Min-based fusion of possibilistic networks

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Motivations

- **Important problem :** databases, expert opinions pooling, preference agregations, etc.
- **Why merging?**
  - exploit complementarities between the sources,
  - get a global and coherent point of view,
  - reduce imprecision,
  - deals with redunduncies, etc.

Motivations

- Different formats for representing uncertain information:
  - Possibilistic logic knowledge bases
  - Possibilistic networks
- General classes of merging operators:
  - Conjunctive merging
  - Disjunctive merging
  - Adaptative merging
- Syntactic fusion defined on possibilistic knowledge bases

Aim

- **Input :**
  - *n* consistent possibilistic networks (provided by *n* experts)
  - Conjunctive operator (minimum)
- **Goal** Compute a new possibilistic network representing the result of merging

Outline

1. **Definitions of Min-based possibilistic networks**
2. **Fusion of same-structure networks**
3. **Fusion of networks with different structures**
   - Case 1 : Union of graphs is acyclic
   - Case 2 : Union of graphs contains cycles

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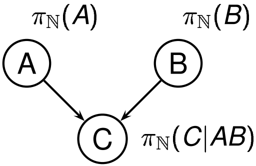
Possibility Distributions

- $\pi : \Omega \to [0, 1]$, $\max_{\omega \in \Omega} \pi(\omega) = 1$.

- Two dual measures :
  - Possibility measure of ϕ:
    $\Pi(\phi) = \max\{\pi(\omega) : \omega \models \phi\}$
    $\Rightarrow$ The compatibility degree of ω with available knowledge.
  - Necessity measure of ϕ:
    $N(\phi) = 1 - \Pi(\neg\phi)$
    $\Rightarrow$ The certainty degree associated with ϕ from available pieces of information encoded by π.

Possibilistic Networks $\mathbb{N} = (\pi_{\mathbb{N}}, G_{\mathbb{N}})$



- $G_{\mathbb{N}}$: Directed Acyclic Graph (DAG)
  - Node $\leftarrow$ variable
  - Arc $\leftarrow$ causal relationship

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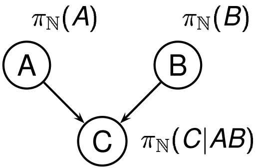

- $G_{\mathbb{N}}$: Directed Acyclic Graph (DAG)
  - Node $\leftarrow$ variable
  - Arc $\leftarrow$ causal relationship
- $\pi_{\mathbb{N}}$: Conditional possibility distributions in the context of each parent

$$\pi_{\mathbb{N}}(ABC) = \min(\pi_{\mathbb{N}}(A), \pi_{\mathbb{N}}(B), \pi_{\mathbb{N}}(C|AB))$$

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Fusion of same-structure networks

Input : Two possibilistic networks having the same graph: $\mathbb{N}1 = (\pi_{\mathbb{N}1}, G_{\mathbb{N}1})$ and $\mathbb{N}2 = (\pi_{\mathbb{N}2}, G_{\mathbb{N}2})$ s.t.

$$G_{\mathbb{N}1} = G_{\mathbb{N}2}$$

- **Result of merging is immediate**
  $\mathbb{N}\oplus = (\pi_{\mathbb{N}\oplus}, G_{\mathbb{N}\oplus})$ such that:
  - $G_{\mathbb{N}\oplus} = G_{\mathbb{N}1} = G_{\mathbb{N}2}$
  - $\forall A$:
    $\pi_{\mathbb{N}\oplus}(A \mid U_A) = min(\pi_{\mathbb{N}1}(A \mid U_A), \pi_{\mathbb{N}2}(A \mid U_A))$.

Fusion of same-structure networks

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- We have :
  $\forall \omega \in \Omega, \pi_{\mathbb{N}\oplus}(\omega) = \min(\pi_{\mathbb{N}1}(\omega), \pi_{\mathbb{N}2}(\omega))$

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Case 1 : Union of graphs is acyclic

Input $\mathbb{N}1 = (\pi_{\mathbb{N}1}, G_{\mathbb{N}1})$ and $\mathbb{N}2 = (\pi_{\mathbb{N}2}, G_{\mathbb{N}2})$ s.t. :

$$G_{\mathbb{N}1} \neq G_{\mathbb{N}2}$$

Result of merging in three steps:

- **step 1 :** Define $G_{\mathbb{N}\oplus}$ the union of $G_{\mathbb{N}1}$ and $G_{\mathbb{N}2}$
- **step 2 :** Expand equivalently $\mathbb{N}1$ and $\mathbb{N}2$ into : $\mathbb{N}1' = (\pi_{\mathbb{N}1'}, G_{\mathbb{N}\oplus})$ and $\mathbb{N}2' = (\pi_{\mathbb{N}2'}, G_{\mathbb{N}\oplus})$
- **step 3 :** Apply Fusion of same structure networks to $\mathbb{N}1'$ and $\mathbb{N}2'$.

Case 1 : Union of graphs is acyclic

Input $\mathbb{N}1 = (\pi_{\mathbb{N}1}, G_{\mathbb{N}1})$ and $\mathbb{N}2 = (\pi_{\mathbb{N}2}, G_{\mathbb{N}2})$ s.t. :

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How to equivalently expand possibilistic networks?

Expanding networks : Adding new variables

- **Input**
  - A possibilistic network $\mathbb{N} = (\pi_{\mathbb{N}}, G_{\mathbb{N}})$ built on a set of variables V
  - A a new variable A
- **Output :** $\mathbb{N}1 = (\pi_{\mathbb{N}1}, G_{\mathbb{N}1})$ such that :
  - $G_{\mathbb{N}1}$ is equal to $G_{\mathbb{N}}$ plus a root node A, and
  - $\pi_{\mathbb{N}1} = \pi_{\mathbb{N}}$ for variables in V, and is uniform on A (i.e., $\forall a \in D_A$, $\pi_{\mathbb{N}1}$(a)=1).

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Then, we have : $\forall \omega \in \times_{A_i \in V} D_{A_i}$,

$$\pi_{\mathbb{N}}(\omega) = \Pi_{\mathbb{N}1}(\omega)$$

Expanding networks : Adding new links

- **Input**
  - A possibilistic network $\mathbb{N} = (\pi_{\mathbb{N}}, G_{\mathbb{N}})$
  - B a variable which is not a parent of A
- **Output :** $\mathbb{N}1 = (\pi_{\mathbb{N}1}, G_{\mathbb{N}1})$ such that :
  - $G_{\mathbb{N}1}$ is obtained from $G_{\mathbb{N}}$ by adding a link from B to A,
  - The new conditionnal possibility distribution of A is:
    $\forall a \in D_a, b \in D_B, u \in D_{Par(A)}$,

$$\pi_{\mathbb{N}1}(a \mid ub) = \pi_{\mathbb{N}}(a \mid u).$$

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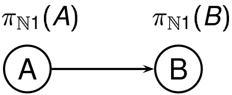
  $$\pi_{\mathbb{N}1}(a \mid ub) = \pi_{\mathbb{N}}(a \mid u).$$

Then, we have :

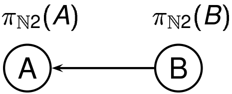
$$\forall \omega, \pi_{\mathbb{N}}(\omega) = \pi_{\mathbb{N}1}(\omega)$$

Case 2 : Union of graphs contains cycles

- **G_1**

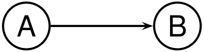


- **G_2**

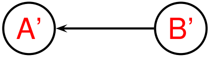


Step 1 rename variables of G_2

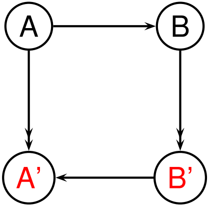
- G_1



- G_2



Step : Relating old and new variables



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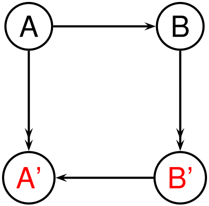
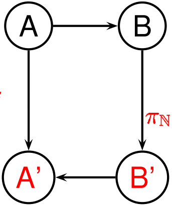


This defines $G_{\mathbb{N}\oplus}$.

Step 3 : Defining conditional possibility distributions

$\pi_{\mathbb{N}\oplus}(A) = \pi_{\mathbb{N}1}(A)$

$\pi_{\mathbb{N}\oplus}(B \mid A) = \pi_{\mathbb{N}1}(B \mid A)$



$$\pi_{\mathbb{N}\oplus}(a'_i \mid a_j b'_k) = \begin{cases} \pi_{\mathbb{N}2}(a_i \mid b'_k) & \textit{if } i = j \\ 0 & \text{other.} \end{cases}$$

$$\pi_{\mathbb{N}\oplus}(b'_i \mid b_j) = \begin{cases} \pi_{\mathbb{N}2}(b_i) & \textit{if } i = j \\ 0 & \text{other.} \end{cases}$$

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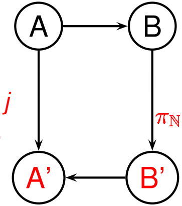

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We have :

$$min(\pi_{\mathbb{N}1}(\omega), \pi_{\mathbb{N}2}(\omega)) = \Pi_{\mathbb{N}\oplus}(\omega).$$

Conclusion

- Fusion of possibilistic networks
- Efficient fusion procedures for networks :
  - having same graphs
  - the union of their graphs is free of cycles
- Investigate other combination operators