Joint Design and Pricing on a Network

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Abstract

In order to optimize revenue, service firms must integrate within their pricing policies the rational reaction of customers to their price schedules. In the airline or telecommunication industry, this process is all the more complex due to interactions resulting from the structure of the supply network. In this paper we consider a streamlined version of this situation where a firm's decision variables involve both prices and investments. We model this situation as a joint design and pricing problem which we formulate as a mixed-integer bilevel program, and whose properties are investigated. In particular, we prove a property of the model that allows the development of an algorithmic framework based on Lagrangian relaxation. This approach is entirely novel, and numerical results show that it is capable of solving problems of significant sizes.

1 Introduction

This paper is devoted to a model that captures the interaction between system design, price setting and consumer choice over a transportation network. The problem involves two decision makers acting non cooperatively and in a sequential way. The upper level (leader) strives to maximize its revenue raised from tariffs imposed on a set of goods or services in its control, while the lower level (follower) optimizes its own objective, taking into account the tariff schedule set by the leader. The leader explicitly incorporates the reaction of the follower in his optimization process. In the field of economics, this fits the principal/agent paradigm (Van Ackere [11]) where the principal, fully aware of the agent's rational behaviour, induces cooperation from the agent through an incentive scheme. In the field of mathematical programming this problem belongs to the class of bilevel optimization problems with bilinear objectives at both levels of decision.

In the current context of deregulation, pricing decisions have become crucial for airline, trucking, telecommunication and service industries where intense price competition and network modifications have occurred. Clearly a profit maximizing firm must consider the trade off between the cost of service and the revenue generated when designing its system and prices.

In the passenger or freight airline industry, a carrier (the leader) selects routing patterns, flight schedules and fares. For instance, Budenbender *et al.* [5] describe a system where freight providers such as express shipment companies operate or rent an aircraft fleet that must provide a high level of service. For consolidation purposes, the freight is first shipped to an airport, next it is flown non-stop to another airport, finally to be loaded on trucks and shipped to its final destination. The problem then consists of determining the terminal to operate, the take-off time, how to transport the freight to an airport, and the ratio to charge. In passenger transportation, the introduction of new flights (direct or through a hub-and-spoke network) must take into account the supply over the entire network of flights, both from the leader airline and its competitors. The decisions are then taken with respect to the incurred costs, the quality of service, the possible influence on demand to other destinations and, most important, the revenues generated by the new services (Lederer [9], [10]).

In the surface freight transportation industry, important structural changes occur as shippers optimize the end-to-end supply through the implementation of web-based portals. In that context, the costs incurred by a carrier is made up of two components: a fixed cost (including trade compliance, trade settlement with country-specific international trading portals, multi-modal aspects, operating resources costs, global handling costs, etc.) and a unit transportation cost (Kerr [6]). Upon reception, a service carrier (the leader) has to decide whether or not to accept a request and, if accepted, to set a price. In reaction to those prices, the shippers (the follower) want their goods be transshipped at minimal cost, hence the bilevel structure of the problem.

In the telecommunication area, a service provider (the leader) has to make network deployment decisions and to set prices for bandwidth usage. The response of users (the follower) to prices induces traffic on the network. In the current deregulated markets, pricing is a fundamental issue for communications carriers. Indeed, as new systems of ever larger capacities are introduced, the marginal cost of data transmission is rapidly decreasing. Exploiting those cost savings and handling increased demand involves the optimization of technology acquisition and pricing processes (Lanning *et al.* [8] and Başar and Srikant [1]).

Design and pricing are also challenging issues for business information service providers [2]. Information agencies such as Reuters and Bloomberg (foreign currency markets) and Aspect Development (component information services) are essentially intermediates between firms that generate and firms that use content. As information service providers (the leader) incur large fixed costs (data entry and updates, software development, database management systems, connections to commercial networks), their problem consists of specifying the size of the database they provide to subscribers (followers) as well as the price they will charge for subscriptions. At the lower level, the subscribers adapt their usage volume according to the level of service and tariffs of the service providers, or yet may select the self-service option whereby they collect and collate information directly from the sources.

Until now, design and pricing issues have mostly been treated separately. However, they are intrinsically linked and have to be addressed jointly. To our knowledge the only papers addressing the joint design and pricing problem are those of Lederer [9], Başar and Srikant [1] and Bashyam [2]. Lederer [9] proposes a Nash equilibrium model of air transport competition where firms select routes and prices. Competition is studied under two different assumptions about consumer choice: either consumers can spread their choice of services on different routes ("bundling"), or they cannot. If bundling is forbidden, the author proves the existence of unique equilibrium prices. Otherwise, a price equilibrium may fail to be unique, or even to exist. At first glance, our work might seem to fit the framework analyzed by Lederer. However it differs in two main respects: bundling is an essential part of our model, and we look for a Stackelberg (leader-follower) equilibrium rather than a Nash equilibrium. Consequently, the focus of this paper is on algorithmic development rather than economic considerations.

Başar and Srikant [1] study the economics of providing large capacity from a telecommunication provider's point of view. Design choices are not modelled using binary decisions but through continuous capacity variables. Each user is charged a fixed price per unit of bandwidth used, and this price is independent from congestion. The transmission rate of each user is assumed to be a function of network congestion and price per unit of bandwidth. The aim of the service provider is to maximize its revenue. The authors show that, as the number of users increases, the optimal price per unit of bandwidth charged by the service provider may increase or decrease depending upon the bandwidth of the link. However, for all values of the link capacity, the overall performance of each user improves and the service provider's revenue per unit of bandwidth increases, thus providing an incentive for the service provider to increase the available bandwidth in proportion to traffic. Although this work provides some theoretically insight into the problem, no computational procedure is described for its solution.

Bashyam [2] analyzes service design and pricing of business information services in a competitive environment, using game-theoretic concepts. The problem consists in determining the optimal size of the database, as well as the subscription price they will fix for subscriptions, taking into account the reaction of subscribers who want to minimize their cost. They consider two types of interactions: monopoly or duopoly, and two types of information delivery technologies: online service that allows subscribers to access information over online networks, and package service that delivers information using physical media such as CD-ROM's. Their analytical approach investigates the differences in price structure associated with the type of provided services. In the case of duopoly, they also analyze the class of consumers (high or low volume consumers) served depending on the size of the database and on prices.

In this paper we focus on a joint design and pricing problem on networks involving multicommodity flows. The upper level is concerned with maximizing profit raised from tariffs set on a subset of arcs which is determined by the leader. This problem can be adequately represented as a bilevel program and constitutes an extension of the model proposed by Brotcorne *et al* [3] and [4] for the determination of optimal tariffs on a single-commodity, respectively multicommodity, transportation network. This research was initiated by a research in telecommunication. The specificity of the problem considered here consists in simultaneously determining which connections are opened and which tariff policy is applied. This differs from our previous work where only tariff schedule was subject to optimization.

The outline of this paper is as follows. In Section 2, we introduce a mixed-integer bilinear formulation for the joint design and pricing problem and discuss its properties. In Section 3, we prove that some lower level constraints can be moved to the first level, thus reducing the size of the problem. In Section 4, we describe a solution algorithm. Finally numerical results are presented and analyzed in Section 5.

2 A joint design and pricing model

Let us consider a network based on the underlying graph $G = (N, A)$, with node set N and arc set A. A node represents either a supply site, a demand site, or the endpoints of an arc on which goods are carried. The set of arcs is partitioned into two subsets A_1 and A_2 where A_1 denotes the set of links operated by the leader and A_2 the set of links operated by its competitors. With each arc $a \in A_1$, we associate a tariff T_a, to be determined by the leader, a fixed opening cost f_a and an operating cost c_a charged to the leader. Arcs in A_2 are tariff-free and only bear a unit cost d_a which is outside the control of the leader. Demand is modelled by a set K of commodities. These may represent distinct physical goods or identical physical goods associated with different points of origin and destination. Each commodity is associated with an origin-destination pair

$(o(k),d(k))$. The demand vector b^k corresponding to commodity k is specified by:

$$b_i^k = \begin{cases} n^k & \text{if } i = o(k), \\ -n^k & \text{if } i = d(k), \\ 0 & \text{otherwise,} \end{cases}$$

where n^k represents the amount of flow of commodity k to be shipped from $o(k)$ to $d(k)$. The variable x_a^k (respectively y_a^k) denotes the flow of commodity k on arc $a \in A_1$ (respectively $a \in A_2$). The binary variable v_a, associated with each arc $a \in A_1$, indicates whether ($v_a = 1$) or not ($v_a = 0$) arc a belongs to the network design.

The leader's variables are either discrete (design variables) or real-valued (tariffs). Lower level variables, i.e., flows, are real-valued. Based on the above notation, the joint design and pricing problem can be formulated as a mixed bilevel program with bilinear objectives and linear constraints. The commodity flows x_a^k and y_a^k correspond to an optimal solution of the lower level linear program parameterized by the upper level tariffs T_a, which is solved on the sub-network resulting from the binary variables v_a:

$$\text{(JDP)} \quad \max_{T,v} \quad \sum_{k \in K} \sum_{a \in A_1} T_a x_a^k - \sum_{a \in A_1} f_a v_a - \sum_{k \in K} \sum_{a \in A_1} c_a x_a^k \tag{1}$$

$$\text{s.t.} \quad v_a \in \{0,1\} \qquad \forall a \in A_1, \tag{2}$$

where (x, y) is an optimal solution of

$$\min_{x,y} \quad \sum_{k \in K} \Big(\sum_{a \in A_1} T_a x_a^k + \sum_{a \in A_2} d_a y_a^k \Big) \tag{3}$$

$$\text{s.t.} \quad A x^k + B y^k = b^k \qquad \forall k \in K, \tag{4}$$

$$x_a^k \leq n^k v_a \qquad \forall k \in K \quad \forall a \in A_1, \tag{5}$$

$$x^k, y^k \geq 0 \qquad \forall k \in K.$$

The upper level objective (1) is to maximize total net revenue and is expressed as the difference between the sum of revenues arising from tariffs T_a and the sum of fixed opening costs and operating costs. The objective of the lower level problem (3) is to minimize the total cost of the paths selected by network users. Constraints (5) state that arcs can only be used if they are open. Constraints (4) represent the flow balance equations.

For specific tariff levels and design variables, the flow repartition for the lower level problem is given by shortest origin-destination paths on the sub-network composed of tariff-free and tariff arcs that are open. We assume that, given the choice between paths of equal cost, the path selected is the one yielding the highest profit for the leader. As in Labbé *et al.* [7], we also assume that:

- there cannot exist a tariff schedule that generates profits and simultaneously creates a negative cost cycle in the network,
- there exists at least one path composed of tariff-free arcs for each origin-destination pair.

These assumptions imply that the lower level optimal solution corresponds to a set of shortest paths, and that the upper level profit is bounded from above. A feasible upper bound on the profit is provided by the following proposition.

Figure 1: Upper bound on the profit not reached at the optimal solution.

Proposition 1. *An upper bound on the leader's profit is the difference between the follower's optimal objective corresponding to infinite tariffs, and the optimum value of the classical network design problem obtained by setting tariffs to* c_a.

Proof

Let us perform the change of variable $T' = T - c$, which is tantamount to increasing the base cost of every tariff link $a \in A_1$ from 0 to c_a at the lower level. For fixed design vector v, the resulting problem is of the form considered by Labbé et al. [7], who derived the valid upper bound $U - L(v)$, where U denotes the cost of a lower level solution when access to tariff arcs is denied (infinite tariffs), and $L(v)$ is the cost of a shortest path solution with $T'_a = 0$ and cost set to c_a on each link $a \in A_1$. It follows that a valid bound for the value of an optimal solution to JDP is given by

$$\max_v \{U - L(v)\} = U - \min_v \{L(v)\}, \tag{6}$$

as claimed. □

Example

The example of Figure 1 shows that the upper bound need not be reached. In this example, demand is set to 2 on origin-destination pair 1-2 and to 4 on pair 3-4, while (5,6) is the sole tariff link. Fixed opening and operating costs for the leader are set, respectively, to 1 and 0. The optimal solution, corresponding to a profit of 11 units is reached for $T_{5,6} = 2$. However the upper bound on the profit is equal to $40 - 23 = 17$. □

Now, taken into account that the entire demand associated with a given OD pair can be assigned to a single shortest path, one can, without loss of generality, reformulate JDP as:

$$\max_{T,v} \quad \sum_{a \in A_1} \sum_{k \in K} n^k T_a x_a^k - \sum_{a \in A_1} f_a v_a - \sum_{a \in A_1} \sum_{k \in K} n^k c_a x_a^k$$

$$
\begin{array}{lll}
\text{s.t.} & v_a \in \{0,1\} & \forall a \in A_1, \\
\\
\min\limits_{x,y} & \sum\limits_{k\in K} n^k (\sum\limits_{a\in A_1} T_a x_a^k + \sum\limits_{a\in A_2} d_a y_a^k) & \\
\text{s.t} & Ax^k + By^k = e^k & \forall k \in K, \\
& x_a^k \le v_a & \forall a \in A_1 \; \forall k \in K, \\
& x^k, y^k \ge 0 & \forall k \in K,
\end{array}
\tag{7}
$$

where

$$
e_i^k = \begin{cases} 1 & \text{if } i = o(k), \\ -1 & \text{if } i = d(k), \\ 0 & \text{otherwise.} \end{cases}
$$

For fixed design vector v, the resulting problem reduces to a multicommodity toll optimization problem that can be reformulated as a mixed integer program (see Brotcorne et al. [4]). This formulation readily extends to a MIP formulation for JDP through incorporation of the design variables v_a. In Section 5 (Numerical Results), this formulation is solved using the commercial software CPLEX and serves as a testbed for our method, on small problem instances.

In the case where there is only one OD pair, JDP reduces to the toll optimization problem analyzed by Brotcorne *et al.* ([3], [4]). Indeed, the binary flow variables x_a can then replace the design variables v_a, and the problem formulation becomes:

$$
\begin{array}{lll}
\max\limits_{T,x} & \sum\limits_{a\in A_1} nT_a x_a - \sum\limits_{a\in A_1} f_a x_a - \sum\limits_{a\in A_1} nc_a x_a & \\
\text{s.t.} & x_a \in \{0,1\} & \forall a \in A_1, \\
\\
\min\limits_{x,y} & n(\sum\limits_{a\in A_1} T_a x_a + \sum\limits_{a\in A_2} d_a y_a) & \\
& Ax + By = e, & \\
& x, y \ge 0. &
\end{array}
$$

Next one defines modified tariffs $\tilde{T}_a = T_a - \frac{1}{n} f_a - c_a$ and obtains the toll optimization problem:

$$
\begin{array}{ll}
\max\limits_{\tilde{T}} & \sum\limits_{a\in A_1} n\tilde{T}_a x_a \\
\\
\min\limits_{x,y} & \sum\limits_{a\in A_1} (\tilde{T}_a + c_a + f_a/n) x_a + \sum\limits_{a\in A_2} d_a y_a \\
\text{s.t.} & Ax + By = e, \\
& x, y \ge 0.
\end{array}
$$

Note that dropping the flow integrality constraints at the upper level is justified by the fact that the lower level constraints are totally unimodular and that it is not in the interest of the leader to induce noninteger (split) flows.

3 Moving constraints from the lower to the upper level

For general bilevel programs, constraints involving both upper and lower level variables cannot be moved freely from one level to the other, without altering both the feasible set and the optimal solution of the bilevel program. Upper level constraints are transparent to the follower, and can only be induced through a proper choice of the leader's tariffs. Even in the simpler case of linear bilevel programming, the feasible set corresponding to joint upper level constraints may be disconnected. This explains why the presence of such constraints may seem like a nuisance from the algorithmic point of view, although the opposite is true for JDP, as we shall see.

By transferring constraints to the lower level, we clearly obtain a relaxation of the original program. It is a remarkable feature of JDP, and the wider class of bilinear bilevel programs to which it belongs, that one can perform this operation, which will be exploited in the design of a solution algorithm.

Proposition 2. *Let x, T, $c \in \mathbb{R}^{m_1}$, y, $d \in \mathbb{R}^{m_2}$, $b^1 \in \mathbb{R}^n$, b^2 a vector of nonnegative components in $\mathbb{R}^r$, $E, F \in \mathbb{R}^{n\times m_2}$ and G a nonnegative matrix in $\mathbb{R}^{r\times m_1}$. Then the following bilinear bilevel programs are equivalent, in the sense that an optimal solution to P1 can be matched to a feasible solution of P2 with the same objective value, and vice versa.*

$$
\begin{array}{llll}
\text{(P1)} & \max\limits_{T} & Tx & \\
& & & \\
& \min\limits_{x,y} & Tx + dy & \\
& \text{s.t.} & Ex + Fy = b^1, & \\
& & Gx \le b^2, & \\
& & x, y \ge 0, &
\end{array}
\qquad
\begin{array}{lll}
\text{(P2)} & \max\limits_{T} & Tx \\
& & Gx \le b^2, \\
& & \\
& \min\limits_{x,y} & Tx + dy \\
& \text{s.t.} & Ex + Fy = b^1, \\
& & x, y \ge 0.
\end{array}
$$

Proof

Let us replace the lower level problems of P1 and P2 by their respective primal-dual optimality conditions, where λ and δ are the dual variables associated with constraints of P1 and their equivalent in P2. This yields

$$
\begin{array}{lll}
\text{(P1}'\text{)} & \max\limits_{T,x,y,\lambda,\delta} & Tx \\
& \text{s.t.} & Ex + Fy = b^1, \\
& & Gx \le b^2, \\
& & \lambda E + \delta G \le T, \\
& & \lambda F \le d, \\
& & (d - \lambda F)y = 0, \\
& & (T - \lambda E - \delta G)x = 0, \\
& & \delta(b^2 - Gx) = 0, \\
& & x, y \ge 0, \\
& & \delta \le 0.
\end{array}
\qquad
\begin{array}{lll}
\text{(P2}'\text{)} & \max\limits_{T,x,y,\lambda} & Tx \\
& \text{s.t.} & Ex + Fy = b, \\
& & Gx \le b^2, \\
& & \lambda E \le T, \\
& & \lambda F \le d, \\
& & (d - \lambda F)y = 0, \\
& & (T - \lambda E)x = 0, \\
& & x, y \ge 0.
\end{array}
$$

Let $(T^\star, x^\star, y^\star, \lambda^\star)$ be an optimal solution of P2$'$. By setting $\delta^\star = 0$, one obtains a solution $(T^\star, x^\star, y^\star, \lambda^\star, 0)$ of P1$'$ with the same objective value.

Conversely, let $s^\star = (T^\star, x^\star, y^\star, \lambda^\star, \delta^\star)$ be an optimal solution of P1$'$, and consider the solution $s' = (T^\star - \delta^\star G, x^\star, y^\star, \lambda^\star, 0)$. By construction (see the sixth constraint of P1$'$), s' is feasible for P1$'$ and also for P2$'$ ($\delta = 0$). Moreover, from the nonnegativity of the matrix G and the negativity of $\delta^\star$, we have that the objective function associated with s' is at least as good as the one associated with $s^\star$, i.e., is also an optimal solution of P1$'$. Thus, moving the constraint $Gx \le b^2$ does not reduce the value of the objective. This concludes the proof. □

Corollary 1. *The capacity constraints (5) of JDP can be moved to the upper level.*

Proof

It is sufficient to show that, for fixed design vector v, the resulting pricing problem can be written in the format P1. This is achieved, very simply, by introducing total arc flow variables

$$x_a = \sum_{k \in K} x_a^k.$$

The resulting bilevel program is:

$$\begin{aligned}
\max_{T} \quad & \sum_{a \in A_1} T_a x_a - \sum_{a \in A_1} c_a x_a \\
\min_{x,y} \quad & \sum_{a \in A_1'} T_a x_a + \sum_{a \in A_2} d_a y_a \\
& Ax^k + By^k = b^k \qquad \forall k \in K, \\
& x_a^k \leq n^k v_a \qquad \forall k \in K \quad \forall a \in A_1, \\
& x_a = \sum_{k \in K} x_a^k \qquad \forall a \in A_1, \\
& y_a = \sum_{k \in K} y_a^k \qquad \forall a \in A_2, \\
& x^k, y^k \geq 0 \qquad \forall k \in K,
\end{aligned}$$

which is in the required format, with the obvious correspondences between vectors and matrices. □

4 A solution procedure for JDP

The difficulty in solving JDP is twofold: the presence of binary variables and the complementarity constraints arising in the optimality conditions of the lower level linear program. In this section, we propose an iterative algorithm that adapts Lagrangian relaxation to the bilevel framework. We treat constraints (7) as the 'complicating' ones; these are appended to the objective to form the usual Lagrangian function. To evaluate the dual function, one has to solve the Lagrangian subproblem, itself an NP-hard toll optimization problem. This latter problem is solved using a variant of the primal-dual algorithm proposed in Brotcorne et al [3]. More precisely, the subproblem is reformulated as a single level bilinear problem through the use of an exact penalty function applied to the lower level complementarity term. Next, we update sequentially the upper and lower level variables, increasing the penalty parameter when no progress is achieved. Whenever a basis change occurs at the lower level, tariffs that are optimal with respect to the new bases are computed. This 'inverse optimization' procedure actually solves a modified multicommodity flow problem.

Let us now detail the procedure. The dual function, for a given nonnegative vector u, is obtained by solving the bilevel program:

$$\begin{aligned}
(\mathrm{LSP}(u)) \qquad L(u) = \max_{T,v,x,y,\lambda} \quad & \sum_{a \in A_1} \sum_{k \in K} n^k T_a x_a^k - \sum_{a \in A_1} f_a v_a - \sum_{a \in A_1} \sum_{k \in K} n^k c_a x_a^k \\
& + \sum_{a \in A_1} \sum_{k \in K} u_a^k (v_a - x_a^k) \\
\text{s.t.} \quad & v_a \in \{0,1\} \qquad \forall a \in A_1,
\end{aligned}$$

$$
\begin{aligned}
\min_{x,y} \quad & \sum_{k\in K} n^k(\sum_{a\in A_1} T_a x_a^k + \sum_{a\in A_2} d_a y_a^k) & \\
\text{s.t} \quad & Ax^k + By^k = e^k & \forall k \in K, \\
& x^k, y^k \geq 0 & \forall k \in K.
\end{aligned}
$$

Since, for each $u \geq 0$, LSP(u) is a relaxation of JDP, the solution $L(u)$ to LSP(u) is an upper bound on the optimal value of JDP. The best upper bound is obtained by solving the Lagrangian Dual Problem:

$$
\text{(DL)} \quad \min\{L(u) : u \geq 0\}. \tag{8}
$$

We solve DL using an algorithm inspired from subgradient optimization with a predetermined stepsize sequence. A subgradient $g(u)$ of $L(u)$ is given by $v(u) - x(u)$, where $(v(u), x(u))$ is an optimal partial solution of LSP(u). Since LSP(u) is nonconvex, the computation of an approximate subgradient is based on a primal-dual algorithm (inner iteration) that will be described later. The resulting algorithm (outer iteration) is outlined below. The number of commodities is $|K|$, γ_j denotes the step size at iteration, Z^*represents the current best leader's objective value and ϵ is a small value which will be precised in the numerical results section.

ALGORITHM JDP (outer iteration)

Step 0 : (initialization)

- $u_a^0 \leftarrow f_a/|K| + \epsilon$; $Z^* \leftarrow -\infty$; $T^0 \leftarrow 0$
- $(x^0,\, y^0) \leftarrow$ an optimal lower level solution corresponding to T^0
- $j \leftarrow 1$

Step j : (outer iteration)

- $(T^j, v^j, x^j, y^j, \lambda^j) \leftarrow$ an approximate solution of LSP(u^{j-1})
- **if** solution improved **then** update Z^*
- $u^j \leftarrow \max\{0, u^{j-1} - \gamma^j(v^j - x^j)\}$
- **if** stopping criterion is met **then** halt
  else $j \leftarrow j + 1$

The primal-dual heuristic procedure used for solving the bilevel Lagrangian subproblem LSP(u) is defined as follows. We first replace the lower level program by its primal-dual optimality conditions:

$$
Z(u) \quad = \quad \max_{T,v,x,y,\lambda} \quad \sum_{a\in A_1}\sum_{k\in K} n^k T_a x_a^k - \sum_{a\in A_1} f_a v_a - \sum_{a\in A_1}\sum_{k\in K} n^k c_a x_a^k
$$

$$
\begin{aligned}
& + \sum_{a\in A_1}\sum_{k\in K} u_a^k(v_a - x_a^k) && \\
\text{s.t.}\quad & v_a \in \{0,1\} && \forall a \in A_1, \\
& Ax^k + By^k = e^k && \forall k \in K, \\
& x^k, y^k \geq 0 && \forall k \in K, \\
& \lambda^k A \leq T && \forall k \in K, \\
& \lambda^k B \leq d && \forall k \in K, \\
& n^k(Tx^k + dy^k - \lambda^k e^k) = 0 && \forall k \in K. \qquad (9)
\end{aligned}
$$

Next we penalize the constraints (9) stating the equality of the primal and dual objectives of the follower's subproblem, whose left-hand-side is nonnegative whenever (x^k, y^k) and λ^k are feasible for the primal and dual problems and each commodity $k \in K$, respectively. This yields the bilinear program:

$$
\begin{aligned}
\text{(PEN)}\quad \max_{T,v,x,y,\lambda}\quad & \sum_{a\in A_1}\sum_{k\in K} n^k T_a x_a^k - \sum_{a\in A_1} f_a v_a - \sum_{a\in A_1}\sum_{k\in K} n^k c_a x_a^k \\
& + \sum_{a\in A_1}\sum_{k\in K} u_a^k(v_a - x_a^k) - M_1 \sum_{k\in K} n^k\Big(\sum_{a\in A_1} T_a x_a^k + \sum_{a\in A_2} d_a y_a^k - \lambda^k e^k\Big) \\
\text{s.t.}\quad & v_a \in \{0,1\} && \forall a \in A_1, \\
& Ax^k + By^k = e^k && \forall k \in K, \\
& x^k, y^k \geq 0 && \forall k \in K, \\
& \lambda^k A \leq T && \forall k \in K, \\
& \lambda^k B \leq d && \forall k \in K,
\end{aligned}
$$

where M_1 is a large positive number. By rewriting the objective, we obtain the equivalent formulation

$$
\begin{aligned}
\text{(PEN')}\quad \max_{T,v,x,y,\lambda}\quad & \sum_{a\in A_1}\sum_{k\in K} ((1-M_1)n^k T_a - u_a^k - n^k c_a)x_a^k - M_1 \sum_{a\in A_2}\sum_{k\in K} n^k d_a y_a^k \\
& + M_1 \sum_{k\in K} n^k \lambda^k e^k + \sum_{a\in A_1}\sum_{k\in K} (u_a^k - f_a) v_a \\
\text{s.t.}\quad & v_a \in \{0,1\} && \forall a \in A_1, \\
& Ax^k + By^k = e^k && \forall k \in K, \\
& x^k, y^k \geq 0 && \forall k \in K, \\
& \lambda^k A \leq T && \forall k \in K, \\
& \lambda^k B \leq d && \forall k \in K.
\end{aligned}
$$

This latter problem is separable in v and T, x, y, λ. Binary variables v_a ($a \in A_1$) are simply set to one if the corresponding term

$$\sum_{k\in K} u_a^k - f_a$$

is positive. The procedure for solving PEN, which is illustrated in Figure 2, iterates between the leader's tariff vector and the follower's commodity flows x^k and y^k. The overall aim of the primal-dual scheme is to induce basis changes for the follower's problem. In this process, extremal flow assignments corresponding to distinct values of the tariff vector T are generated, and we expect one of these combinations to be of high quality for JDP.

At a given iteration, the tariff vector T solves the penalized problem PEN for fixed flow vectors x^k, y^k (Step 1). Next the flow variables on both the tariff and tariff-free arcs solve PEN for fixed tariff vector T (Step 2); this is achieved by computing shortest paths for all OD pairs. These solutions can be improved by noting that, for a given lower level flow vector (x, y), one can derive the profit-maximizing tariff vector that is compatible with (x, y) by solving a simple linear program (Step 3). The main components of the primal-dual algorithm are made explicit

Figure 2: Primal-dual algorithm for the Lagrangian Subproblem.

below. At Step 0, flows on tariff and tariff-free arcs are initialized at values that achieved the best leader profit obtained at the previous main iterations. The design vector v is then set to the optimal solution solution of the problem:

$$\begin{aligned} (\text{PEN1}(v)) \quad \max_{v} \quad & \sum_{a \in A_1} \sum_{k \in K} (u_a^k - f_a) v_a \\ \text{s.t.} \quad & v_a \in \{0, 1\} \qquad \forall a \in A_1. \end{aligned}$$

At Step 1, for fixed commodity flows x^k, let T and λ be solutions of the problem:

$$\begin{aligned} (\text{PEN2}(T, \lambda)) \quad \max_{T,\lambda} \quad & (1 - M_1) \sum_{a \in A_1} \sum_{k \in K} n^k x_a^k T_a + M_1 \sum_{k \in K} n^k \lambda^k e^k \\ \text{s.t.} \quad & \lambda^k A \leq T && \forall k \in K, \\ & \lambda^k B \leq d && \forall k \in K. \end{aligned}$$

This linear program can be easily solved using a linear programming software such as CPLEX. Its dual is a multicommodity flow problem. At Step 2, the multicommodity flows x^k and y^k solve the lower level problem, for fixed tariff vector T.

$$\begin{aligned} (\text{PEN3}(x, y, \lambda)) \quad \max_{x,y,\lambda} \quad & \sum_{a \in A_1} \sum_{k \in K} ((1 - M_1) n^k T_a - u_a^k - n^k c_a) x_a^k \\ & - M_1 \sum_{a \in A_2} \sum_{k \in K} n^k d_a y_a^k + M_1 \sum_{k \in K} n^k \lambda^k e^k \\ \text{s.t.} \quad & A x^k + B y^k = e^k && \forall k \in K, \\ & x^k, y^k \geq 0 && \forall k \in K, \\ & \lambda^k A \leq T && \forall k \in K, \\ & \lambda^k B \leq d && \forall k \in K. \end{aligned}$$

This problem can be decomposed into a shortest path problem to determine the arc flows x^k, y^k and a linear program to obtain λ^k.

$$
\begin{aligned}
(\text{PEN4}(x,y)) \quad \max_{x,y} \quad & \sum_{a\in A_1}\sum_{k\in K}((1-M_1)n^kT_a - u_a^k - n^kc_a)x_a^k \\
& -M_1 \sum_{a\in A_2}\sum_{k\in K} n^k d_a y_a^k \\
\text{s.t.} \quad & Ax^k + By^k = e^k && \forall k \in K, \\
& x^k, y^k \geq 0 && \forall k \in K.
\end{aligned}
$$

$$
\begin{aligned}
(\text{PEN5}(\lambda)) \quad \max_{\lambda} \quad & M_1 \sum_{k\in K} n^k \lambda^k e^k \\
\text{s.t.} \quad & \lambda^k A \leq T && \forall k \in K, \\
& \lambda^k B \leq d && \forall k \in K.
\end{aligned}
$$

At Step 3, the algorithm computes a common tariff vector that maximizes the total profit of the leader while maintaining the lower level optimality of the current commodity flows. The structure of this program is that of an uncapacitated multicommodity network flow problem, and is thus ‘easy’.

$$
\begin{aligned}
(\text{T-OPT}) \quad \max_{T,\lambda} \quad & \sum_{a\in A_1}\sum_{k\in K} n^k T_a x_a^k - \sum_{a\in A_1}\sum_{k\in K} n^k c_a x_a^k \\
\text{s.t.} \quad & \lambda^k A \leq T && \forall k \in K, \\
& \lambda^k B \leq d && \forall k \in K, \\
& n^k(Tx^k + dy^k - \lambda^k e^k) = 0 && \forall k \in K.
\end{aligned}
$$

The algorithm is outlined below, where j denotes the index of the outer iteration, α is a relaxation factor $\alpha \in (0,1)$ and Z^* represents the current best objective value.

PRIMAL-DUAL ALGORITHM (inner iteration)

Step 0 : (initialization)

- $x_0 \leftarrow x^{j-1}$; $\gamma \in [0,1]$
- **if** $\sum_{k\in K}(u_a^k - f_a) \geq 0$ **then** $v_a^j \leftarrow 1$ **else** $v_a^j \leftarrow 0$
- $l \leftarrow 1$ (minor iteration index)

Step 1 : (computation of T_l and λ_l)

- for fixed x_{l-1}^k and y_{l-1}^k, $(T_l, \lambda_l) \leftarrow$ solution of PEN2(T,λ)

Step 2 : (computation of x_l and y_l)

- solve (PEN4(x,y)) for the tariff vector $(1-\alpha)T_l + \alpha T_{l-1}$

Step 3 : (computation of optimal tariffs for given flows)

if flows are identical to those of some previous iteration
  then go to Step 4
  else

- $\tilde{T} \leftarrow$ optimal solution of T-OPT

- **if** $x_{la} = 1$ **then** $\tilde{v}_a \leftarrow 1$ **else** $\tilde{v}_a \leftarrow 0$

- let $\tilde{Z}(\tilde{T}, \tilde{v}) = \tilde{T}x_l - f\tilde{v} - cx_l$.

- **if** $\tilde{Z} > Z^*$ then $Z^* \leftarrow \tilde{Z}$ and $(T^*, v^*, x^*, y^*, \lambda^*) \leftarrow (\tilde{T}, \tilde{v}, x_l, y_l, \lambda_l)$

Step 4 : (Stopping criterion)

if stopping criterion met **then** $(T^j, v^j, x^j, y^j, \lambda^j) \leftarrow (T^*, v^*, x^*, y^*, \lambda^*)$

else $l \leftarrow l + 1$, increase M_1 and go to Step 1

5 Numerical Results

The heuristic developed was tested on sets of randomly generated grid networks with 60 nodes (5×12), 208 two-way arcs, 10 , 20 and 40 origin-destination pairs, and where the proportion of tariff arcs varies from 5% to 20%. Two cost structures, symmetric and asymmetric, are considered for two-way arcs. The random generation process is described in Brotcorne *et al.* [3], while the algorithm is coded in C and implemented on an Enterprise 10 000 workstation.

For the heuristic, the Lagrange multipliers u_a^0 are initialized to $(f_a/|K|) + \epsilon$ where ϵ is fixed to 0.01. Such values result in the opening of all tariff arcs at the initial subgradient iteration. The step length along the subgradient direction is set to 5 and the algorithm is halted as soon as the profit value of JDP corresponding to $(T^*, v^*, x^*, y^*, \lambda^*)$ is not improved after 30 subgradient iterations. In solving the Lagrangian subproblem, the penalty factor M_1 is initialized to 1.3 and incremented by 0.05 at the end of each primal-dual iteration. The number of primal-dual iterations is set to 20. The setting of these parameters achieves a trade-off between two conflicting objectives: maximizing the number of bases visited and reducing the time-consuming process of optimization with respect to each basis. In order to induce basis changes, the parameter γ is set to a 'high' value, namely $\gamma = 0.5$.

The numerical results are summarized in Tables 1 to 8. In Tables 1, 2, 5 and 7, 8, the opening costs are commensurate to link cost and usage. With respect to these basic scenarios, sensitivity analyses are performed with respect to fixed costs. In Tables 3, 4, 6, each line corresponds to an average taken over 5 problem instances.

The first column '%TA' of each table provides the percentage of tariff arcs. Label '#TA' refers to the number of tariff arcs with nonzero flow in the final solution. Label 'DI' refers to the index of the subgradient iteration at which the solution was reached. Labels '#BAS' and 'BOPT' refer respectively to the number of follower basis met during the iterative process and the basis number associated with the heuristic solution. Label '%OPT' refers to the ratio of the heuristic objective over the optimal solution achieved by the mixed integer programming code CPLEX 8.1, which was halted whenever either a time limit of 8 hours was reached, a node limit of 400 000 was reached, or memory requirements exceeded one gigabyte. In the case of premature termination, the optimum value is replaced by the best lower bound achieved. This is indicated by a star (*) in the tables' sixth column. The two 'CPU' labels refer to running times expressed in seconds. The label 'GAP' refers to the integrality gap; if the optimal solution is not available, then GAP is computed with respect to the best integer solution found by CPLEX. Finally, in Tables 4 and 6, the label 'NOPT' refers to the number of problems solved to optimality. For larger instances, Tables 7, 8, only columns related to the heuristic are reported.

As a general rule, the Lagrangian relaxation scheme produces high-quality solutions quite rapidly and consistently. Typically, the solutions lie within 5% of optimality. With the exception of the smallest problems (10 commodities, 5% or 10% tariff arcs) the proposed heuristic is much faster than the exact MIP algorithm. It has been observed that even if the CPU time required by the heuristic increases with the percentage of tariff arcs and the number of commodities, this increase is more modest than for CPLEX.

All 10-commodity instances (except instances with nul fixed cost and 20% of tariff arcs) could be solved by CPLEX, despite high duality gaps (up to 76.67%). However, running times grow fast and in an unstable fashion as as the number of tariff arcs is increased from 5% to 20%. In contrast, Table 1 shows moderate CPU times for the Lagrangian algorithm, for which both symmetric and asymmetric 20-commodity instances are solved, with no significant decrease in solution quality (see Table 2). As a general rule, the symmetric instances prove more difficult. Beyond 20 commodities, these problems could not be solved to optimality by CPLEX.

The sensitivity analyses confirm some intuitive results. For instance, when the ratio of opening to operating costs is high, most tariff arcs are closed. In this case the combinatorial structure is 'weak' and it is not surprising to observe that CPLEX can solve easily this class of problems. The converse conclusion holds when this ratio is low.

Upper level envelopes on profit function are given in Tables 4 and 3 for instances with respectively 40 commodities and 20 commodities. These functions have a similar shape. They increase sharply at the beginning of the process, and flatten out in the middle and at the end of the algorithmic process.

6 Conclusion

In this paper, we presented an algorithm for solving a mixed continuous-discrete design problem that arises naturally when modelling pricing decisions over transportation networks. The algorithm is based on the novel application of Lagrangian relaxation within a bilevel programming framework, and solved to near-optimality randomly generated instances involving more than 4 000 variables. These encouraging results comfort our belief that this methodology can be generalized to problems of the same type involving capacities on the links of the network.

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% TA	#TA	DI	#BAS	BOPT	%OPT	CPU	GAP	CPU
			Heuristic				**Cplex**	
	5	14	44	22	1.00	23	41.73	21
	3	16	31	31	1.00	21	22.52	4
5	2	10	75	60	1.00	20	38.11	7
	1	19	11	10	1.00	19	13.30	2
	3	12	24	23	1.00	18	11.17	3
average	2.8	14.2	37.0	29.2	1.00	20.2	25.37	7.4
	4	26	122	120	1.00	31	30.06	23
	8	12	93	52	0.92	22	40.49	45
10	4	18	31	29	1.00	21	7.22	5
	4	55	333	249	1.00	58	76.67	67
	5	11	71	37	1.00	20	69.66	43
average	5.0	24.4	130.0	97.4	0.98	30.4	44.82	36.6
	7	29	114	104	1.00	35	9.35	49
	4	20	63	57	1.00	23	19.70	64
15	12	24	85	85	1.00	29	3.93	32
	6	18	60	49	1.00	23	4.91	8
	7	25	114	92	1.00	30	11.31	24
average	7.2	23.2	87.2	77.4	1.00	28.0	9.84	35.4
	5	35	167	132	0.95	42	8.27	14
	8	44	139	138	1.00	43	13.25	234
20	10	58	343	236	1.00	74	22.88	1509
	8	29	160	160	1.00	42	14.97	137
	2	41	173	169	1.00	45	35.11	5
average	6.6	41.6	392.8	334.0	0.99	49.2	18.90	379.8

Table 1: Asymmetric networks, 10 commodities

	Heuristic						Cplex	
% TA	#TA	DI	#BAS	BOPT	%OPT	CPU	GAP	CPU
	6	13	175	78	0.95	55	67.49	921
5	5	11	46	40	1.00	37	27.06	15
	4	37	469	454	0.95	119	116.57	247
	4	13	26	16	1.00	41	52.89	30
	5	24	82	80	1.00	58	43.74	44
average	4.8	19.8	159.6	133.6	0.98	62.0	61.55	251.4
	7	53	298	284	0.88	133	82.86	2015
10	7	32	118	106	0.97	77	18.82	855
	3	9	234	219	1.00	83	72.80	1048
	4	14	23	15	0.99	40	33.96	35
	6	27	264	232	1.00	91	56.66	445
average	5.4	27.0	187.4	171.0	0.97	84.8	53.02	879.6
	7	41	234	219	1.00	137	58.43	14911
15	9	46	263	230	* 1.24	148	75.2	28983
	7	22	139	134	1.00	82	22.36	3209
	12	18	105	89	1.00	90	27.99	4842
	14	37	239	234	1.00	135	14.30	20455
average	9.8	32.8	196.0	181.2	1.05	118.4	39.65	14480.0
	18	44	415	367	1.00	455	25.10	29358
20	10	22	79	74	1.00	79	29.10	908
	15	32	258	196	* 1.04	181	24.64	28865
	12	37	533	251	* 0.97	392	28.69	29184
	8	32	456	294	* 1.11	336	37.89	28919
average	12.6	33.4	348.2	236.4	1.02	268.6	29.08	23446.8

Table 2: Asymmetric networks, 20 commodities

		Heuristic							Cplex	
f_a	% TA	NOPT	#TA	DI	#BAS	BOPT	LB/OPT	CPU	GAP	CPU
0	5	5	8.8	0.2	7.6	3.8	0.99	12.0	14.07	14.2
0	10	5	18.4	0.2	8.6	5	0.99	12.8	10.95	56.8
0	15	5	23.6	0.4	6.8	5.2	0.99	13.2	8.58	489.8
0	20	1	31	5.2	6.8	8.0	1.02	17.4	14.30	5070.4
15	5	5	5.0	4.6	30.8	17.6	1.00	15.8	22.97	12.4
15	10	5	9.8	16.0	80.8	51.5	0.99	27.0	22.91	54.17
15	15	5	10.6	22.4	76.4	63.4	0.99	27.2	14.87	198.20
15	20	5	15.0	22.0	119.8	81.4	0.99	38.4	11.54	951.4
30	5	5	2.8	14.2	37.0	29.2	1.00	20.2	25.37	7.4
30	10	5	5.0	24.4	130.0	94.4	0.98	30.4	44.82	36.6
30	15	5	7.2	23.2	87.2	77.4	1.00	28.0	9.84	35.4
30	20	5	6.6	41.6	392.8	334.0	0.99	49.2	18.90	379.8
60	5	5	1.2	11.0	20.6	18.8	1.00	16.2	15.00	10.9
60	10	5	2.4	31.8	137.6	123.4	31.4	65.01	6.8	
60	15	5	3.0	43.6	109.6	108	0.99	43.6	24.18	14.6
60	20	5	3.6	33.2	179.4	137	0.98	43.4	35.79	60.4

Table 3: Fixed cost sensitivity: asymmetric networks, 10 commodities

					Heuristic				Cplex	
f_a	% TA	NOPT	#TA	DI	#BAS	BOPT	LB/OPT	CPU	GAP	CPU
0	5	5	9.0	0.6	16.0	12.0	0.99	28.4	23.04	1959.6
0	10	3	15.6	0.4	17.6	8.4	1.03	31.8	27.00	8682.0
0	15	0	27.0	16.0	18.2	17.0	1.03	53.4	19.88	15974.0
0	20	0	34.4	0.8	9.8	7.4	1.05	45.6	24.19	16453.4
30	5	5	4.8	19.8	159.6	133.60	0.98	6.02	61.55	251.4
30	10	5	5.4	27.0	187.4	171.2	0.97	84.8	53.02	879.6
30	15	4	9.8	32.8	196.0	181.2	1.05	118.4	39.65	14480.0
30	20	1	12.6	33.4	348.2	236.4	1.02	268.6	29.08	23446.8
60	5	5	2.0	29.2	177.0	165.6	1.00	67.8	89.02	29.6
60	10	5	3.0	33.2	232.4	187.8	0.97	86.6	271.19	35.8
60	15	5	5.6	37.2	317.6	263.6	1.00	140.8	39.87	3500.2
60	20	4	7.0	49.4	466.8	426.4	1.02	314.6	33.04	6561.0

Table 4: Fixed cost sensitivity, asymmetric networks, 20 commodities

			Heuristic				Cplex	
% TA	#TA	DI	#BAS	BOPT	%OPT	CPU	GAP	CPU
	6	10	46	46	1.00	55	42.90	580
	10	2	6	2	1.00	32	45.12	1667
5	6	7	39	34	1.00	42	53.6	961
	8	1	63	13	0.97	75	42.97	9220
	4	24	55	50	0.93	32	15.30	12
average	6.8	8.8	41.8	29	0.98	47.5	39.98	2488.0
	6	16	43	43	0.98	62	26.66	502
	9	14	177	106	* 1.00	97	48.59	29057
10	6	30	66	66	1.00	82	35.01	11373
	3	16	154	74	0.98	90	41.30	29
	9	19	205	123	* 2.38	128	142.96	28951
average	6.6	19	129	82.4	1.27	92.0	58.90	13982.4
	6	15	70	63	* 1.00	69	47.60	10560
	4	58	284	278	1.00	111	41.08	11094
15	10	28	191	170	* 1.03	98	46.03	29196
	13	80	660	645	* 1.22	543	75.12	29285
	10	24	238	231	* 1.12	155	51.38	29199
average	8.6	41	288.6	277.4	1.07	195.5	52.242	21866.8
	9	32	239	215	* 1.11	223	44.54	29207
	8	26	258	168	* 1.53	223	120.86	29301
20	5	26	375	357	1.00	196	18.24	203
	9	51	442	397	* 1.09	396	51.32	28917
	8	53	383	359	* 0.99	171	24.71	30138
average	7.8	37.6	339.4	299.2	1.14	241.9	51.934	23553.2

Table 5: Symmetric networks, 20 commodities

					Heuristic				Cplex	
f_a	% TA	NOPT	#TA	DI	#BAS	BOPT	LB/OPT	CPU	GAP	CPU
0	5	4	9.6	0.0	9.2	5.6	0.99	27.6	18.99	375
0	10	1	18.6	3.4	10.8	7.4	1.03	3.0	30.79	11941.8
0	15	0	25.4	8.2	14.8	13.6	1.19	47.6	50.07	20458.2
0	20	1	35.8	1.6	19.6	15.8	1.15	52.0	35.86	21470.4
30	5	5	6.8	8.8	41.8	29.0	0.98	47.5	39.98	2488.0
30	10	3	6.6	19.0	129.0	82.4	1.27	92.0	58.90	13982.4
30	15	1	8.6	41.0	288.6	277.4	1.07	195.5	52.242	21866.8
30	20	1	7.8	37.6	339.4	299.2	1.14	241.9	51.934	23553.2
60	5	5	4.2	22.4	86.0	82.8	1.00	63.6	53.39	158.6
60	10	5	3.8	27.0	168.4	154.8	0.98	79.4	40.41	543.8
60	15	3	5.4	38.6	356.2	271.0	0.96	176.0	145.67	12750.8
60	20	4	4.8	46.2	335.6	295.4	1.00	184.0	45.42	6695.6

Table 6: Fixed cost sensitivity: symmetric networks, 20 commodities;

	Heuristic					
% TA	#TA	DI	#BAS	BOPT	OPT	CPU
	2	30	239	231	118	101
	1	17	205	196	92	73
5	4	18	235	234	551	340
	5	31	248	209	753	179
	6	15	307	259	491	256
average	3.6	22.2	246.8	225.8	401.0	189.8
	5	47	172	166	352	112
	13	7	455	102	1028	419
10	4	36	711	449	644	331
	9	21	428	272	1081	327
	10	39	368	364	491	263
average	8.2	30	426.8	270.6	719.2	290.5
	9	94	1775	1572	2036	2649
	8	46	514	501	1582	466
15	6	42	742	556	573	661
	3	57	457	452	270	459
	10	23	597	258	931	600
average	7.2	52.4	817	667.8	1078.4	966.8
	9	60	1303	928	1294	2322
	12	48	980	628	1236	1686
20	13	81	1259	1007	1486	1909
	11	76	1236	846	942	2214
	2	35	484	484	491	282
average	9.4	60	1052.4	778.6	1089.8	1682.8

Table 7: Asymmetric networks, 40 commodities

			Heuristic			
% TA	#TA	DI	#BAS	BOPT	OPT	CPU
	6	15	307	259	753	211
	4	24	425	321	505	248
5	3	24	288	287	180	196
	5	26	355	318	282	228
	3	37	259	230	450	180
average	4.2	25.2	326.8	283	434	212.6
	3	21	338	319	364	229
	4	14	136	95	748	170
10	8	55	873	676	653	640
	2	23	412	303	86	300
	7	69	1090	829	467	1316
average	4.8	36.4	569.8	444.4	463.6	531.0
	6	36	387	378	578	540
	3	24	420	345	82	350
15	9	66	1303	1043	819	2053
	4	49	858	715	203	741
	4	38	464	446	330	406
average	5.2	42.6	686.4	585.4	402.4	818.0
	9	44	602	441	1292	1918
	6	46	558	512	567	649
20	12	67	1780	1279	2047	4495
	14	40	781	470	1324	2409
	14	50	830	718	1958	2092
average	11.0	49.4	910.2	684.0	1437.6	2312.6

Table 8: Symmetric networks, 40 commodities

Figure 3: Network with 20 commodities

Figure 4: Network with 40 commodities