \Lambda(\mathbf{u} - \mathbf{v}) \|^2 = | \langle \ell, \mathbf{u} - \mathbf{v} \rangle + (\mathcal{A}\Lambda\mathbf{v} - \mathbf{y}, \Lambda(\mathbf{u} - \mathbf{v})) + \langle \Lambda^*\mathbf{y}, \mathbf{u} - \mathbf{v} \rangle | .$$

Therefore, we find that

$$\begin{aligned} \|\Lambda(\mathbf{u}-\mathbf{v})\|^2 &\leq |\langle \boldsymbol{\ell} + \Lambda^* \mathbf{y}, \mathbf{u}-\mathbf{v} \rangle| + |(\mathcal{A}\Lambda\mathbf{v} - \mathbf{y}, \Lambda(\mathbf{u}-\mathbf{v}))| \leq \\ &\leq \|\boldsymbol{\ell} + \Lambda^* \mathbf{y}\| \|\Lambda(\mathbf{u}-\mathbf{v})\| + \|\mathcal{A}\Lambda\mathbf{v} - \mathbf{y}\|_* \|\Lambda(\mathbf{u}-\mathbf{v})\|, \end{aligned}$$

where

$$\|\mu\| := \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{\langle \mu, \mathbf{w} \rangle}{\|\Lambda\mathbf{w}\|}$$

denotes the norm of the functional $\mu : \mathbf{V}_0 \rightarrow \mathbb{R}$.

To prove that the value of $\|\boldsymbol{\ell} + \Lambda^* \mathbf{y}\|$ is finite we note that

$$\begin{aligned} \langle \boldsymbol{\ell} + \Lambda^* \mathbf{y}, \mathbf{w} \rangle &\leq \|\boldsymbol{\ell}\|_{\mathbf{V}_0^*} \|\mathbf{w}\|_{\mathbf{V}} + \|\mathbf{y}\| \|\Lambda\mathbf{w}\| \leq (\mathbf{c}_3^{-1} \|\boldsymbol{\ell}\|_{\mathbf{V}_0^*} + \|\mathbf{y}\|) \|\Lambda\mathbf{w}\| \leq \\ &\leq \mathbf{c}_1^{-1/2} (\mathbf{c}_3^{-1} \|\boldsymbol{\ell}\|_{\mathbf{V}_0^*} + \|\mathbf{y}\|) \|\Lambda\mathbf{w}\|. \end{aligned}$$

General estimate

As a result we obtain the general form of a functional a posteriori estimate for an elliptic type problem:

$$\| \Lambda(\mathbf{u} - \mathbf{v}) \| \leq [\ell + \Lambda^* \mathbf{y}] + \| \mathcal{A} \Lambda \mathbf{v} - \mathbf{y} \|_* . \quad (25)$$

Denote

$$\begin{aligned} \mathbf{D}(\mathbf{y}_1, \mathbf{y}_2) &:= \frac{1}{2}(\mathcal{A}\mathbf{y}_1, \mathbf{y}_1) + \frac{1}{2}(\mathcal{A}^{-1}\mathbf{y}_2, \mathbf{y}_2) - (\mathbf{y}_1, \mathbf{y}_2) = \\ &= \frac{1}{2}(\mathcal{A}(\mathbf{y}_1 - \mathcal{A}^{-1}\mathbf{y}_2), \mathbf{y}_1 - \mathcal{A}^{-1}\mathbf{y}_2) = \frac{1}{2} \|\mathbf{y}_1 - \mathcal{A}^{-1}\mathbf{y}_2\|_*^2 = \\ &= \frac{1}{2}(\mathcal{A}^{-1}(\mathbf{y}_2 - \mathcal{A}\mathbf{y}_1), \mathbf{y}_2 - \mathcal{A}\mathbf{y}_1) = \frac{1}{2} \|\mathbf{y}_2 - \mathcal{A}\mathbf{y}_1\|_*^2 \end{aligned}$$

Then

$$\|\mathcal{A}\Lambda\mathbf{v} - \mathbf{y}\|_*^2 = 2\mathbf{D}(\Lambda\mathbf{v}, \mathbf{y})$$

and we obtain

$$\| \Lambda(\mathbf{u} - \mathbf{v}) \| \leq \mathbf{I} \ell + \Lambda^* \mathbf{y} \mathbf{I} + \sqrt{2\mathbf{D}(\Lambda \mathbf{v}, \mathbf{y})}.$$

Square both sides and use Young's inequality

$$\| \Lambda(\mathbf{u} - \mathbf{v}) \|^2 \leq \left(1 + \frac{1}{\beta}\right) \mathbf{I} \ell + \Lambda^* \mathbf{y} \mathbf{I}^2 + 2(1 + \beta) \mathbf{D}(\Lambda \mathbf{v}, \mathbf{y}) \quad (26)$$

This estimate holds for any $\mathbf{y} \in \mathbf{Y}^*$ and $\beta > 0$. Denote its right-hand side by $2\mathcal{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y})$

$2\mathcal{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y})$ is a sharp upper bound

Proposition

For any $\beta > 0$ there exists $\mathbf{y} \in \mathbf{Y}^*$ such that

$$2\mathcal{M}_{\oplus}(\beta, \mathbf{v}, \mathbf{y}) = \|\Lambda(\mathbf{u} - \mathbf{v})\|^2 .$$

Proof. Set $\mathbf{y}_1 = \frac{1}{1+\beta}(\mathbf{p} + \beta\mathcal{A}\Lambda\mathbf{v})$ where $\mathbf{p} = \mathcal{A}\Lambda\mathbf{u}$. Note that $\langle \ell + \Lambda^*\mathbf{y}_1, \mathbf{w} \rangle = (-\mathbf{p} + \mathbf{y}_1, \Lambda\mathbf{w})$.

$$\begin{aligned} \mathbf{I} \ell + \Lambda^*\mathbf{y}_1 \mathbf{I} &= \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{(\mathbf{y}_1 - \mathbf{p}, \Lambda\mathbf{w})}{\|\Lambda\mathbf{w}\|} = \\ &= \frac{\beta}{1+\beta} \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{(\mathcal{A}\Lambda(\mathbf{v} - \mathbf{u}), \Lambda\mathbf{w})}{\|\Lambda\mathbf{w}\|} = \frac{\beta}{1+\beta} \|\Lambda(\mathbf{v} - \mathbf{u})\| , \end{aligned}$$

Similarly

$$\mathbf{D}(\Lambda \mathbf{v}, \mathbf{y}_1) = \frac{1}{2(1+\beta)^2} \|\Lambda(\mathbf{v} - \mathbf{u})\|^2 .$$

Therefore,

$$\begin{aligned} \left(1 + \frac{1}{\beta}\right) \|\ell + \Lambda^* \mathbf{y}\|^2 + 2(1 + \beta) \mathbf{D}(\Lambda \mathbf{v}, \mathbf{y}) &= \\ \left(1 + \frac{1}{\beta}\right) \left(\frac{\beta}{1+\beta} \|\Lambda(\mathbf{v} - \mathbf{u})\|\right)^2 + 2(1 + \beta) \left(\frac{1}{2(1+\beta)^2} \|\Lambda(\mathbf{v} - \mathbf{u})\|^2\right) &= \\ = \frac{\beta}{1+\beta} \|\Lambda(\mathbf{v} - \mathbf{u})\|^2 + \frac{1}{1+\beta} \|\Lambda(\mathbf{v} - \mathbf{u})\|^2 = \|\Lambda(\mathbf{v} - \mathbf{u})\|^2 . \end{aligned}$$

We replace $\|\ell + \Lambda^* \mathbf{y}\|$ by the norm in a Hilbert space \mathbf{U} provided that ℓ belongs to a narrower set. Assume that

$$\begin{aligned} \ell &\in \mathbf{U} \subset \mathbf{V}_0^*, \\ \mathbf{y} \in \mathbf{Q}^* &:= \{\mathbf{z}^* \in \mathbf{Y}^* \mid \Lambda^* \mathbf{z}^* \in \mathbf{U}\}. \end{aligned}$$

Note that \mathbf{Q}^* can be endowed with the norm

$$\|\mathbf{y}\|_{\mathbf{Q}^*}^2 := \|\mathbf{y}\|_*^2 + \|\Lambda^* \mathbf{y}\|_{\mathbf{U}}^2.$$

If $\ell \in \mathbf{U}$, then \mathbf{Q}^* contains the exact solution \mathbf{p} of the dual problem! This fact is important for the proof of the sharpness of the Majorant.

Then

$$\langle \ell + \Lambda^* \mathbf{y}, \mathbf{w} \rangle = (\ell + \Lambda^* \mathbf{y}, \mathbf{w}) \quad \mathbf{w} \in \mathbf{V}_0.$$

$$\begin{aligned} \|\ell + \Lambda^* \mathbf{y}\| &= \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{\langle \ell + \Lambda^* \mathbf{y}, \mathbf{w} \rangle}{\|\Lambda \mathbf{w}\|} \leq \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{\|\ell + \Lambda^* \mathbf{y}\| \|\mathbf{w}\|}{\|\Lambda \mathbf{w}\|} \leq \\ &\leq \|\ell + \Lambda^* \mathbf{y}\| \mathbf{c}_1^{-1} \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{\|\mathbf{w}\|}{\|\Lambda \mathbf{w}\|} \leq \mathbf{c}_1^{-1} \mathbf{c}_3^{-1} \|\ell + \Lambda^* \mathbf{y}\|. \end{aligned}$$

Denote $\mathbf{c}^2 = \mathbf{c}_1^{-2} \mathbf{c}_3^{-2}$.

General form of the computable Majorant

Now, the Majorant \mathcal{M}_{\oplus} is replaced by \mathbf{M}_{\oplus} , namely we arrive at the estimate

$$\begin{aligned} \frac{1}{2} \|\Lambda(\mathbf{v} - \mathbf{u})\|^2 &\leq \mathbf{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}) := \\ &:= (\mathbf{1} + \beta) \mathbf{D}(\Lambda \mathbf{v}, \mathbf{y}) + \frac{\mathbf{1} + \beta}{2\beta} \mathbf{c}^2 \|\ell + \Lambda^* \mathbf{y}\|^2. \quad (27) \end{aligned}$$

Relationships with other methods

$$\mathbf{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y})$$

involves an arbitrary function \mathbf{y} . We are aimed to show that some special choices of it lead to known error estimates.

We assume that $\langle \ell, \mathbf{w} \rangle = (\mathbf{g}, \mathbf{w})$, where $\mathbf{g} \in \mathbf{U}$, so that $\mathbf{p} \in \mathbf{Q}^* \subset \mathbf{Q}_{\ell}^*$ and

$$\mathbf{Q}_{\ell}^* := \{\mathbf{y} \in \mathbf{Q}^* \mid (\Lambda^* \mathbf{y} + \mathbf{g}, \mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathbf{V}_0\}.$$

Let us first define the function

$$\mathbf{y}_0 = \mathcal{A}\mathbf{v}. \quad (28)$$

A variety of options comes from the relation

$$\mathbf{y} = \mathbf{\Pi}\mathbf{y}_0, \quad (29)$$

where $\mathbf{\Pi}$ is a certain continuous mapping.

Residual based estimate

If Π is the **identity mapping**, i.e., $\mathbf{y} = \mathbf{y}_0 := \mathcal{A}\Lambda\mathbf{v}$, then

$$\mathbf{D}(\Lambda\mathbf{v}, \mathbf{y}_0^*) = \mathbf{0}.$$

Take the majorant with such a \mathbf{y} :

$$\frac{1}{2} \|\Lambda(\mathbf{v} - \mathbf{u})\|^2 \leq (1 + \beta)\mathbf{D}(\Lambda\mathbf{v}, \mathbf{y}) + \frac{1 + \beta}{2\beta} \|\ell + \Lambda^*\mathbf{y}\|^2.$$

Now, it contains only the second term, which after the minimization with respect to β gives

$$\|\Lambda(\mathbf{v} - \mathbf{u})\| \leq \|\ell + \Lambda^*\mathcal{A}\Lambda\mathbf{v}\| = \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{(\ell, \mathbf{w}) + (\mathcal{A}\Lambda\mathbf{v}, \Lambda\mathbf{w})}{\|\Lambda\mathbf{w}\|}. \quad (30)$$

If \mathbf{v} is obtained by FEM and $\mathbf{v} = \mathbf{u}_h \in \mathbf{V}_h := \mathbf{V}_{0h} + \mathbf{u}_0$, then we arrive at the following estimate:

$$\| \Lambda(\mathbf{u}_h - \mathbf{u}) \| \leq \| \ell + \Lambda^* \mathcal{A} \Lambda \mathbf{u}_h \| = \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{(\ell, \mathbf{w} - \mathbf{w}_h) + (\mathcal{A} \Lambda \mathbf{u}_h, \Lambda(\mathbf{w} - \mathbf{w}_h))}{\| \Lambda \mathbf{w} \|}.$$

We find an upper bound of the right-hand side by the arguments accepted in the classical residual method.

Conclusion

If in the functional a posteriori error estimate is applied to a FEM solution \mathbf{u}_h then we may select the variable \mathbf{y} in the simplest way as

$$\mathbf{y} = \Lambda \mathbf{u}_h.$$

Then, if \mathbf{u}_h is a Galerkin approximation, we can use this fact and obtain an upper bound given by the *residual type a posteriori error estimate* that involves integral terms associated with finite elements and interelement jumps.

Estimates using post-processing of the dual variable

In $\mathbf{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y})$ the best choice is $\mathbf{y} = \mathbf{p} \in \mathbf{Q}^*$. Therefore, if $\mathbf{y}_0 \notin \mathbf{Q}^*$ then its mapping \mathbf{Q}^* could be a better approximation of \mathbf{p} . Let us denote such a mapping by Π_1 . We obtain

$$\mathbf{y}_1 = \Pi_1 \mathbf{y}_0 \in \mathbf{Q}^* \quad (31)$$

and the quantity $\mathbf{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}_1)$, which leads to the error majorant

$$\mathbf{M}_{\oplus}^{(1)}(\mathbf{v}) = \inf_{\beta \in \mathbb{R}_+} \left\{ (1 + \beta) \mathbf{D}(\Lambda \mathbf{v}, \Pi_1(\mathcal{A} \Lambda \mathbf{v})) + \frac{1 + \beta}{2\beta} \mathbf{c}^2 \|\ell + \Lambda^* \Pi_1(\mathcal{A} \Lambda \mathbf{v})\|^2 \right\}. \quad (32)$$

Particular case

In the simplest case associated with the problem

$$\Delta \mathbf{u} + \mathbf{f} = \mathbf{0}, \quad \mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega$$

we have

$$\begin{aligned} \mathbf{M}_{\oplus}^{(1)}(\mathbf{u}_h) &= \\ &= \inf_{\beta \in \mathbb{R}_+} \left\{ (\mathbf{1} + \beta) \|\nabla \mathbf{u}_h - \Pi_1(\nabla \mathbf{u}_h)\|^2 + \frac{(\mathbf{1} + \beta) \mathbf{C}_{\Omega}^2}{2\beta} \|\mathbf{f} + \mathbf{div} \Pi_1(\nabla \mathbf{u}_h)\|^2 \right\}. \end{aligned}$$

If Π_1 is a gradient averaging operator, then the first term in the right-hand side is **the difference between the original and averaged gradient**, i.e. it coincides with a **gradient averaging indicator**. However, as we have seen in previous lectures, such an indicator cannot provide a reliable upper bound of the error. The second term in the right-hand side shows what is necessary to add **in order to provide the reliability**.

Estimates based on the "equilibration" of the dual variable

Let Π_2 maps \mathbf{Y}^* to the set \mathbf{Q}_ℓ^* . Define

$$\mathbf{y}_2 = \Pi_2 \mathbf{y}_0 \in \mathbf{Q}_\ell^*. \quad (33)$$

Then,

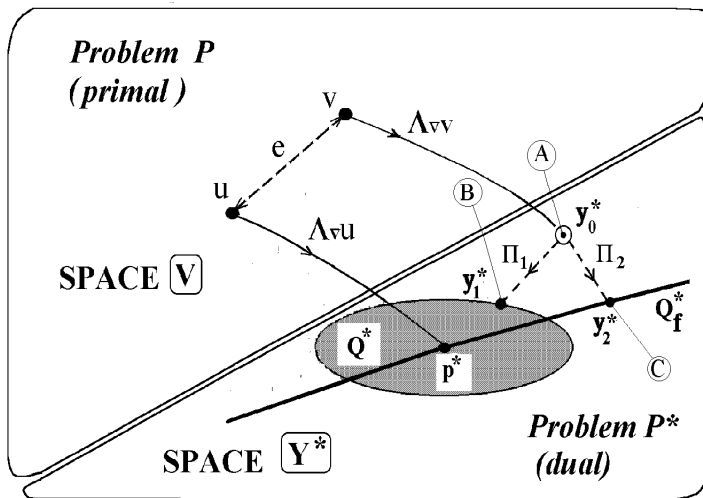
$$\Lambda^* \mathbf{y}_2 + \ell = \mathbf{0},$$

so that the Majorant has only the first term:

$$\mathbf{M}_{\oplus}^{(2)}(\mathbf{v}) = \mathbf{D}(\Lambda \mathbf{v}, \mathbf{y}_2).$$

Π_2 is natural to call an **equilibration operator**. In general, it is rather difficult to construct an "exact mapping" Π_2 to \mathbf{Q}_ℓ^* . One may use an operator $\tilde{\Pi}_2$, which provides an approximate "equilibration". In this case, the **second term of the Majorant does not vanish and should be taken into account**.

Various choices of the dual variable lead to certain a posteriori methods



A priori projection type error estimates

As an exercise, we now will derive classical a priori projection type error estimates from a functional a posteriori estimate. Let $\mathbf{u}_h \in \mathbf{V}_h$ be a Galerkin approximation of \mathbf{u} . We have

$$\|\Lambda(\mathbf{u} - \mathbf{u}_h)\|^2 \leq 2(1 + \beta) \mathbf{D}(\Lambda \mathbf{u}_h, \mathbf{y}) + \left(1 + \frac{1}{\beta}\right) \mathbf{I} \Lambda^* \mathbf{y} + \ell \mathbf{I}^2$$

Set here $\mathbf{y} = \mathcal{A} \Lambda \mathbf{v}_h$, where \mathbf{v}_h is an arbitrary element of \mathbf{V}_h . Then,

$$\begin{aligned} \mathbf{I} \Lambda^* \mathbf{y} + \ell \mathbf{I} &= \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{(\mathbf{y} - \mathbf{p}, \Lambda \mathbf{w})}{\|\Lambda \mathbf{w}\|} = \\ &= \sup_{\mathbf{w} \in \mathbf{V}_0} \frac{(\mathcal{A} \Lambda(\mathbf{v}_h - \mathbf{u}), \Lambda \mathbf{w})}{\|\Lambda \mathbf{w}\|} \leq \|\Lambda(\mathbf{v}_h - \mathbf{u})\|. \end{aligned}$$

It is easy to see that

$$\mathbf{D}(\Lambda \mathbf{u}_h, \mathcal{A} \Lambda \mathbf{v}_h) = \mathbf{J}(\mathbf{v}_h) - \mathbf{J}(\mathbf{u}_h).$$

Indeed,

$$\begin{aligned} \mathbf{D}(\Lambda \mathbf{u}_h, \mathcal{A} \Lambda \mathbf{v}_h) &= \frac{1}{2}(\mathcal{A} \Lambda \mathbf{v}_h, \Lambda \mathbf{v}_h) + \langle \ell, \mathbf{v}_h \rangle - \\ &\quad - \frac{1}{2}(\mathcal{A} \Lambda \mathbf{u}_h, \Lambda \mathbf{u}_h) - \langle \ell, \mathbf{u}_h \rangle + \\ &\quad + (\mathcal{A} \Lambda \mathbf{u}_h, \Lambda(\mathbf{u}_h - \mathbf{v}_h)) + \langle \ell, \mathbf{u}_h - \mathbf{v}_h \rangle. \end{aligned}$$

Since $\mathbf{u}_h \in \mathbf{V}_h$ is a Galerkin approximation, the last two terms vanish and we obtain the relation.

We know that

$$\begin{aligned} \|\Lambda(\mathbf{u}_h - \mathbf{u})\|^2 &= 2(\mathbf{J}(\mathbf{u}_h) - \mathbf{J}(\mathbf{u})), \\ \|\Lambda(\mathbf{v}_h - \mathbf{u})\|^2 &= 2(\mathbf{J}(\mathbf{v}_h) - \mathbf{J}(\mathbf{u})). \end{aligned}$$

Therefore,

$$\begin{aligned} 2\mathbf{D}(\Lambda \mathbf{u}_h, \Lambda \mathbf{v}_h) &= 2(\mathbf{J}(\mathbf{v}_h) - \mathbf{J}(\mathbf{u})) - 2(\mathbf{J}(\mathbf{u}_h) - \mathbf{J}(\mathbf{u})) = \\ &= \|\Lambda(\mathbf{v}_h - \mathbf{u})\|^2 - \|\Lambda(\mathbf{u}_h - \mathbf{u})\|^2. \end{aligned}$$

Now, the error estimate comes in the form

$$\begin{aligned} \|\Lambda(\mathbf{u} - \mathbf{u}_h)\|^2 &\leq (\mathbf{1} + \beta)(\|\Lambda(\mathbf{v}_h - \mathbf{u})\|^2 - \|\Lambda(\mathbf{u}_h - \mathbf{u})\|^2) + \\ &\quad + \left(\mathbf{1} + \frac{\mathbf{1}}{\beta}\right) \|\Lambda(\mathbf{v}_h - \mathbf{u})\|^2. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} (\mathbf{2} + \beta) \|\Lambda(\mathbf{u} - \mathbf{u}_h)\|^2 &\leq \\ &\leq (\mathbf{1} + \beta) \|\Lambda(\mathbf{v}_h - \mathbf{u})\|^2 + \left(\mathbf{1} + \frac{\mathbf{1}}{\beta}\right) \|\Lambda(\mathbf{v}_h - \mathbf{u})\|^2, \end{aligned}$$

We see that

$$\| \Lambda(\mathbf{u} - \mathbf{u}_h) \| \|^2 \leq \left(1 + \frac{1}{\beta(2 + \beta)} \right) \| \Lambda(\mathbf{u} - \mathbf{v}_h) \| \|^2 .$$

Since β is an arbitrary positive number, we arrive at the projection type error estimate

$$\| \Lambda(\mathbf{u} - \mathbf{u}_h) \| \leq \inf_{\mathbf{v}_h \in \mathbf{V}_h} \| \Lambda(\mathbf{u} - \mathbf{v}_h) \| .$$

Diffusion equation

Let \mathcal{A} is produced by a matrix $\mathbf{A} = \{\mathbf{a}_{ij}\} = \{\mathbf{a}_{ji}\}$, $\mathbf{V} = \mathbf{H}^1(\Omega)$, where Ω is a Lipschitz domain, $\mathbf{U} = \mathbf{L}^2(\Omega, \mathbb{R}^n)$, and $\mathbf{A}\mathbf{w} = \nabla\mathbf{w}$. Let the entries of \mathbf{A} be bounded at almost all points of Ω and such that

$$\mathbf{c}_1|\xi|^2 \leq \mathbf{a}_{ij}\xi_i\xi_j \leq \mathbf{c}_2|\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \quad (34)$$

Then, the spaces \mathbf{Y} and \mathbf{Y}^* have the norms

$$\|\mathbf{y}\|^2 = \int_{\Omega} \mathbf{A}\mathbf{y} \cdot \mathbf{y} \, dx, \quad \|\mathbf{y}\|_*^2 = \int_{\Omega} \mathbf{A}^{-1}\mathbf{y} \cdot \mathbf{y} \, dx.$$

Dirichlet boundary conditions

We begin with the problem

$$\mathbf{div} \mathbf{A} \nabla \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \quad (35)$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \partial\Omega. \quad (36)$$

In this case, $\mathbf{V}_0 = \mathring{\mathbf{H}}^1(\Omega)$ and \mathbf{u} meets the integral identity

$$\int_{\Omega} \mathbf{A} \nabla \mathbf{u} \cdot \nabla \mathbf{w} \, dx + \langle \mathbf{f}, \mathbf{w} \rangle = 0, \quad \forall \mathbf{w} \in \mathbf{V}_0. \quad (37)$$

The relation $(\mathbf{y}, \mathbf{\Lambda} \mathbf{w}) = \langle \mathbf{\Lambda}^* \mathbf{y}, \mathbf{w} \rangle$ has the form

$$\int_{\Omega} \mathbf{y} \cdot \nabla \mathbf{w} \, dx = \langle -\mathbf{div} \mathbf{y}, \mathbf{w} \rangle,$$

where $\mathbf{\Lambda}^* = -\mathbf{div}$ and $\mathbf{div} \mathbf{y}$ is in $\mathbf{H}^{-1}(\Omega)$.

The operator \mathbf{A} satisfies the required inequality

$$c_{\Omega} \|\nabla \mathbf{w}\| \geq \|\mathbf{w}\|, \quad \forall \mathbf{w} \in \mathring{\mathbf{H}}^1(\Omega).$$

Upper estimates of $\|\mathbf{v} - \mathbf{u}\|$ for an approximation $v \in \mathbf{V}_0 + \mathbf{u}_0$ follow from the general estimate presented in Lecture 5. We have

$$\frac{1}{2} \int_{\Omega} \mathbf{A} \nabla(\mathbf{v} - \mathbf{u}) \cdot \nabla(\mathbf{v} - \mathbf{u}) \, dx \leq \mathbf{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}),$$

where

$$\begin{aligned} \mathbf{M}_{\oplus}(\mathbf{v}, \beta, \mathbf{y}) = & \\ \frac{1 + \beta}{2} \int_{\Omega} \left(\nabla \mathbf{v} - \mathbf{A}^{-1} \mathbf{y} \right) \cdot \left(\mathbf{A} \nabla \mathbf{v} - \mathbf{y} \right) \, dx &+ \frac{1 + \beta}{2\beta} \frac{c_{\Omega}^2}{c_1^2} \|\operatorname{div} \mathbf{y} - \mathbf{f}\|^2 \end{aligned} \quad (38)$$

Mixed boundary conditions

Let $\partial\Omega$ consist of two measurable nonintersecting parts $\partial_1\Omega$ and $\partial_2\Omega$, on which different boundary conditions are given:

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \partial_1\Omega,$$

$$\boldsymbol{\nu} \cdot \mathbf{A}\nabla\mathbf{u} + \mathbf{F} = \mathbf{0} \quad \text{on} \quad \partial_2\Omega.$$

Set

$$\mathbf{V}_0 := \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial_1\Omega \right\}$$

and

$$\langle \boldsymbol{\Lambda}^* \mathbf{y}, \mathbf{w} \rangle = \int_{\Omega} \mathbf{y} \cdot \nabla \mathbf{w} \, dx, \quad \forall \mathbf{w} \in \mathbf{V}_0.$$

Assume that

$$\mathbf{f} \in \mathbf{L}^2(\Omega), \quad \mathbf{F} \in \mathbf{L}^2(\partial_2\Omega).$$

and \mathbf{y} possesses an extra regularity, namely,

$$\mathbf{y} \in \mathbf{Q}^*(\Omega) := \left\{ \mathbf{y} \in \mathbf{Y}^* \mid \operatorname{div} \mathbf{y} \in \mathbf{L}^2(\Omega), \mathbf{y} \cdot \boldsymbol{\nu} \in \mathbf{L}^2(\partial_2\Omega) \right\}.$$

Then, for any $\mathbf{w} \in \mathbf{V}_0$, we have

$$\langle \ell + \boldsymbol{\Lambda}^* \mathbf{y}, \mathbf{w} \rangle = \int_{\Omega} (\operatorname{div} \mathbf{y} - \mathbf{f}) \mathbf{w} \, d\mathbf{x} + \int_{\partial_2\Omega} (\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}) \mathbf{w} \, d\mathbf{s},$$

Note that $\mathbf{p} \in \mathbf{Q}^*(\Omega)$!

Now, we obtain

$$|\langle \ell + \Lambda^* \mathbf{y}, \mathbf{w} \rangle| \leq \|\mathbf{div} \mathbf{y} - \mathbf{f}\|_{2,\Omega} \|\mathbf{w}\|_{2,\Omega} + \|\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}\|_{2,\partial_2\Omega} \|\mathbf{w}\|_{2,\partial_2\Omega}.$$

Let γ and γ_* be two numbers such that $\gamma > \mathbf{1}$, $\gamma_* > \mathbf{1}$, $\frac{\mathbf{1}}{\gamma} + \frac{\mathbf{1}}{\gamma_*} = \mathbf{1}$. Use the algebraic inequality

$$\mathbf{ab} + \mathbf{cd} \leq \sqrt{\gamma \mathbf{a}^2 + \gamma_* \mathbf{c}^2} \sqrt{\frac{\mathbf{1}}{\gamma} \mathbf{b}^2 + \frac{\mathbf{1}}{\gamma_*} \mathbf{d}^2}.$$

Then

$$|\langle \ell + \Lambda^* \mathbf{y}, \mathbf{w} \rangle| \leq \left(\gamma \|\mathbf{div} \mathbf{y} - \mathbf{f}\|_{2,\Omega}^2 + \gamma_* \|\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}\|_{2,\partial_2\Omega}^2 \right)^{1/2} \times \left(\frac{\mathbf{1}}{\gamma} \|\mathbf{w}\|_{2,\Omega}^2 + \frac{\mathbf{1}}{\gamma_*} \|\mathbf{w}\|_{2,\partial_2\Omega}^2 \right)^{1/2}.$$

Since (Friederichs type inequality)

$$\|\mathbf{w}\|_{2,\Omega}^2 \leq \mathbf{C}_F^2(\Omega) \|\nabla \mathbf{w}\|_{2,\Omega}^2, \quad \forall \mathbf{w} \in \mathbf{V}_0,$$

and (trace inequality)

$$\|\mathbf{w}\|_{2,\partial_2\Omega}^2 \leq \mathbf{C}_{\text{tr}}^2(\Omega, \partial_2\Omega) \|\mathbf{w}\|_{1,2,\Omega}^2, \quad \forall \mathbf{w} \in \mathbf{V}_0,$$

we find that

$$\begin{aligned} \frac{1}{\gamma} \|\mathbf{w}\|_{2,\Omega}^2 + \frac{1}{\gamma_*} \|\mathbf{w}\|_{2,\partial_2\Omega}^2 &\leq \\ &\leq \mathbf{C}_F^2 \frac{1}{\gamma} \|\nabla \mathbf{w}\|^2 + \mathbf{C}_{\text{tr}}^2 \frac{1}{\gamma_*} \left(\|\mathbf{w}\|_{2,\Omega}^2 + \|\nabla \mathbf{w}\|_{2,\Omega}^2 \right) \leq \\ &\leq \left(\mathbf{C}_F^2 \frac{1}{\gamma} + \mathbf{C}_{\text{tr}}^2 \frac{1}{\gamma_*} \left(\mathbf{1} + \mathbf{C}_F^2 \right) \right) \|\nabla \mathbf{w}\|_{2,\Omega}^2. \end{aligned}$$

Therefore, there exist a positive constant \mathbf{C}_γ such that

$$\frac{1}{\mathbf{C}_\gamma^2} = \inf_{\mathbf{w} \in \mathbf{V}_0} \frac{\int_{\Omega} \mathbf{A} \nabla \mathbf{w} \cdot \nabla \mathbf{w} \, dx}{\frac{1}{\gamma} \|\mathbf{w}\|_{2,\Omega}^2 + \frac{1}{\gamma_*} \|\mathbf{w}\|_{2,\partial_2\Omega}^2}.$$

The value of this constant can be estimated numerically by minimizing the above quotient on a sufficiently representative finite dimensional subspace. Besides, if \mathbf{C}_F and \mathbf{C}_{tr} are estimated, then

$$\mathbf{C}_\gamma^2 \leq \widehat{\mathbf{C}}_\gamma^2 := \left(\mathbf{C}_F^2 \frac{1}{\gamma} + \mathbf{C}_{tr}^2 (1 + \mathbf{C}_F^2) \frac{1}{\gamma_*} \right) \mathbf{c}_1^{-1},$$

so that an upper bound of \mathbf{C}_γ is directly computed. Now,

$$\begin{aligned} |\langle \ell + \mathbf{\Lambda}^* \mathbf{y}, \mathbf{w} \rangle| &\leq \\ &\leq \widehat{\mathbf{C}}_\gamma \left(\gamma \|\operatorname{div} \mathbf{y} - \mathbf{f}\|_{2,\Omega}^2 + \gamma_* \|\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}\|_{2,\partial_2\Omega}^2 \right)^{1/2} \|\nabla \mathbf{w}\|. \end{aligned}$$

From this estimate, we obtain

$$\|\ell + \Lambda^* \mathbf{y}\|^2 \leq \widehat{\mathbf{C}}_\gamma^2 \left(\gamma \|\operatorname{div} \mathbf{y} - \mathbf{f}\|_{2,\Omega}^2 + \frac{\gamma}{\gamma - 1} \|\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}\|_{2,\partial_2\Omega}^2 \right).$$

Consider first the case, in which we simply set $\gamma = \gamma^* = 2$. Then

$$\widehat{\mathbf{C}}_{(\gamma=2)}^2 := \widehat{\mathbf{C}}_2^2 = \frac{1}{2} \left(\mathbf{C}_F^2 + \mathbf{C}_{\operatorname{tr}}^2 (\mathbf{1} + \mathbf{C}_F^2) \right) \mathbf{c}_1^{-1},$$

$$\|\ell + \Lambda^* \mathbf{y}\|^2 \leq 2\widehat{\mathbf{C}}_2^2 \left(\|\operatorname{div} \mathbf{y} - \mathbf{f}\|_{2,\Omega}^2 + \|\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}\|_{2,\partial_2\Omega}^2 \right).$$

and we find that

$$\begin{aligned} \mathbf{M}_\oplus(\mathbf{v}, \boldsymbol{\beta}, \mathbf{y}) &= \frac{1 + \boldsymbol{\beta}}{2} \int_\Omega (\nabla \mathbf{v} - \mathbf{A}^{-1} \mathbf{y}) \cdot (\mathbf{A} \nabla \mathbf{v} - \mathbf{y}) \, dx + \\ &+ \frac{1 + \boldsymbol{\beta}}{2\boldsymbol{\beta}} \widehat{\mathbf{C}}_2^2 \left(\|\operatorname{div} \mathbf{y} - \mathbf{f}\|_{2,\Omega}^2 + \|\mathbf{y} \cdot \boldsymbol{\nu} + \mathbf{F}\|_{2,\partial_2\Omega}^2 \right). \quad (39) \end{aligned}$$

This Majorant gives an upper bound of the deviation for any $\mathbf{v} \in \mathbf{V}_0 + \mathbf{u}_0$, $\mathbf{y} \in \mathbf{Q}^*$, and $\boldsymbol{\beta} > 0$.

For the generalized diffusion equation it is as follows:

$$\| \nabla(\mathbf{u} - \mathbf{v}) \|^2 \leq \sqrt{\mathbf{D}(\nabla \mathbf{v}, \mathbf{y})} + \mathbf{C}(\Omega, \mathbf{A}) \|\mathbf{div} \mathbf{v} + \mathbf{f}\|,$$

where

$$\begin{aligned} \|\eta\|^2 &:= \int_{\Omega} \mathbf{A} \nabla \eta \cdot \nabla \eta \, \mathbf{d}\mathbf{x}, \\ \mathbf{D}(\nabla \mathbf{v}, \mathbf{y}) &:= \int_{\Omega} (\mathbf{A} \nabla \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{A}^{-1} \mathbf{y} \cdot \mathbf{y} - 2 \nabla \mathbf{v} \cdot \mathbf{y}) \, \mathbf{d}\mathbf{x} = \\ &= \int_{\Omega} (\mathbf{A} \nabla \mathbf{v} - \mathbf{y}) \cdot (\nabla \mathbf{v} - \mathbf{A}^{-1} \mathbf{y}) \, \mathbf{d}\mathbf{x}, \\ \|\mathbf{w}\| &\leq \mathbf{C}(\Omega, \mathbf{A}) \|\nabla \mathbf{w}\|. \end{aligned}$$

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