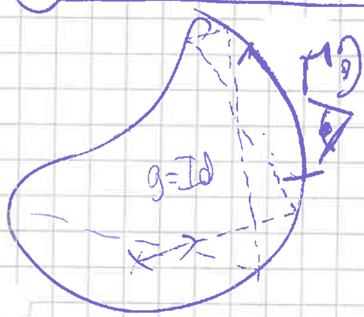
Quantitative unique continuation for waves, and the cost of approximate controls (with Camille Laurent)

(0) Intro: The BLR thm:



$\Omega \subset \mathbb{R}^n$, $\partial\Omega$ C^∞, $(g \in C^\infty(\bar\Omega; \mathbb{R}^{n\times n}))$

$$(*)\begin{cases} \partial_t^2 u - \Delta_g u + (Vu) = 0 & (0,T)\times\Omega \\ u|_{\partial\Omega} = 0 & (0,T)\times\partial\Omega \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \Omega \end{cases}$$

Thm (BLR) Given $T>0$, $\Gamma \subset \partial\Omega$

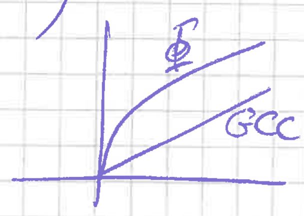
(i) Observability: $\exists C>0$, $\forall (u_0,u_1) \in H_0^1 \times L^2$, u sol $(*)$
$$\|(u_0,u_1)\|^2_{H_0^1\times L^2} \leq C \int_0^T \|\partial_n u\|^2_{L^2(\Gamma)}\, dt$$

"$\Updownarrow$"

(ii) (Γ, T) GCC: every ray of geometric optics (wrt g), traveling at speed 1, enters Γ in $(0,T)$.

Question (1): What if (Γ,T) does not satisfy GCC (i.e. $\exists$ ray not entering Γ in $(0,T)$):

- Do we have $\partial_n u = 0$ on $(0,T)\times\Gamma \Rightarrow (u_0,u_1) = 0$?
- If so, do we have
$$\|(u_0,u_1)\| \leq \Phi\left(\int_0^T \|\partial_n u\|^2_{L^2(\Gamma)}\, dt\right)$$
$$\left(\text{with } \Phi(x) \gg x,\ x \text{ small},\ \Phi(0)=0\right)$$



Proof of BLR: 2 Nonconstructive steps

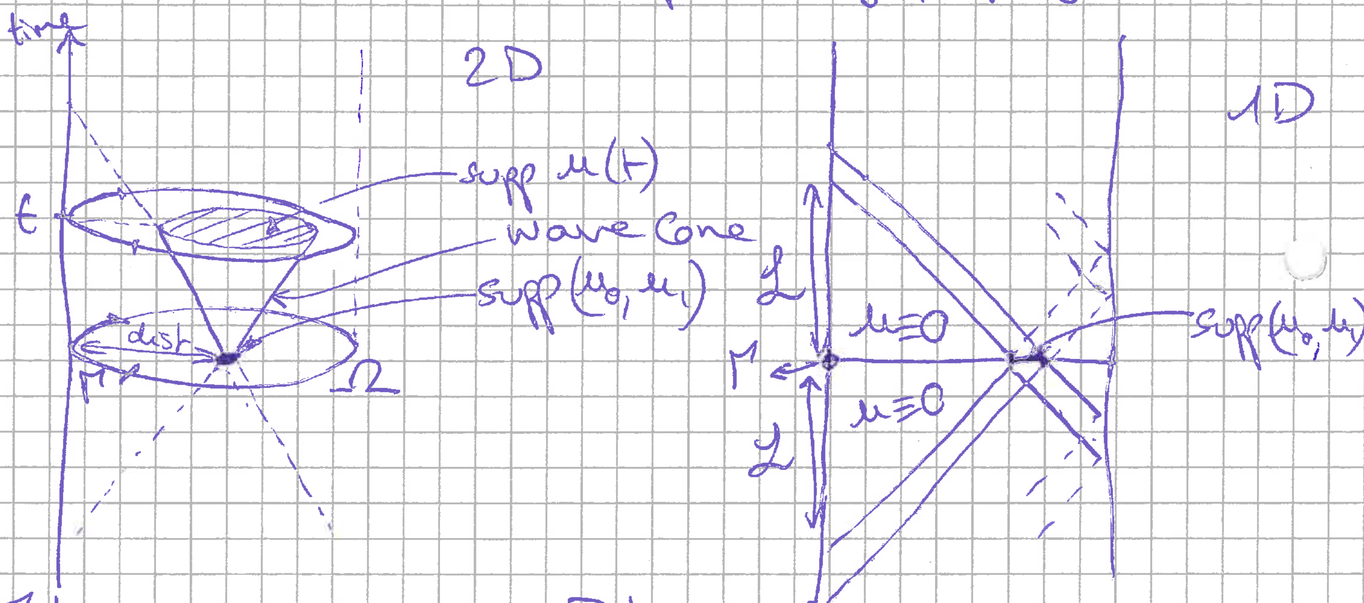
Question (3): Give a constructive proof of BLR?

① Quantitative Unique Continuation :

Unique Continuation ?

$$\left\{\begin{array}{l} u \text{ sol of } (*) \\ \partial_n u \equiv 0 \text{ on } (0,T)\times\Gamma \end{array}\right\} \underset{\text{UC}}{\overset{?}{\Longrightarrow}} (u_0,u_1)\equiv(0,0) \quad \text{i.e. } u\equiv 0$$

Obstruction : Finite speed of propagation :



Γ being fixed , UC False if

$$T < 2\mathcal{L}(\Gamma,\Omega) = 2\sup_{x\in\Omega} \operatorname{dist}(\Gamma,x)$$

Conversely (for any Γ open $\neq\emptyset$)

- Analytic setting (g analytic , l.o.t. analytic)
  ↳ Holmgren–John '49 $\quad T > 2\mathcal{L}(\Gamma,\Omega) \Longrightarrow$ UC

smooth setting $g\in C^\infty$ / $V\in L^\infty(\Omega)$

- { Rauch–Taylor 73 / Lerner 88) $\quad T = +\infty \longrightarrow$ UC
- Robbiano 91 $\quad T > C_0\mathcal{L}(\Gamma,\Omega)$
- Hörmander 92 $\quad T > 2\sqrt{\frac{27}{23}}\,\mathcal{L}(\Gamma,\Omega)$
- Tataru 95 $\quad T > 2\mathcal{L}(\Gamma,\Omega)$!
  (Robbiano–Zuily 98 , Hörmander 97)

Rem : • Cohérence : (Γ, T) GCC $\Rightarrow T > 2\mathcal{L}(\Gamma, \Omega)$

• Holmgren : l.o.t Analytic

↳ Here $\Box + V(t,x)$

Analytic wrt t (only)

↓

[Tat_Rob_Zui_Hör]

Our result : Stability in this thm.

Thm (Lau_Léau 15) : $\forall T > 2\mathcal{L}(\Gamma, \Omega)$, $\exists C > 0$, st $\forall U_0 = (u_0, u_1) \in H_0^1 \times L^2$, and u sol $(*)$, we have :

• $\|U_0\|_{L^2 \times H^{-1}} \leq C e^{C\mu} \|\partial_n u\|_{L^2((0,T)\times\Gamma)} + \frac{1}{\mu} \|U_0\|_{H_0^1 \times L^2}$, $\forall \mu \geq \mu_0$

• $\|U_0\|_{H_0^1 \times L^2} \leq C e^{C\Lambda} \|\partial_n u\|_{L^2((0,T)\times\Gamma)}$, $\Lambda = \frac{\|U_0\|_{H_0^1 \times L^2}}{\|U_0\|_{L^2 \times H^{-1}}}$

• $$\|U_0\|_{L^2 \times H^{-1}} \leq C \frac{\|U_0\|_{H^1 \times L^2}}{\log\left(\pi + \frac{\|U_0\|_{H^1 \times L^2}}{\|\partial_n u\|_{L^2((0,T)\times\Gamma)}}\right)}$$

$$\begin{cases} \frac{1}{\log(1+\frac{1}{x})} & x > 0 \\ 0 & x = 0 \end{cases}$$

Rem : • the 3 ineq are equiv.

• Λ = typical frequency of the data (u_0, u_1), e.g. if $\begin{cases} u_0 = \varphi_\lambda \\ u_1 = 0 \end{cases}$ $-\Delta \varphi_\lambda = \lambda \varphi_\lambda$, this is $\Lambda = \sqrt{\lambda}$

$$\|\varphi_\lambda\|_{L^2(\Omega)} \leq C e^{C\sqrt{\lambda}} \|\partial_n \varphi_\lambda\|_{L^2(\Gamma)}$$

↳ tunelling of eigenfcts

• Explicit depdence wrt lot $V(x) \in L^\infty$, $b(x)\partial_t$, $W(x) \cdot \nabla_x$

History:
- Analytic setting Lebeau 92
  + Optimal as soon as GCC fails
- Robbiano 95 $\dfrac{1}{\log\left(1+\frac{1}{\varepsilon}\right)^{1/2}}$ for T large
- Phung 10 $\dfrac{1}{\log^{1-\varepsilon}}$
- Tataru 99 a "strategy" to obtain $\dfrac{1}{\log^{1-\varepsilon}}$ for $T > 2\mathcal{L}(\Gamma, \Omega)$ (without boundary)

Corollary (Approx Control, LL15)

$$(**)\quad \begin{cases} (\partial_t^2 - \Delta_g + V)\, v = 0 \\ v|_{\partial\Omega} = \mathbb{1}_\Gamma\, g \\ (v, \partial_t v) = (v_0, v_1) \end{cases}$$

$\forall \varepsilon > 0 \quad \forall (v_0, v_1) \in H^1_0 \times L^2, \quad \exists\, g \in L^2((0,T)\times\Gamma)$

with $\|g\|_{L^2((0,T)\times M)} \leq \underbrace{C e^{C/\varepsilon}}_{\boxed{\text{Cost}}} \|(v_0, v_1)\|_{H^1_0 \times L^2}$

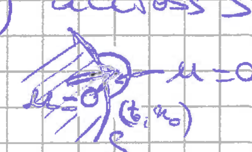
st $\|(v, \partial_t v)|_{t=T}\|_{L^2 \times H^{-1}} \leq \varepsilon\, \|(v_0, v_1)\|_{H^1 \times L^2}$

for u sol (**)

② An idea of the proof

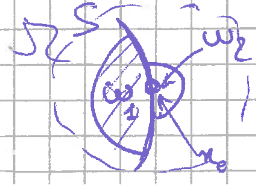
PU: 1) local uniqueness near (t_0, x_0) accross S

2) Propagation of zero / smallness



step 2) Propagation of smallness:

In the elliptic case (Hörmander th, Bahouri, Robbiano)



Prove $\|u\|_{\omega_2} \leq e^{c\mu}\|u\|_{\omega_1} + e^{-\kappa\mu}\|u\|_\Omega \quad \forall \mu \geq \mu_0$

$\left(\leq \|u\|^{\delta}_{\omega_1}\|u\|^{1-\delta}_{\Omega}\right)$

Propagate:

$$\|u\|_{\omega_3} \leq e^{c\tilde\mu}\|u\|_{\omega_2} + e^{-\kappa\tilde\mu}\|u\|_\Omega \qquad \tilde\mu \geq \mu_0$$

$$\leq e^{c\tilde\mu}\left(e^{c\mu}\|u\|_{\omega_1} + e^{-\kappa\mu}\|u\|_\Omega\right) + e^{-\kappa\tilde\mu}\|u\|_\Omega$$

$$\leq e^{c(\tilde\mu+\mu)}\|u\|_{\omega_1} + \left(e^{c\tilde\mu-\kappa\mu} + e^{-\kappa\tilde\mu}\right)\|u\|_\Omega$$

take $\mu = \frac{2c}{\kappa}\tilde\mu$

$$\leq e^{\hat c\tilde\mu}\|u\|_{\omega_1} + e^{-\hat\kappa\tilde\mu}\|u\|_\Omega \quad \underline{\text{Same form}}$$

Here, we expect:

$$\|u\|_{\omega_2} \leq e^{c\mu}\|u\|_{\omega_1} + \frac{1}{\mu}\|u\|_{\tilde\Omega}$$

$$\|u\|_{\omega_3} \leq e^{c\tilde\mu}\|u\|_{\omega_2} + \frac{1}{\tilde\mu}\|u\|_\Omega$$

$$\leq e^{c\tilde\mu}\left(e^{c\mu}\|u\|_{\omega_1} + \frac{1}{\mu}\|u\|_\Omega\right) + \frac{1}{\tilde\mu}\|u\|_{\tilde\Omega}$$

$$\leq e^{c(\tilde\mu+\mu)}\|u\|_{\omega_1} + \underbrace{\left(\frac{e^{c\tilde\mu}}{\mu} + \frac{1}{\tilde\mu}\right)}_{\frac{c}{\tilde\mu} \text{ if } \mu = e^{c\tilde\mu}\tilde\mu}\|u\|_{\tilde\Omega}$$

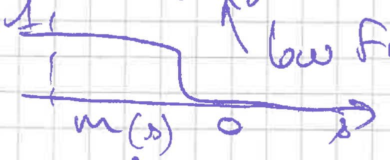
$$\leq \underbrace{e^{ce^{c\tilde\mu}}}_{\text{Very Bad!}}\|u\|_{\omega_1} + \frac{1}{\tilde\mu}\|u\|_\Omega$$

Idea: Tataru: Prove/propagate low-freq estimates only:

$$(**) \quad \left\|m\left(\frac{D_t}{\mu}\right)\chi_{\omega_2}u\right\| \leq e^{\kappa\mu}\left\|m\left(\frac{D_t}{\mu}\right)\chi_{\omega_1}u\right\| + e^{-\kappa'\mu}\|u\|_{\tilde\Omega}$$

low freq

expo remainder



Then: issue: patch these estimates together

Remainder terms $\mathcal{O}(\mu^{-\infty})$ if C^∞ $\left(1-m\left(\frac{D_t}{2\mu}\right)\right)\chi\, m\left(\frac{D_t}{\mu}\right)$

$\mathcal{O}\left(e^{-\mu^{1/\alpha}}\right)$ if G^α $\alpha > 1$

To obtain $\mathcal{O}(e^{-\kappa\mu}) \longrightarrow$ Analytic cutoff fcts!

$$m\left(\frac{D_t}{\mu}\right) \longrightarrow m_\delta\left(\frac{D_t}{\mu}\right)$$

Analytic Regularizat° $\quad m_\delta(s) = e^{-\frac{D_t^2}{\delta}} m(s) = \dots e^{-\frac{\delta|s|^2}{4}} * m(s)$ Analytic

Right regularization parameter $\delta \simeq \mu$.

then replace "Compactly supported in K" by "exponentially decaying outside K"

Step 1) Prove (**) !!! Starting point

Carleman estimates with [Tat-Rob-Zuilhof] Analytic localizat° localizat° near $\xi_t = 0$

$$\| e^{-\frac{\varepsilon D_t^2}{\tau}} e^{\tau\psi} u \| \leq \| e^{-\frac{\varepsilon D_t^2}{\tau}} e^{\tau\psi} \Box u \| + e^{-d\tau} \| e^{\tau\psi} u \|$$

large parameter ($\tau \simeq \mu$)

Appropriate local weight fct.

$$\left(\frac{\tau}{4\pi\varepsilon}\right)^{1/2} e^{-\frac{\tau}{4\varepsilon}t^2} * v$$

↳ Reminiscent to the FBI transform

$$\partial_t^2 - \Delta \longrightarrow -\partial_t^2 - \Delta$$

$$\partial_t \longrightarrow i\partial_t$$

- Good starting point: Carleman = qtitative
- then: A complex analytic argument (maximum pciple)

③ Constructive proof of the BLR thm

Proof of BLR: 2 steps:

Step 1: Prove $\|(u_0, u_1)\|^2_{H^1 \times L^2} \leq C_0 \|\partial_\nu u\|^2_{L^2((0,T)\times\Gamma)} + \underbrace{C_1 \|(u_0, u_1)\|^2_{L^2 \times H^{-1}}}_{\text{Remainder}}$

↳ "High frequency step"

↳ closed graph thm + propagat° of singularities along rays of geometric optics.

Step 2: Get rid of $\|(u_0, u_1)\|_{L^2 \times H^{-1}}$

↳ By contradiction → Reduction to uniq continuat°

Two non constructive arguments !!! (4)

↳ Use our th to perform Step 2 :

$$\|U_0\|^2_{H^1_0 \times L^2} \leq C_0 \, Obs + C_1 \underbrace{\|U_0\|^2_{L^2 \times H^{-1}}}$$

$$\underset{\text{Laurent 15}}{\leq} K e^{K\mu} Obs + \frac{1}{\mu^2} \|U_0\|^2_{H^1_0 \times L^2}$$

if $\frac{C_1}{\mu^2} \leq \frac{1}{2}$

$$\frac{1}{2} \|U_0\|^2_{H^1 \times L^2} \leq \left(C_0 + K e^{K\sqrt{2C_1}} \right) Obs \quad \blacksquare$$

Application 1 : Potential $V(x)$

Thm : (Γ, T) GCC Then

$$\|(u, \partial_t u)|_{t=0}\|^2_{H^1_0 \times L^2} \leq C e^{C\|V\|_\infty} \int_0^T \|\partial_n u\|^2_{L^2(\Gamma)} dt$$

$\forall V \in L^\infty$ and u sol of
V Real valued
$V \geq 0$

$$\Box u + V(x) u = 0$$

Previously : • Uniformity : Dehman Ervedoza 14

• Duyckaerts Zhang Zuazua 08

$C \exp(C \|V\|_\infty^{2/3})$

if a multiplier Condition.

Conjecture this is optimal

Application 2 : On a compact manifold M with internal observat

Complete Constructive proof and

$$\frac{C}{k(T)} \leq C_{obs}(T, \omega) \leq C e^{C/k(T)}$$

$$k(T) = \inf_{\rho \in T^*M} \frac{1}{T} \int_0^T \chi_\omega^2 \circ \phi_t(\rho) \, dt$$

control fct $\simeq 1_\omega$ smooth

geodesic flow on T^*M

as $T \to T_{GCC}(\omega)^+$

$T_{GCC}(\omega) = \inf\{T > 0, (\omega, T) \text{ satisfies GCC}\}$